

Nonlinear partial differential equations and logarithmic singularities

By

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Abstract

We study logarithmic singularities of solutions to some nonlinear partial differential equations near noncharacteristic hypersurfaces. It is a logarithmic analogue of the Painlevé PDE test for integrable equations.

§ 1. Painlevé PDE test and WTC expansions

Weiss, Tabor and Carnevale ([7]) constructed a family of meromorphic solutions to some integrable equations. Let us review their calculation in the case of the KdV equation:

$$(1.1) \quad u_{ttt} - 6uu_t + u_x = 0 \quad (t, x \in \mathbb{R}).$$

For any real-analytic function $\psi(x)$, set $T = t - \psi(x)$. Then it has been proved in [7] that the equation (1.1) has a family of meromorphic solutions of the form

$$(1.2) \quad u = \frac{2}{T^2} - \frac{1}{6}\psi_x + gT^2 - \frac{1}{36}\psi_{xx}T^3 + hT^4 - \frac{1}{24}g_xT^5 + \dots,$$

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where $g = g(x)$ and $h = h(x)$ are arbitrary real-analytic functions. Since the function ψ is arbitrary, the noncharacteristic surface $T = t - \psi(x) = 0$ is a multi-dimensional analogue of moving singularities of the Painlevé equations.

Similar solutions exist for other integrable equations and their expansions are called WTC expansions. If an equation has a family of solutions which are expressed by WTC expansions, then it is said to pass the Painlevé PDE test. (See [1] and [7] for details).

When non-integrable equations are considered, different kinds of expansions have to be introduced. The solutions are generally multi-valued as is proved in [2] and [3]. We mainly follow the latter in the present paper.

§ 2. Kobayashi's theory ([3])

We introduce some notation following Kobayashi and explain his result. Let $(t, x) = (t, x_1, \dots, x_n) \in \mathbb{C} \times \mathbb{C}^n$, fix $m \in \mathbb{N}^*$ and set $I_m = \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n; j + |\alpha| \leq m \text{ and } j < m\}$, $N =$ the cardinal of I_m , and $U = (U_{j,\alpha})_{(j,\alpha) \in I_m} \in \mathbb{C}^N$. Moreover, we set $\partial_t = \partial/\partial t$, $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$.

We study nonlinear PDEs of the form

$$(2.1) \quad \partial_t^m u = f\left(t, x, (\partial_t^j \partial_x^\alpha u)_{(j,\alpha) \in I_m}\right).$$

Here $f(t, x, U)$ is holomorphic in $\{(t, x) \in \mathbb{C}_t \times \mathbb{C}_x^n; |t| < r_0, |x| < R_0\} \times \mathbb{C}_U^N$, where r_0 and R_0 are positive constants. (Although f is assumed to be a polynomial in [3], this condition can be relaxed).

The function f can be expanded in the variable U :

$$(2.2) \quad f(t, x, U) = \sum_{\mu \in \mathcal{M}} f_\mu(t, x) U^\mu, \quad \mu = (\mu_{j,\alpha})_{(j,\alpha) \in I_m}, \quad U^\mu = \prod_{(j,\alpha) \in I_m} U_{j,\alpha}^{\mu_{j,\alpha}}$$

for some subset $\mathcal{M}$ of $\mathbb{N}^N$, the set of $\mathbb{N}$ -valued functions on I_m . We assume that $f_\mu(t, x)$ does not vanish identically.

Next we expand $f_\mu(t, x)$ in t :

$$(2.3) \quad f_\mu(t, x) = t^{k_\mu} \sum_{k=0}^{\infty} f_{\mu,k}(x) t^k.$$

We assume that $f_{\mu,0}(x)$ does not vanish identically.

