

RESIDUES OF HOLOMORPHIC VECTOR FIELDS RELATIVE TO SINGULAR INVARIANT SUBVARIETIES

Daniel LEHMANN and Tatsuo SUWA

1- Introduction

Let $\mathcal{F}$ be a holomorphic foliation with singularities in a smooth complex manifold W , and V an analytic subvariety (not necessarily everywhere smooth), invariant by $\mathcal{F}$ ("invariant", or equivalently "saturated" means: if a point of V belongs to the regular part of $\mathcal{F}$, then the whole leaf through this point is included in V). We shall assume furthermore that the normal bundle to the regular part of V in W has a natural extension ν to the whole V , and even a smooth extension $\tilde{\nu}$ to a germ of neighborhood of V in W , making us able to use connections on $\tilde{\nu}$ and to integrate associated differential forms on compact pieces of V . [For instance, as we shall see, such a natural extension $\tilde{\nu}$ always exists for complex hypersurfaces, for algebraic subsets of $\mathbf{CP}^{p+q}$ defined by q global equations, or for "strongly" locally complete intersections (SLCI: see definition below)].

Denote by p (resp. $p+q$, resp. s) the complex dimension of V (resp. W , resp. of the leaves of $\mathcal{F}$). Then, it is easy to prove that the characteristic classes of ν in dimension $> 2(p-s)$ will "localize" near $\Sigma = [\text{Sing}(\mathcal{F}) \cap V] \cup \text{Sing}(V)$, and give rise to a residue for each connected component Σ_α of Σ : in fact, once we know $\tilde{\nu}$ to exist, the definition and the proof of the existence of this residue work exactly in the same way as in the case where V is smooth (see theorem 3, p.227, in [L]), and we shall omit the theory for $s > 1$. We will concentrate ourselves to the computation of the residue at an isolated point of $[\text{Sing}(\mathcal{F}) \cap V] \cup \text{Sing}(V)$, for Chern numbers, when $s = 1$: we get then formulas generalizing the ones in [LN₁] and [Su] and also, in the spirit of Baum-Bott ([BB₁],[BB₂]), the Grothendieck residues already known when V is smooth ([L]) (see the theorem 1 below, and its third particular case with theorem 2).

This residue has first been defined by C.Camacho and P.Sad ([CS]) when $p = q = s = 1$, V smooth and Σ_α an isolated point. When the invariant curve V may have singularities, the theory has then been generalized by A.Lins Neto [LN₁] for $W = \mathbf{CP}^2$, by M.Soaes [So] when the surface W is a complete intersection in $\mathbf{CP}^n$, and in [Su]

for arbitrary complex surfaces. It has also been studied in higher dimensions when V is smooth, first in the case $s = p$, $q = 1$ by B.Gmira [G], J.P.Brasselet (unpublished) and A.Lins Neto [LN₂], and then in [L] for the general case with more precise formulas when $s = 1$.

All these results extend by taking, instead of $\tilde{\nu}$, any C^∞ vector bundle on a germ of neighborhood of V in W , the restriction of which to the regular part of V being holomorphic and equipped with an action of a holomorphic vector field X_0 tangent to this regular part (see theorem 1' below). In particular, if we take $T(W)$, with the action $[X_0, \cdot]$ on $T(W)|_V$, we get a formula for computing the index defined in the theorem 8 of [L]. (We were wrong when claiming that the index there defined was the same as the index of [LN₁] for $p = q = s = 1$: there was a mistake in the proof of part (iv) of this theorem, the 3 first parts remaining correct).

Many thanks to F. Hidaka, Y. Miyaoka, P. Molino, A. Rayman, R. Silhol and M. Soares for helpful conversations.

2- Background on locally complete intersections (LCI and SLCI)

Let W be a complex manifold of complex dimension $n = p + q$, and V an analytic irreducible subvariety of pure complex dimension p . We shall call "reduced locally defining function" for V every holomorphic map $f : U \rightarrow \mathbb{C}^q$ defined on an open set U of W , such that:

- (i) $V \cap U = f^{-1}(0)$,
- (ii) the q components of f generate the ideal $I(V \cap U)$ of holomorphic functions which vanish on $V \cap U$; (for instance, if $q = 1$, this condition implies that f may not have factors which are powers).

