

A table of coherent band-Gordian distances between knots

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Abstract

We introduce some criteria for two links, which are related by a coherent band surgery, using the determinant, and the Jones, HOMFLYPT, and Q polynomials. We give a table of coherent band-Gordian distances between two knots with up to seven crossings.

1 Introduction

There are several criterion for two links, which are related by a band surgery or crossing change. In this paper, we introduce further criteria using the determinant, and the Jones, HOMFLYPT, and Q polynomials. A band surgery and a crossing change are local changes in a link diagram as shown in Figure. 1. If we consider oriented links, there are two types for a band surgery according to an orientation; a coherent band surgery (Fig 2) and an incoherent one. In particular, an incoherent band surgery between two knots is called an $H(2)$ -move [14] (Figure. 3). Recently, these local moves are studied in connection with an application to the study of DNA site-specific recombination; see [5, 6, 9].

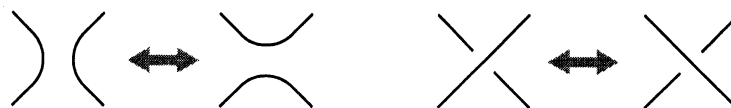


Figure 1: A band surgery and a crossing change.

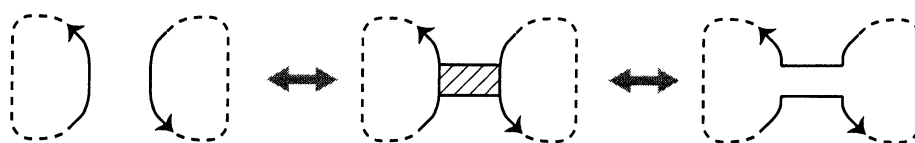


Figure 2: A coherent band surgery.

Given two links L and L' , we want to decide whether they are related by a band surgery or a crossing change. The signature and Arf invariant are most useful tools for this problem (Propositions 2.2 and 2.3). There are also several other methods to deal

with this problem: for a coherent band surgery, see [19, 21]; for a crossing change, see [30, 32, 35, 40, 41, 42]; for an $H(2)$ -move, see [20, 23, 26]; see also [1].

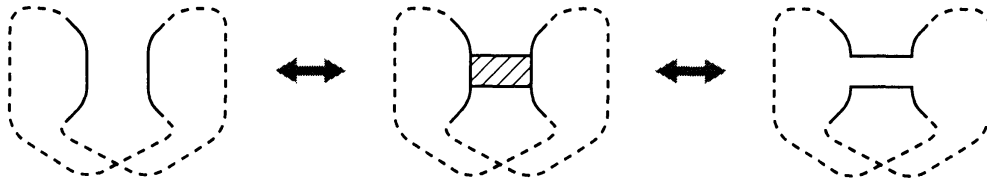


Figure 3: An $H(2)$ -move.

Our main results are two criteria: The first one is a condition on the determinant of a link or knot which is obtained from a 2-bridge knot by a coherent band surgery or $H(2)$ -move (Theorem 3.2), which is easily obtained by using a condition on the determinant of a knot obtained from a 2-bridge knot by a crossing change due to Hitoshi Murakami [32] (Proposition 3.1).

The second one uses some special values of the polynomial invariants. For the Jones polynomial, we have a criterion on two links which are related by a coherent band surgery [19, Theorem 2.2] (Theorem 4.2). Developing this, we obtain Theorem 4.6. In a similar way, for the HOMFLYPT polynomial we obtain Theorem 5.4 developing Proposition 5.1, and for the Q polynomial Theorem 6.2 developing Proposition 6.1. We give some examples for each of these criteria, which display the efficiency of them. In a forthcoming paper [24] we will make a detailed report on these criteria.

Notation. For knots and links with up to 9 crossings we use Rolfsen notations [38, Appendix C]. For a knot or link L , we denote by $L!$ its mirror image. For an oriented 2-component link with c crossings we use the notations c_n^2 and $c_n^{2'}$, where we usually suppose that linking number of c_n^2 is negative and that of $c_n^{2'}$ is positive as in Table 2 in [21]; more precisely, c_n^2 denotes an oriented link with negative linking number with diagram as in the table of [38] and $c_n^{2'}$ denotes one of the oriented links obtained from c_n^2 by reversing the orientation of one component.

2 Some invariants

The *Conway polynomial* $\nabla(L; z) \in \mathbb{Z}[z]$ [4], the *Jones polynomial* $V(L; t) \in \mathbb{Z}[t^{\pm 1/2}]$ [17], and the *HOMFLYPT polynomial* $P(L; v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ [10, 17, 36] are invariants of the isotopy type of an oriented link L , which are defined by the following formulas:

$$\nabla(U; z) = 1; \quad (1)$$

$$\nabla(L_+; z) - \nabla(L_-; z) = z\nabla(L_0; z); \quad (2)$$

$$V(U; t) = 1; \quad (3)$$

$$t^{-1}V(L_+; t) - tV(L_-; t) = (t^{1/2} - t^{-1/2})V(L_0; t); \quad (4)$$

$$P(U; v, z) = 1; \quad (5)$$

$$v^{-1}P(L_+; v, z) - vP(L_-; v, z) = zP(L_0; v, z), \quad (6)$$

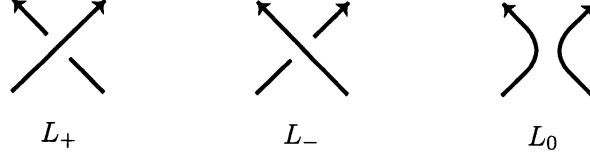


Figure 4: A skein triple.

where U is the unknot and (L_+, L_-, L_0) is a skein triple.