We set

$$|\mu| = \sum_{(j,\alpha) \in I_m} \mu_{j,\alpha}, \quad \gamma(\mu) = \sum_{(j,\alpha) \in I_m} j \mu_{j,\alpha}, \quad r(\mu) = \frac{\gamma(\mu) - m - k_\mu}{|\mu| - 1} \quad (\mu \in \mathcal{M}, |\mu| \geq 2).$$

Kobayashi ([3]) defined his exponent σ_c by

$$\sigma_c = \sup_{\mu \in \mathcal{M}, |\mu| \geq 2} r(\mu).$$

He assumed that σ_c was a rational number and imposed a kind of nonresonance condition on it. In particular, it must avoid the values $0, 1, 2, \dots, m-2$. Under some additional assumptions, he constructed solutions of the form

$$(2.4) \quad u = t^{\sigma_c} \sum_{j=0}^{\infty} u_j(x) t^{j/p},$$

where u_n 's are holomorphic in a common neighborhood of the origin, $u_0 \not\equiv 0$ and p is the smallest positive integer such that $p\sigma_c \in \mathbb{Z}$. A general noncharacteristic surface $t = \psi(x)$ can be transformed to $t = 0$ by a change of coordinates and the WTC expansion (1.2) for the KdV equation is just an example of (2.4).

Kichenassamy-Srinivasan ([2]) chose a different formulation and introduced expansions involving logarithms in the higher order terms.

§ 3. Logarithmic singularities

We consider the resonant case where $\sigma_c \in \{0, 1, 2, \dots, m-2\}$. The expansion is radically different from the one in (2.4) in that the leading term involves a logarithm.

We assume the following:

- (A0) $\sigma_c = l \in \{0, 1, 2, \dots, m-2\}$.
- (A1) $\mathcal{M}_0 \stackrel{\text{def}}{=} \{\mu \in \mathcal{M}; |\mu| \geq 2, r(\mu) = l (= \sigma_c)\}$ is non-empty.
- (A2) If $\mu \in \mathcal{M}_0$ and $\mu_{j\alpha} \neq 0$, then $j \geq l+1$ and $\alpha = 0$.
- (A3) For a sufficiently small positive constant $C > 0$, we have

$$m - l + k_\mu - \gamma(\mu) + l|\mu| \geq C \sum_{\substack{(j,\alpha) \in I_m \\ j \leq l}} \mu_{j,\alpha}$$

for any $\mu \in \mathcal{M} \setminus \mathcal{M}_0$. (This is the case if f is a polynomial).

We use the following notation:

- $\mathcal{R}(\mathbb{C} \setminus \{0\})$, the universal covering space of $\mathbb{C} \setminus \{0\}$,
- $S_\theta = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |\arg t| < \theta\}$,
- $S(\varepsilon(y)) = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}); 0 < |t| < \varepsilon(\arg t)\}$, where $\varepsilon(y)$ is a positive continuous function on $\mathbb{R}_y$,
- $D_r = \{x = (x_1, \dots, x_n) \in \mathbb{C}^n; |x_i| < r \text{ for } i = 1, \dots, n\}$.

Definition 3.1. $\tilde{\mathcal{O}}_+$ denotes the set of all $v(t, x)$ satisfying the following two conditions:

- i) $v(t, x)$ is a holomorphic function on $S(\varepsilon(y)) \times D_r$ for some positive continuous function $\varepsilon(y)$ on $\mathbb{R}_y$ and some constant $r > 0$.
- ii) There exists a constant $a > 0$ such that for any $\tilde{r} \in]0, r[$ and $\theta > 0$ we have

$$\max_{x \in D_{\tilde{r}}} |v(t, x)| = O(|t|^a) \quad (\text{as } t \rightarrow 0 \text{ in } S_\theta).$$

Our main result is the following:

Theorem 3.2. Assume (A0)–(A3) and set $\beta_{j,l} = (-1)^{j-l-1} l!(j-l-1)!$ for $j \geq l+1$. Let $A = a(x)$ be a holomorphic solution to

$$(3.1) \quad \sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) \left(\prod_{j=l+1}^{m-1} \beta_{j,l}^{\mu_{j,0}} \right) A^{|\mu|-1} = \beta_{m,l}$$

in a neighborhood of $x = 0$. Then, for any holomorphic function $b(x)$ in a neighborhood of $x = 0$, there exists a function $v(t, x) \in \tilde{\mathcal{O}}_+$ such that

$$\begin{aligned} u(t, x) &= a(x)t^l \log t + t^l b(x) + t^l v(t, x) \\ &= t^l \{a(x) \log t + b(x) + v(t, x)\} \end{aligned}$$

is a solution to (2.1).