If $U \supset V$, we say that f is a "reduced defining function", insisting sometimes "globally defined" near V .

The subvariety V is said to be a "locally complete intersection" (briefly: LCI) if the following condition holds: there exists a family $(f_h : U_h \rightarrow \mathbb{C}^q)_h$ of reduced locally defining functions for V , such that $\bigcup_h U_h \supset V$. Such a family will be called a "system of reduced equations" for V . Recall the following proposition, well known to the specialists:

Proposition 1

(i) Let $f_1 : U \rightarrow \mathbb{C}^q$ and $f_2 : U \rightarrow \mathbb{C}^q$ be two reduced locally defining functions for V defined on the same open set U . Then, there exists an holomorphic map $\tilde{g} : U \rightarrow gl(q, \mathbb{C})$ taking values in the set $gl(q, \mathbb{C})$ of $q \times q$ matrices with complex coefficients, satisfying $f_1 = \langle \tilde{g}, f_2 \rangle$, such that the restriction g of $\tilde{g}$ to $V \cap U$ is uniquely defined and takes values in the group $GL(q, \mathbb{C})$ of invertible matrices.

(ii) If V is LCI, and if $(f_h : U_h \rightarrow \mathbb{C}^q)_h$ denotes a system of reduced equations for V , let $\tilde{g}_{hk} : U_h \cap U_k \rightarrow gl(q, \mathbb{C})$ such that $f_h = \langle \tilde{g}_{hk}, f_k \rangle$ on $U_h \cap U_k$, and denote by g_{hk} the restriction of $\tilde{g}_{hk}$ to $V \cap U_h \cap U_k$. The family (g_{hk}) is then a system of transition functions for a holomorphic q vector bundle $\nu \rightarrow V$. This vector bundle is well defined (it does not depend on the choice of the given system of reduced equations for V).

(iii) The bundle ν is an extension to V of the (holomorphic) bundle normal to $V - \text{Sing}(V)$ in W : more precisely, there exists a natural bundle map $\pi : T_{\mathbb{C}}(W)|_V \rightarrow \nu$ which, over the regular part of V , has rank q and the complex tangent bundle to this regular part for kernel (we may therefore identify the restriction of ν to this regular part with the usual normal bundle).

Proof:

Let f_1 and f_2 be such as in (i). Since the components $f_{1,\lambda}$ ($1 \leq \lambda \leq q$) of f_1 and $f_{2,\lambda}$ of f_2 generate the ideal $I(V \cap U)$, there exists $q \times q$ matrices $\tilde{g}$ and $\tilde{h}$ with holomorphic coefficients such that $f_1 = \langle \tilde{g}, f_2 \rangle$ and $f_2 = \langle \tilde{h}, f_1 \rangle$. Furthermore, since f_1 and f_2 vanish on $U \cap V$, we get also on $U \cap V$: $df_1 = \langle g, df_2 \rangle$ and $df_2 = \langle h, df_1 \rangle$ (where g and h denote the restrictions of $\tilde{g}$ and $\tilde{h}$ to $U \cap V$). Since $df_1 = \langle g \circ h, df_1 \rangle$ on $V \cap U$, $g \circ h = Id$ on the regular part of $V \cap U$. By continuity, since this regular part is everywhere dense in $V \cap U$, one still has $g \circ h = Id$ on the whole $V \cap U$: g takes values in $GL(q, \mathbb{C})$. The uniqueness of g is obvious since $g = h^{-1}$. This proves part (i) of the proposition.