For a skein triple (L_+, L_-, L_0) , the link L_+ is obtained from L_- by changing a crossing, and vice versa, and the link L_0 is obtained from L_+ or L_- by a coherent band surgery, and vice versa. Conversely, it is easy to see the following:

Lemma 2.1. *If a c -component link L and a $(c+1)$ -component link M are related by a coherent band surgery, then there exist c -component links L_+ , L_- and $(c+1)$ -component links M_+ , M_- such that each of the following is a skein triple: (L_+, L, M) , (L, L_-, M) , (M_+, M, L) , (M, M_-, L) .*

For a c -component link L , $i^{c-1}V(L; -1)$ is an integer and the determinant $\det L$ is given by $\det L = |V(L; -1)|$. Putting $t = -1$ in Eq. (4), we obtain

$$-V(L_+; -1) + V(L_-; -1) = 2iV(L_0; -1); \quad (7)$$

Let (L_+, L_-, L_0) be a skein triple. Then Murasugi [34, Lemma 7.1] has shown:

$$|\sigma(L_\pm) - \sigma(L_0)| \leq 1. \quad (8)$$

Since we may consider the link L_+ or L_- as obtained from L_0 by a coherent band surgery, and vice versa, we have the following.

Proposition 2.2. (i) *If two oriented links L and L' are related by a coherent band surgery, then*

$$|\sigma(L) - \sigma(L')| \leq 1. \quad (9)$$

(ii) *If two oriented links L and L' are related by a crossing change, then*

$$|\sigma(L) - \sigma(L')| \leq 2. \quad (10)$$

The Arf invariant (or Robertello invariant) [37] of a knot K , $\text{Arf}(K)$, is given by

$$\text{Arf}(K) = a_2(K) \in \mathbf{Z}_2, \quad (11)$$

where $a_2(K)$ is the coefficient of z^2 of the Conway polynomial of K . Whenever an equality in this paper contains an Arf invariant it is to be understood in the sense of mod 2. We say that an oriented link L is related (in the sense of Robertello [37]) to a knot K if there exists a smooth embedding of a planar surface F in $S^3 \times I$ such that F meets $S^3 \times \{0, 1\}$ transversely in K and L , respectively. Let L be a proper link, that is, the sum of the linking numbers of any component of L with all the other components is even. We may define its Arf invariant to be the Arf invariant of any knot K related to it. In particular, we have:

Proposition 2.3. *If a knot K is obtained from a proper 2-component link L by a coherent band surgery, then $\text{Arf}(K) = \text{Arf}(L)$.*

3 Determinant of a link obtained from a 2-bridge knot by a band surgery

For relatively prime integers p, q with $p > q > 0$ and p odd, we let $S_{p,q}$ denote the 2-bridge knot for which the lens space of type (p, q) is the 2-fold branched cover of S^3 . More explicitly, let $a_1, a_2, a_3, \dots, a_n$ be positive integers obtained from the continued fraction

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots + \frac{1}{a_n}}}}. \quad (12)$$

Then $S_{p,q}$ is isotopic to a 2-bridge knot in Conway's normal form $C(a_1, a_2, a_3, \dots, a_{n-1}, a_n)$ as shown in Figure. 5, where the box containing an integer a or $-a$, $a > 0$, denotes a 2-braid as shown in Figure. 6. Also, $S_{p,-q}$ presents the mirror image of $S_{p,q}$; cf. [25, Sec. 2.1].

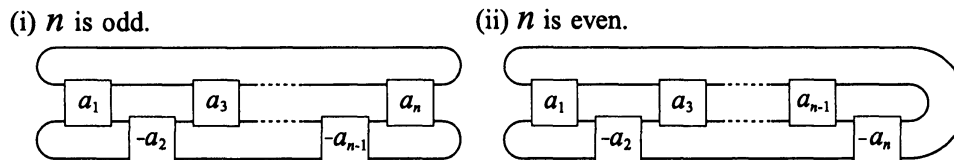


Figure 5: The 2-bridge knot $C(a_1, a_2, a_3, \dots, a_{n-1}, a_n)$.

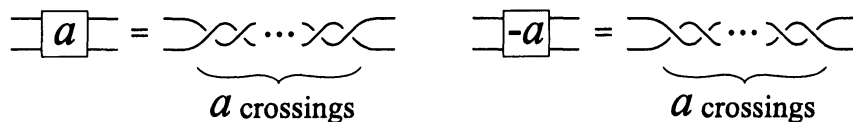


Figure 6: 2-braids.