Remark. The left hand side of (3.1) is entire in A . Picard's theorem in value distribution theory assures that there exists a solution to (3.1) in a generic case.

§ 4. Proof of Theorem 3.2

Set $u(t, x) = a(x)t^l \log t + t^l b(x) + t^l v(t, x)$. We shall derive an equation with a new unknown function $v(t, x)$. It is a nonlinear Fuchsian equation. Its coefficients are singular because they involve logarithms. We refer the reader to [6] for details.

§ 5. Moving singularities: nonlinear wave equation

In our main theorem, we claimed the existence of solutions with logarithmic singularities along $t = 0$. By a change of coordinates, we can construct solutions which are singular along other noncharacteristic hypersurfaces.

We consider

$$(5.1) \quad \square u(s, y) = g(s, y; u, \partial_s u, \nabla_y u)$$

in an open set of $\mathbb{C}^{n+1} = \mathbb{C}_s \times \mathbb{C}_y^n$. Here $\square = \partial^2/\partial s^2 - \sum_{i=1}^n \partial^2/\partial y_i^2$, $\nabla_y u = (\partial u/\partial y_1, \dots, \partial u/\partial y_n)$. We assume that $g(s, y; z, \sigma, \eta)$ is a holomorphic function in all its arguments and is entire in (z, σ, η) . Moreover we assume that it is a polynomial of degree 2 in (σ, η) . Its homogeneous part of degree 2 is denoted by g_2 .

Let $\psi(y)$ be a holomorphic function with

$$(5.2) \quad 1 - \{\nabla_y \psi(y)\}^2 \neq 0,$$

where $\nabla_y \psi(y) = (\psi_1(y), \dots, \psi_n(y))$, $\psi_i(y) = \partial \psi(y)/\partial y_i$ ($i = 1, 2, \dots, n$), $\{\nabla_y \psi(y)\}^2 = \sum_{i=1}^n \psi_i(y)^2$. Moreover we assume that

$$(5.3) \quad g_2(\psi(y), y; 0, 1, -\nabla_y \psi(y)) \neq 0.$$

Theorem 5.1. *Assume (5.2) and (5.3). Then, in a neighborhood of the hypersurface $\Sigma = \{s = \psi(y)\}$, there exists a family of solutions $u(s, y)$ to (5.1) with the asymptotic behavior*

$$u(s, y) \sim -\frac{1 - \{\nabla_y \psi(y)\}^2}{g_2(\psi(y), y; 0, 1, -\nabla_y \psi(y))} \log(s - \psi(y)) \quad \text{as } s \rightarrow \psi(y).$$

Proof. Set $t = s - \psi(y)$, $x = y$, $\Psi = 1 - \{\nabla_y \psi(y)\}^2 (\neq 0)$. Then, we have $\partial_s = \partial_t$, $\partial_{y_i} = -\psi_i \partial_t + \partial_{x_i}$. We can apply Theorem 3.2 near $t = 0$. The assumption (5.3) corresponds to $k_\mu = 0$. Details are explained in [6]. $\square$

§ 6. Moving singularities: third order case

Next we consider a third order equation. Let $P(s, y; \partial_s, \partial_y)$ be a linear partial differential operator of third order. Its principal symbol is denoted by $\sigma(P)$. Its coefficients are assumed to be holomorphic near $s = \psi(y)$. We consider

$$(6.1) \quad Pu = g(s, y; (\partial_s^j \partial_y^\alpha u)_{j+|\alpha| \leq 2}).$$

We assume that $g(s, y; (Y_{j,\alpha})_{j+|\alpha| \leq 2})$ is entire in $(Y_{j,\alpha})_{j+|\alpha| \leq 2}$. Moreover, we assume that g is a polynomial of degree 2 in $(Y_{j,\alpha})_{j+|\alpha|=1,2}$ and we denote its homogeneous part of degree 2 by g_2 .