From the uniqueness of g in part (i), we deduce immediately that the (g_{hk}) given in part (ii) satisfy the cocycle condition, and are therefore a system of transition functions for a holomorphic vector bundle $\nu \rightarrow V$. Let (g'_{hk}) denotes the system of transition functions arising from another system (f'_h) of reduced equations for V (with the same open covering (U_h) for the moment): after part (i), there exists a family $(\tilde{g}_h)$ such $f_h = \langle \tilde{g}_h, f'_h \rangle$. Denoting (g_h) the induced family on V , the uniqueness in part

(i) implies that the 2 cocycles (g_{hk}) and (g'_{hk}) differ by the coboundary of (g_h) : they define therefore isomorphic bundles. If we change the covering (U_h) , we can use a common refinement to both coverings, for coming back to the case of a same covering.

Notice that the sections σ of ν may be identified with the families $(\sigma_h : U_h \rightarrow \mathbf{C}^q)_h$ of maps such that $\sigma_h = \langle g_{hk}, \sigma_k \rangle$ on $V \cap U_h \cap U_k$. On the other hand we get also there: $df_h = \langle g_{hk}, df_k \rangle$. Therefore the family of $(df_h : T_{\mathbf{C}}(W)|_{V \cap U_h} \rightarrow \mathbf{C}^q)$ defines a bundle map $\pi : T_{\mathbf{C}}(W)|_V \rightarrow \nu$. Furthermore, the kernel of df_h on the regular part of $U_h \cap V$ is exactly the tangent space to this regular part. This achieves the proof of part (iii).

By continuity and reducing the open sets U_h to smaller ones if necessary, we may assume that the functions $\tilde{g}_{hk}$ take themselves values in $GL(q, \mathbf{C})$. However it is not clear that the cocycle condition remains true off V . This justifies the following definition: a LCI subvariety V of W will be said a “strongly” locally complete intersection (shortly SLCI), if there exists a smooth C^∞ vector bundle $\tilde{\nu} \rightarrow U$, defined over some neighborhood U of V in W , the restriction of which to V being ν .

Assuming V to be SLCI, and given an extension $\tilde{\nu} \rightarrow U$ of ν , we shall call “ C^∞ ” any section of ν which is the restriction of a C^∞ section of $\tilde{\nu}$. Local sections over U_h are given by maps $U_h \rightarrow \mathbf{C}^q$, and in particular the q constant functions corresponding to the canonical base of $\mathbf{C}^q$ make a local trivialization of $\tilde{\nu}$ over U_h (or of ν over $V \cap U_h$) called the “trivialization associated” to f_h .

Remarks:

1) Notice that the singular foliations $df_h = 0$ on U_h and $df_k = 0$ on U_k do not coincide in general on $U_h \cap U_k$.

2) We can define a virtual tangent bundle τ to V in the $\tilde{K}U$ theory by

$$[\tau] = [T_{\mathbf{C}}(W)|_V] - [\nu].$$

3) We do not know if LCI implies automatically SLCI. However, there are many examples of SLCI.

4) Let $\mathcal{O}_W$ be the sheaf of holomorphic functions on W , and $\mathcal{I}$ the sheaf of ideals defining the subvariety V in W . Thus $\mathcal{O}_V = \mathcal{O}_W/\mathcal{I}$ is the sheaf of holomorphic functions on V . If V is LCI, then the sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free and the sheaf of germs of holomorphic functions of the bundle $\nu \rightarrow V$ above is identified with the dual sheaf

$\mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_V)$. Furthermore, the bundle map $\pi : T_{\mathbf{C}}(W)|_V \rightarrow \nu$ corresponds, on the sheaf level, to the morphism dual to the one $\mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_W \otimes_{\mathcal{O}_W} \mathcal{O}_V$ induced by $f \rightarrow df \otimes 1$, where $\Omega_W = \mathcal{O}_W(T_{\mathbf{C}}^*(W))$ denotes the cotangent sheaf of W .

Example 1: Any hypersurface V of W (pure complex codimension 1) is SLCI. In fact, if we set $\tilde{g}_{hk} = \frac{f_h}{f_k}$ where (f_h) denotes a family of local defining functions without factors which are powers, then the system $(\tilde{g}_{hk})$ satisfies the cocycle condition and it defines an holomorphic extension $\tilde{\nu}$ of ν defined on the union of the domains U_h of f_h .