The following criteria is due to H. Murakami [32, Corollary 2.8].

Proposition 3.1. *Suppose that a knot K is obtained from a 2-bridge knot $S_{p,q}$ by a crossing change. Then there exists an integer s such that:*

$$|\det K - p|/2 \equiv \pm qs^2 \pmod{p}. \quad (13)$$

Using this, we may deduce the following.

Theorem 3.2. *Suppose that a link L is obtained from a 2-bridge knot $S_{p,q}$ by a coherent or incoherent band surgery. Then there exists an integer s such that:*

$$\det L \equiv \pm qs^2 \pmod{p}. \quad (14)$$

Proof. Suppose that L and $S_{p,q}$ are related by a coherent band surgery. Then by Lemma 2.1 there exists a knot K such that $(K, S_{p,q}, L)$ is a skein triple. From Eq. (7) we have

$$-V(K; -1) + V(S_{p,q}; -1) = 2iV(L; -1), \quad (15)$$

which implies

$$2 \det L = |2iV(L; -1)| = |-V(K; -1) + V(S_{p,q}; -1)|. \quad (16)$$

Since K and $S_{p,q}$ are related by a crossing change, by Proposition 3.1 there exists an integer s such that Eq. (13) holds, which implies

$$\det K + p \equiv \det K - p \equiv \pm 2qs^2 \pmod{2p}, \quad (17)$$

Since $\det K = |V(K; -1)|$ and $p = |V(S_{p,q}; -1)|$, combining Eqs. (16) and (17), we obtain Eq. (14). □

By Theorem 3.2 a 2-bridge knot may have some condition on the values of $\det L$, where L is either a 2-component link with $d_{cb}(S_{p,q}, L) = 1$ or a knot with $d_2(S_{p,q}, L) = 1$. For 2-bridge knots with up to 8 crossings, Table 1 lists these values; the remaining 2-bridge knots $3_1, 5_2, 6_2, 7_1, 7_2, 7_6, 8_4, 8_6, 8_7, 8_{14}$ have no such restrictions.

Table 1: Values which $\det L$ does not take with $d_{cb}(S_{p,q}, L) = 1$ or $d_2(S_{p,q}, L) = 1$

$S_{p,q}$	$\neq \det L$
$4_1 = S_{5,2}$	1, 4 (mod 5)
$5_1 = S_{5,1}$	2, 3 (mod 5)
$6_1 = S_{9,2}$	3, 6 (mod 9)
$6_3 = S_{13,5}, 8_1 = S_{13,6}$	1, 3, 4, 9, 10, 12 (mod 13)
$7_3 = S_{13,3}$	2, 5, 6, 7, 8, 11 (mod 13)
$7_4 = S_{15,4}$	2, 3, 7, 8, 12, 13 (mod 15)
$7_5 = S_{17,5}, 8_2 = S_{17,6}$	1, 2, 4, 8, 9, 13, 15, 16 (mod 17)
$7_7 = S_{21,8}$	1, 4, 5, 16, 17, 20 (mod 21)
$8_3 = S_{17,4}$	3, 5, 6, 7, 10, 11, 12, 14 (mod 17)
$8_8 = S_{25,9}$	2, 3, 5, 7, 8, 10, 12, 13, 15, 17, 18, 20, 22, 23 (mod 25)
$8_9 = S_{25,7}$	1, 4, 5, 6, 9, 10, 11, 14, 15, 16, 19, 20, 21, 24 (mod 25)
$8_{11} = S_{27,10}$	3, 6, 12, 15, 21, 24 (mod 27)
$8_{12} = S_{29,12}, 8_{13} = S_{29,11}$	1, 4, 5, 6, 7, 9, 13, 16, 20, 22, 23, 24, 25, 28 (mod 29)

Example 3.3. Table 2 shows 2-component links which are not obtained from the 2-bridge knots in Table 1 by a coherent band surgery. The symbol $\times$ means that the link in the row is not obtained from the 2-bridge knot in the column by a coherent band surgery. For example, the knot 6_1 and the link 6_1^2 are not related by a coherent band surgery; moreover this implies that $K \in \{6_1, 6_1!\}$ and $L \in \{6_1^2, 6_1^{2'}, 6_1^{2!}, 6_1^{2'!}\}$ are not related by a coherent band surgery; cf. [15, Table 2], [16, Table II].

Table 2: Links and 2-bridge knots which are not related by a single coherent band surgery.