Set $Y_{j,\alpha}^{(0)} = (-\nabla_y \psi(y))^\alpha$ if $j + |\alpha| = 2$ and $Y_{j,\alpha}^{(0)} = 0$ otherwise. We assume

$$(6.2) \quad g_2(\psi(y), y; (Y_{j,\alpha}^{(0)})_{j+|\alpha| \leq 2}) \neq 0$$

$$(6.3) \quad \sigma(P)(s, y; 1, -\nabla_y \psi(y)) \neq 0$$

Theorem 6.1. *Assume (6.2) and (6.3). Then, in a neighborhood of the hypersurface $\Sigma = \{s = \psi(y)\}$, there exists a family of solutions $u(s, y)$ to (6.1) with the asymptotic behavior*

$$u(s, y) \sim a(y)t \log t, \quad t = s - \psi(y), \quad a(y) = \frac{-\sigma(P)(s, y; 1, -\nabla_y \psi(y))}{g_2(\psi(y), y; (Y_{j,\alpha}^{(0)}))}$$

as $s \rightarrow \psi(y)$.

Proof. We set $t = s - \psi(y)$, $x = y$ again. By using (6.3), we can rewrite (6.1) as

$$\partial_t^3 u = [\text{linear part}] + \sigma(P)^{-1} g\left(t + \psi(x), x; \left(\partial_t^j (-(\nabla_x \psi) \partial_t + \nabla_x)^\alpha u\right)_{j+|\alpha| \leq 2}\right).$$

Here $\sigma(P) = \sigma(P)(t + \psi(x), x; 1, -\nabla_x \psi(x))$ and $-(\nabla_x \psi) \partial_t + \nabla_x$ is the n -tuple of vector fields whose i -th component is $-(\partial \psi / \partial x_i) \partial_t + \partial_{x_i}$. Expanding the right hand side as in (2.2) and (2.3), we find the term

$$\sigma(P)^{-1} g_2\left(\psi(x), x; (Y_{j,\alpha}^{(0)})_{j+|\alpha| \leq 2}\right) U_{2,0}^2, \quad U_{2,0} = \partial_t^2 u.$$

This term is $f_{\bar{\mu},0}(x) U^{\bar{\mu}}$ ($\mu = \bar{\mu}$, $k = 0$) in the notation of (2.2) and (2.3), where we define $\bar{\mu}$ by

$$\bar{\mu}_{2,0} = 2, \quad \bar{\mu}_{j,\alpha} = 0 \text{ (otherwise)}.$$

The assumption (6.2) implies $k_{\bar{\mu}} = 0$, and we have $\gamma(\bar{\mu}) = 4$, $|\bar{\mu}| = 2$, $r(\bar{\mu}) = 1$.

We claim that $\mathcal{M}_0 = \{\bar{\mu}\}$. First, the assumption on g means $\sum_{j+|\alpha|=1,2} \mu_{j,\alpha} \leq 2$ for each $\mu \in \mathcal{M}$. If $\mu_{2,0} = 2$, then $\mu_{j,\alpha} = 0$ ($j + |\alpha| = 1, 2$). It follows that $\gamma(\mu) = \gamma(\bar{\mu})$. Since $|\mu| \geq |\bar{\mu}|$ and $k_\mu \geq k_{\bar{\mu}} = 0$, we obtain the estimate $r(\mu) \leq r(\bar{\mu}) = 1$, the equality being true if and only if $\mu = \bar{\mu}$. On the other hand, if $\mu_{2,0} \leq 1$, then we have $\gamma(\mu) \leq |\mu|$ and $r(\mu) < 1$.

We have $f_{\bar{\mu},0}(x) = \sigma(P)^{-1} g_2\left(\psi(x), x, (Y_{j,\alpha}^{(0)})\right)$. So (3.1) becomes

$$\sigma(P)^{-1} g_2(\psi, x, (Y_{j,\alpha}^{(0)})) A = -1.$$

Hence we have $A = a(x) = -\sigma(P)/g_2(\psi(x), x, (Y_{j,\alpha}^{(0)}))$, $u \sim a(x)t \log t$. □

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