L	$\det L$	4_1	5_1 7_4	6_1 8_{11}	6_3 8_1	7_3	7_5 8_2	7_7	8_3	8_8	8_9	8_{12} 8_{13}
U^2	0											
$2_1^2 = H_-$	2		×			×	×			×		
$4_1^2, 7_7^2$	4	×			×		×	×			×	×
$3_1 \# H_-, 6_1^2$	6	×		×		×			×		×	×
$5_1^2, 7_8^2, 8_1^2$	8		×			×	×			×		
$6_2^2, 4_1 \# H_-, 5_1 \# H_-$	10				×				×	×	×	
$6_3^2, 3_1 \# 4_1^2$	12		×	×	×				×	×		
$7_1^2, 5_2 \# H_-$	14	×			×				×		×	
$7_3^2, 7_4^2$	16	×			×		×	×			×	×
7_2^2	18		×			×	×			×		
7_5^2	20					×		×	×	×	×	×
	22		×		×			×	×	×		×
7_6^2	24	×		×		×			×		×	×

4 Coherent band-Gordian distance

The following is Proposition 2.3 in [22]:

Proposition 4.1. *If two knots K and K' are related by a sequence of two coherent band surgeries, then they are related by a single $SH(3)$ -move, and vice versa. Thus $d_{cb}(K, K') = 2sd_3(K, K')$ and $u_{cb}(K) = 2su_3(K)$.*

The following is Theorem 2.2 in [19].

Theorem 4.2. *If two links L and L' are related by a coherent band surgery, $d_{cb}(L, L') = 1$, then*

$$V(L; \omega)/V(L'; \omega) \in \left\{ \pm i, -\sqrt{3}^{\pm 1} \right\}. \quad (18)$$

Then we have the following, which is given in [22, Theorem 3.1].

Corollary 4.3. *If two knots K and K' are related by a single $SH(3)$ -move, $sd_3(K, K') = 1$, then*

$$V(K; \omega)/V(K'; \omega) \in \left\{ \pm 1, \pm i\sqrt{3}^{\pm 1}, 3^{\pm 1} \right\}. \quad (19)$$

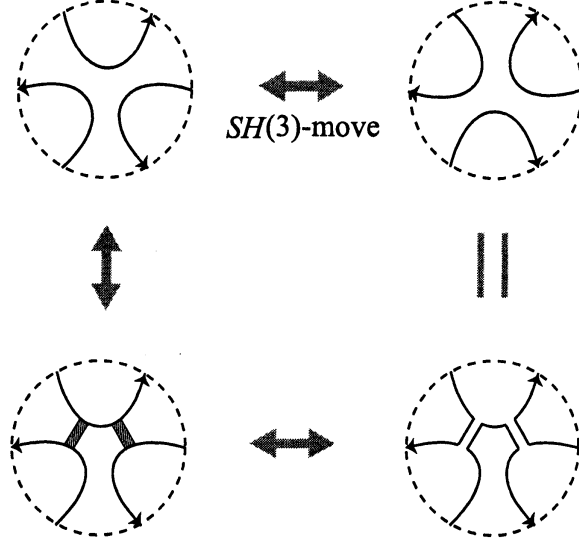


Figure 7: An $SH(3)$ -move is correspond to two coherent band surgeries.

Example 4.4. Let $K = 4_1$ and $K' = 3_1! \# 3_1$. Then $sd_3(K, K') > 1$; see [15, Table 1]. Since $\sigma(K) = \sigma(K') = 0$, the signature cannot show $sd_3(K, K') > 1$. However, since $V(K; \omega) = -1$, $V(K'; \omega) = 3$, we can prove by using Corollary 4.3. In Table 3 we list all such pairs of knots with up to 7 crossings.

Table 3: Pairs of knots K and K' with $|\sigma(K) - \sigma(K')| \leq 2$ and $sd_3(K, K') > 1$.

K	K'	$\sigma(K)$	$\sigma(K')$	$V(K; \omega)$	$V(K'; \omega)$
4_1	$3_1! \# 3_1$	0	0	-1	3
5_2	$3_1! \# 3_1$	2	0	-1	3
7_6	$3_1! \# 3_1$	2	0	-1	3
6_2	$3_1 \# 3_1$	2	4	1	-3
7_2	$3_1 \# 3_1$	2	4	1	-3
$7_3!$	$3_1 \# 3_1$	4	4	1	-3

The following is Theorem 5.2 in [24].

Theorem 4.5. Suppose that a $(c+1)$ -component link L' is obtained from a c -component link L by a coherent band surgery. If $V(L'; \omega) = \eta i V(L; \omega) = \pm i^c (i\sqrt{3})^\delta$, $\eta = \pm 1$, then $i^c V(L'; -1) \equiv \eta i^{c-1} V(L; -1) \pmod{3^{\delta+1}}$.

Theorem 4.6. Suppose that two links L and L' are related by a sequence of two coherent band surgeries, $d_{cb}(L, L') = 2$. Let L be a c -component link. If $V(L; \omega) = -V(L'; \omega) = \pm i^{c-1} (i\sqrt{3})^\delta$, then

$$i^{c-1} V(L; -1) \equiv -i^{c-1} V(L'; -1) \pmod{3^{\delta+1}} \quad (20)$$

By Proposition 4.1, we have:

Corollary 4.7. *If two knots K and K' are related by a single $SH(3)$ -move, $\text{sd}_3(K, K') = 1$, and $V(K; \omega) = -V(K'; \omega) = \pm(i\sqrt{3})^\delta$, then*

$$V(K; -1) \equiv -V(K'; -1) \pmod{3^{\delta+1}} \quad (21)$$

Example 4.8. Let $K = 6_1$ and $K' = 3_1$. Then $\text{sd}_3(K, K') > 1$. Since $\sigma(K) = 0$, $\sigma(K') = 2$, the signature cannot show $\text{sd}_3(K, K') > 1$. However, since $V(K; \omega) = i\sqrt{3}$, $V(K'; \omega) = -i\sqrt{3}$, $V(K; -1) = 9$, $V(K'; -1) = -3$, we can prove by using Corollary 4.7. In Table 4 we list all such pairs of knots with up to 7 crossings.

Table 4: Pairs of knots K and K' with $|\sigma(K) - \sigma(K')| \leq 2$ and $\text{sd}_3(K, K') > 1$.

K	K'	$\sigma(K)$	$\sigma(K')$	$V(K; \omega)$	$V(K'; \omega)$	$V(K; -1)$	$V(K'; -1)$
6_1	3_1	0	2	$i\sqrt{3}$	$-i\sqrt{3}$	9	-3
6_1	7_4	0	-2	$i\sqrt{3}$	$-i\sqrt{3}$	9	-15
6_1	7_7	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	9	21
6_1	$3_1 \# 4_1$	0	-2	$i\sqrt{3}$	$-i\sqrt{3}$	9	-15
$7_4!$	7_7	2	0	$i\sqrt{3}$	$-i\sqrt{3}$	-15	21
$7_7!$	7_7	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	21	21
$3_1 \# 4_1$	7_7	2	0	$i\sqrt{3}$	$-i\sqrt{3}$	-15	21

Similarly, we have:

Corollary 4.9. *If two 2-component links L and L' are related by a sequence of two coherent band surgeries, $\text{d}_{\text{cb}}(L, L') = 2$, and $V(L; \omega) = -V(L'; \omega) = \pm i(i\sqrt{3})^\delta$, then*

$$V(L; -1)/i \equiv -V(L'; -1)/i \pmod{3^{\delta+1}} \quad (22)$$

In Table 4 we list all pairs of 2-component links with up to 6 crossings, which can be shown to have coherent band-Gordian distance > 2 by Corollary 4.9 but cannot be shown by using the signature. Thus by Table 3 in [15] we can conclude they have coherent band-Gordian distance 4.

Table 5: Pairs of links L and L' with $|\sigma(L) - \sigma(L')| \leq 2$ and $\text{d}_{\text{cb}}(L, L') = 4$.

L	L'	$\sigma(L)$	$\sigma(L')$	$V(L; \omega)$	$V(L'; \omega)$	$V(L; -1)/i$	$V(L'; -1)/i$
$3_1 \# H_+$	6_3^2	1	3	$-\sqrt{3}$	$\sqrt{3}$	6	-12
$3_1 \# H_+$	$6_3^{2'}$	1	-1	$-\sqrt{3}$	$\sqrt{3}$	6	-12
$T_6!$	6_3^2	1	3	$-\sqrt{3}$	$\sqrt{3}$	6	-12
$T_6!$	$6_3^{2'}$	1	-1	$-\sqrt{3}$	$\sqrt{3}$	6	-12

5 The HOMFLYPT polynomial

Let $\Sigma_k(L)$ be the k -fold cyclic covering space of S^3 branched over a link L . Lickorish and Millett [27, Theorem 2] have shown:

$$P(L; i, i) = (-2)^{\tau/2}, \quad (23)$$

where $\tau = \dim H_1(\Sigma_3(L); \mathbf{Z}_2)$. Putting $v = z = i$ in Eq. (6), we obtain

$$P(L_+; i, i) + P(L_-; i, i) + P(L_0; i, i) = 0, \quad (24)$$

where (L_+, L_-, L_0) is a skein triple. Using this, we have a criterion on the HOMFLYPT polynomials of two links which are related by a crossing change [29, Theorem 1.1] or a coherent band surgery [21, Proposition 2.4].

Proposition 5.1. *If two links L and L' are related by either a crossing change or a coherent band surgery, then*

$$P(L; i, i)/P(L'; i, i) \in \{1, -2^{\pm 1}\}. \quad (25)$$

The Conway polynomial $\nabla(L; z)$ of a c -component link L may be written $\nabla(L; z) = z^{c-1}\varphi(z)$, where $\varphi(z)$ is an integer polynomial in z^2 . Then we obtain a symmetric integer polynomial $\tilde{\Delta}_L(t)$ by

$$\tilde{\Delta}_L(t) = \varphi(t^{1/2} - t^{-1/2}), \quad (26)$$

which is called the *Hosokawa polynomial* [12]; cf. [33, pp.120]. Then Hosokawa and Kinoshita [13] have shown the following; cf. [28, Corollary 9.8]:

Proposition 5.2. *The order of the first homology group of the k -fold cyclic covering space of S^3 branched over a c -component link L , $H_1(\Sigma_k(L); \mathbf{Z})$, is given by*

$$k^{c-1} \prod_{j=1}^{k-1} \tilde{\Delta}_L(\xi^j), \quad (27)$$

where ξ is a primitive k th root of unity.

Using Proposition 5.2, we obtain:

Lemma 5.3. *Let L be a c -component link. If $P(L; i, i) = (-2)^h$, then*

$$[\nabla(L; z)/z^{c-1}]_{z^2=-3} \equiv 0 \pmod{2^h}. \quad (28)$$

Using this lemma, we obtain the following.

Theorem 5.4. *Suppose that a $(c+1)$ -component link L' is obtained from a c -component link L by a coherent band surgery. If $P(L; i, i) = P(L'; i, i) = (-2)^h$, then*

$$\left[\frac{\nabla(L; z) + z\nabla(L'; z)}{z^{c-1}} \right]_{z^2=-3} \equiv \left[\frac{\nabla(L; z) - z\nabla(L'; z)}{z^{c-1}} \right]_{z^2=-3} \equiv 0 \pmod{2^{h+1}}. \quad (29)$$

6 The Q polynomial

The *Q polynomial* $Q(L; z) \in \mathbf{Z}[z^{\pm 1}]$ [3, 11] is an invariant of the isotopy type of an unoriented link L , which is defined by the following formulas:

$$Q(U; z) = 1; \quad (30)$$

$$Q(L_+; z) + Q(L_-; z) = z(Q(L_0; z) + Q(L_\infty; z)), \quad (31)$$

where U is the unknot and $(L_+, L_-, L_0, L_\infty)$ is an unoriented skein quadruple.

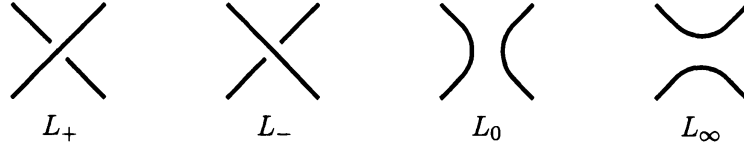


Figure 8: An unoriented skein quadruple.

Let $\rho(L) = Q(L; (\sqrt{5} - 1)/2)$. Then Jones [18] has shown

$$\rho(L) = \pm \sqrt{5}^r \quad (32)$$

where $r = \dim H_1(\Sigma(L); \mathbf{Z}_5)$.

Furthermore, Rong [39] has shown that there are six cases for the ratios among $\rho(L_-)$, $\rho(L_+)$, $\rho(L_0)$, $\rho(L_\infty)$ as in Table 6.

Table 6: The values of the Q polynomials at $z = (\sqrt{5} - 1)/2$.

Cases	$\rho(L_-)/\rho(L_\infty)$	$\rho(L_0)/\rho(L_\infty)$	$\rho(L_+)/\rho(L_\infty)$	$\rho(L_+)/\rho(L_-)$
(a)	1	$\sqrt{5}$	1	1
(b)	$\sqrt{5}$	1	-1	$-\sqrt{5}^{-1}$
(c)	1	-1	-1	-1
(d)	-1	-1	1	-1
(e)	-1	1	$\sqrt{5}$	$-\sqrt{5}$
(f)	$\sqrt{5}^{-1}$	$\sqrt{5}^{-1}$	$\sqrt{5}^{-1}$	1

Using Table 6, we have criteria on the Q polynomials of two links which are related by a crossing change [40, Theorem 4.1] or a band surgery [19, Theorem 3.1].

Proposition 6.1. (i) *If two links L and L' are related by a crossing change, then*

$$\rho(L)/\rho(L') \in \left\{ \pm 1, -\sqrt{5}^{\pm 1} \right\}. \quad (33)$$

(ii) If two links L and L' are related by a band surgery, then

$$\rho(L)/\rho(L') \in \left\{ \pm 1, \sqrt{5}^{\pm 1} \right\}. \quad (34)$$

Moreover, using Table 6, we have the following.

Theorem 6.2. Suppose that two links L and L' are related by either a crossing change or a band surgery and that $\rho(L) = \rho(L') = \pm\sqrt{5}^r$. Then

$$\det L + \det L' \equiv 0 \text{ or } \det L - \det L' \equiv 0 \pmod{5^{r+1}}. \quad (35)$$

Example 6.3. $d_{cb}(9_{39!}, 6_2^2) > 1$. Since $\rho(9_{39!}) = \rho(6_2^2) = -\sqrt{5}$, $\det(9_{39!}) = 55$, and $\det(6_2^2) = 10$, the result follows by Theorem 6.2. Note that since $\sigma(9_{39!}) = 2$, $\sigma(6_2^2) = 3$, we cannot use Proposition 2.2.

7 Table of $d_{cb}(K, K')$

We give a table of coherent band-Gordian distances between two knots (cf: [15, Table 1])

Table 7: Coherent band-Gordian distances between two knots with up to 6 crossings.

	U	3_1	$3_1!$	4_1	5_1	$5_1!$	5_2	$5_2!$	6_1	$6_1!$	6_2	$6_2!$	6_3	$3_1\#3_1$	$3_1!\#3_1!$	$3_1!\#3_1$
U	0	2	2	2	4	4	2	2	2	2	2	2	2	4	4	2
3_1		0	4	2	2	6	2	4	4 [†]	2	2	4	2	2	6	2
$3_1!$			0	2	6	2	4	2	2	4 [†]	4	2	2	6	2	2
4_1				0	4	4	2	2	2	2	2	2	2	4	4	4
5_1					0	8	2	6	4	4	2	6	4	2	8	4
$5_1!$						0	6	2	4	4	6	2	4	8	2	4
5_2							0	4	2	2	2	4	2	2	6	4
$5_2!$								0	2	2	4	2	2	6	2	4
6_1									0	2	2	2	2	4	4	2
$6_1!$										0	2	2	2	4	4	2
6_2											0	4	2	4	6	2
$6_2!$												0	2	6	4	2
6_3													0	4	4	2
$3_1\#3_1$														0	8	4
$3_1!\#3_1!$															0	4
$3_1!\#3_1$																0

†: corrected

Table 8: Coherent band-Gordian distances between two knots with up to 7 crossings.

	7_1	$7_1!$	7_2	$7_2!$	7_3	$7_3!$	7_4	$7_4!$	7_5	$7_5!$	7_6	$7_6!$	7_7	$7_7!$	$3_1\#4_1$	$3_1!\#4_1$
U	6	6	2	2	4	4	2	2	4	4	2	2	2	2	2	2
3_1	4	8	2	4	6	2	4	2	2	6	2	4	2	2	2	4
$3_1!$	8	4	4	2	2	6	2	4	6	2	4	2	2	2	4	2
4_1	6	6	2	2	4	4	2/4	2/4	4	4	2	2	2	2	2	2
5_1	2	10	2	6	8	2	6	2	2	8	2	6	4	4	2	6
$5_1!$	10	2	6	2	2	8	2	6	8	2	6	2	4	4	6	2
5_2	4	8	2	4	6	2	4	2	2	6	2	4	2	2/4	2	4
$5_2!$	8	4	4	2	2	6	2	4	6	2	4	2	2/4	2	4	2
6_1	6	6	2/4	2	4	4	4	2	4	4	2	2	4	2	2	4
$6_1!$	6	6	2	2/4	4	4	2	4	4	4	2	2	2	4	4	2
6_2	4	8	2	4	6	2	4	2/4	2/4	6	2	4	2	2	2/4	4
$6_2!$	8	4	4	2	2	6	2/4	4	6	2/4	4	2	2	2	4	2/4
6_3	6	6	2	2	4	4	2/4	2/4	4	4	2	2	2	2	2	2
$3_1\#3_1$	2	10	4	6	8	4	6	2	2	8	2	6	4	4	2	6
$3_1!\#3_1!$	10	2	6	4	4	8	2	6	8	2	6	2	4	4	6	2
$3_1!\#3_1$	6	6	2/4	2/4	4	4	2/4	2/4	4	4	4	4	2	2	2	2
7_1	0	12	4	8	10	2	8	4	2	10	4	8	6	6	4	8
$7_1!$		0	8	4	2	10	4	8	10	2	8	4	6	6	8	4
7_2			0	4	6	2	4	2	2	6	2	4	2	2/4	2/4	4
$7_2!$				0	2	6	2	4	6	2	4	2	2/4	2	4	2/4
7_3					0	8	2	6	8	2	6	2	4	4	6	2/4
$7_3!$						0	6	2	2	8	2	6	4	4	2/4	6
7_4							0	4	6	2	4	2	2/4	4	4	2
$7_4!$								0	2	6	2	4	4	2/4	2	4
7_5									0	8	2	6	4	4	2	6
$7_5!$										0	6	2	4	4	6	2
7_6											0	4	2	2	2	4
$7_6!$												0	2	2	4	2
7_7													0	4	4	2
$7_7!$														0	2	4
$3_1\#4_1$															0	4
$3_1!\#4_1$																0

The symbol 2/4 means $d_{cb}(K, K') = 2$ or 4.

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References

- [1] Y. Abe and T. Kanenobu, *Unoriented band surgery on knots and links*, Kobe J. Math. (to appear).
- [2] J. S. Birman and T. Kanenobu, *Jones' braid-plat formula and a new surgery triple*, Proc. Amer. Math. Soc. **102** (1988), 687–695.
- [3] R. D. Brandt, W. B. R. Lickorish, and K. C. Millett, *A polynomial invariant for unoriented knots and links*, Invent. Math. **84** (1986), no. 3, 563–573.
- [4] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), Pergamon, Oxford, 1970, pp. 329–358.
- [5] Isabel D. Darcy and D. W. Sumners, *A strand passage metric for topoisomerase action*, KNOTS '96 (Tokyo), World Sci. Publ., River Edge, NJ, 1997, pp. 267–278.
- [6] Isabel K. Darcy, *Biological distances on DNA knots and links: applications to XER recombination*, J. Knot Theory Ramifications **10** (2001), no. 2, 269–294, Knots in Hellas '98, Vol. 2 (Delphi).
- [7] Isabel K. Darcy, Kai Ishihara, Ram K. Medikonduri, and Koya Shimokawa, *Rational tangle surgery and Xer recombination on catenanes*, Algebr. Geom. Topol. **12** (2012), no. 2, 1183–1210.
- [8] Isabel K. Darcy and De Witt Sumners, *Rational tangle distances on knots and links*, Math. Proc. Cambridge Philos. Soc. **128** (2000), no. 3, 497–510.
- [9] Yuanan Diao, Claus Ernst, and Anthony Montemayor, *Nullification of knots and links*, J. Knot Theory Ramifications **21** (2012), no. 6, 1250046, 24 pp.
- [10] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 2, 239–246.
- [11] C. F. Ho, *A polynomial invariant for knots and links—preliminary report*, Abstracts Amer. Math. Soc. **6** (1985), 300.
- [12] Fujitsugu Hosokawa, *On ∇ -polynomials of links*, Osaka Math. J. **10** (1958), 273–282.
- [13] Fujitsugu Hosokawa and Shin'ichi Kinoshita, *On the homology group of branched cyclic covering spaces of links*, Osaka Math. J. **12** (1960), 331–355.

- [14] J. Hoste, Y. Nakanishi, and K. Taniyama, *Unknotting operations involving trivial tangles*, Osaka J. Math. **27** (1990), no. 3, 555–566.
- [15] Kai Ishihara and Dorothy Buck, *Nullification distance between links with small crossing numbers*, Proceedings of “Mathematics of Knots IV”, 2011, pp. 182–188.
- [16] Kai Ishihara and Koya Shimokawa, *Band surgeries between knots and links with small crossing numbers*, Prog. Theor. Phys. Suppl. **191** (2011), 245–255.
- [17] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), no. 2, 335–388.
- [18] V. F. R. Jones, *On a certain value of the Kauffman polynomial*, Comm. Math. Phys. **125** (1989), no. 3, 459–467.
- [19] T. Kanenobu, *Band surgery on knots and links*, J. Knot Theory Ramifications **19** (2010), no. 12, 1535–1547.
- [20] T. Kanenobu, *$H(2)$ -Gordian distance of knots*, J. Knot Theory Ramifications **20** (2011), no. 6, 813–835.
- [21] T. Kanenobu, *Band surgery on knots and links, II*, J. Knot Theory Ramifications **21** (2012), no. 9, 1250086, 22 pp.
- [22] T. Kanenobu, *$SH(3)$ -move and other local moves on knots*, preprint.
- [23] T. Kanenobu and Y. Miyazawa, *$H(2)$ -unknotting number of a knot*, Commun. Math. Res. **25** (2009), no. 5, 433–460.
- [24] T. Kanenobu and H. Moriuchi, *Links which are related by a band surgery or crossing change*, submitted.
- [25] Akio Kawauchi, *A survey of knot theory*, Birkhäuser Verlag, Basel, 1996, Translated and revised from the 1990 Japanese original by the author.
- [26] W. B. R. Lickorish, *Unknotting by adding a twisted band*, Bull. London Math. Soc. **18** (1986), no. 6, 613–615.
- [27] W. B. R. Lickorish and K. C. Millett, *Some evaluations of link polynomials*, Comment. Math. Helv. **61** (1986), no. 3, 349–359.
- [28] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997.
- [29] Y. Miyazawa, *Gordian distance and polynomial invariants*, J. Knot Theory Ramifications **20** (2011), 895–907.
- [30] Yasuyuki Miyazawa, *The Jones polynomial of an unknotting number one knot*, Topology Appl. **83** (1998), no. 3, 161–167.

- [31] Anthony Montemayor, *On nullification of knots and links*, Master's thesis, Western Kentucky Univ., 2012.
- [32] H. Murakami, *Some metrics on classical knots*, Math. Ann. **270** (1985), no. 1, 35–45.
- [33] K. Murasugi, *Knot theory & its applications*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008, Translated from the 1993 Japanese original by Bohdan Kurpita, Reprint of the 1996 translation.
- [34] Kunio Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. **117** (1965), 387–422.
- [35] Yasutaka Nakanishi, *A note on unknotting number*, Math. Sem. Notes Kobe Univ. **9** (1981), no. 1, 99–108.
- [36] Józef H. Przytycki and Paweł Traczyk, *Invariants of links of Conway type*, Kobe J. Math. **4** (1988), no. 2, 115–139.
- [37] R. A. Robertello, *An invariant of knot cobordism*, Comm. Pure Appl. Math. **18** (1965), 543–555.
- [38] Dale Rolfsen, *Knots and links*, AMS Chelsea Press, Providence, RI, 2003, Originally published: Berkeley, CA: Publish or Perish Inc. 1976.
- [39] Y. W. Rong, *The Kauffman polynomial and the two-fold cover of a link*, Indiana Univ. Math. J. **40** (1991), no. 1, 321–331.
- [40] A. Stoimenow, *Polynomial values, the linking form and unknotting numbers*, Math. Res. Lett. **11** (2004), no. 5-6, 755–769.
- [41] Ichiro Torisu, *The determination of the pairs of two-bridge knots or links with Gordian distance one*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1565–1571.
- [42] Paweł Traczyk, *A criterion for signed unknotting number*, Low-dimensional topology (Funchal, 1998), Contemp. Math., vol. 233, Amer. Math. Soc., Providence, RI, 1999, pp. 215–220.

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