Example 2: Any algebraic set V in $W = \mathbf{CP}^n$ which is globally a complete intersection is SLCI. In fact, denote by $(X_0, X_1, \dots, X_n)$ homogeneous coordinates in $\mathbf{CP}^n$, and $F_1, F_2, \dots, F_q$ homogeneous polynomials in the variables $(X_0, X_1, \dots, X_n)$ of respective degree $d_1, d_2, \dots, d_q$ such that V has pure complex codimension q , and is defined by the q equations $F_\lambda = 0$ ($1 \leq \lambda \leq q$). In the affine open subset U_i of $\mathbf{CP}^n$ defined by $X_i \neq 0$, $V \cap U_i$ has for equation with respect to the affine coordinates $(\frac{X_j}{X_i})_{j, j \neq i}$: $\frac{1}{(X_i)^{d_\lambda}} F_\lambda = 0$, ($1 \leq \lambda \leq q$). Therefore, on $U_i \cap U_j$ the change of equations $\tilde{g}_{ij}$ is equal to the diagonal $q \times q$ matrix $\left(\frac{X_j}{X_i}\right)^{d_1}, \dots, \left(\frac{X_j}{X_i}\right)^{d_q}$. [In fact, in this case, it is not necessary to assume that the components $\frac{1}{(X_i)^{d_\lambda}} F_\lambda$ ($1 \leq \lambda \leq q$) generate the ideal $I(V \cap U_i)$]. Denoting by $\tilde{L} \rightarrow \mathbf{CP}^n$ the hyperplane bundle (dual of the tautological bundle), $\tilde{\nu}$ is defined on the whole $\mathbf{CP}^n$ by the formula

$$\tilde{\nu} = \bigoplus_{\lambda=1}^q (\tilde{L})^{\otimes d_\lambda}.$$

Hence: $1 + c_1(\tilde{\nu}) + \dots + c_q(\tilde{\nu}) = \prod_{\lambda=1}^q (1 + d_\lambda c)$, with $c = c_1(\tilde{L})$.

3- Statement of the theorems 1 and 1'

Assume from now on that V is invariant by a holomorphic vector field with singularities X_0 on U . Let θ_{X_0} the $\mathbf{C}$ -linear operator defined for any section $\pi(Y)$ over the regular part of V by: $\theta_{X_0}(\pi(Y)) = \pi([X_0, \tilde{Y}]|_V)$, $\tilde{Y}$ denoting some local extension of Y near V .

In case V is LCI, let $f_h = 0$ be a local reduced equation of V : each component $(df_h(X_0))_\lambda$ ($1 \leq \lambda \leq q$) of the derivative $df_h(X_0)$ has to vanish on $V \cap U_h$, and must be therefore a linear combination with holomorphic coefficients of the components $(f_h)_\lambda$ of f_h : there exists a $q \times q$ matrix $\tilde{C}_h$ with holomorphic coefficients such that: $df_h(X_0) = \langle \tilde{C}_h, f_h \rangle$. Denote by $C_h = ((C_{h,\lambda}^\mu))$ the restriction of $\tilde{C}_h$ to $V \cap U_h$.

Lemma 1

- (i) $\theta_{x_0}(\pi(Y))$ depends only on $\pi(Y)$, not on Y nor on $\tilde{Y}$.
- (ii) $\theta_{x_0}(u\sigma) = u\theta_{x_0}(\sigma) + (X_0.u)\sigma$, for any function u on V which is the restriction of a C^∞ function $\tilde{u} : U \rightarrow \mathbb{C}$.
- (iii) If V is LCI, and $f_h = 0$ a local reduced equation, we have, denoting $(\sigma_1, \dots, \sigma_q)$ the trivialization associated to f_h :

$$\theta_{x_0}(\sigma_\lambda) = - \sum_{\mu} C_{h,\lambda}^\mu \sigma_\mu.$$

(In particular, over the regular part of $V_h = V \cap U_h$, C_h depends only on f_h , not on the choice of $\tilde{C}_h$).

Parts (i) and (ii) of the lemma are proved in lemma 2-1 p.220 of [L]. For proving part (iii), take a partition $\{i_1, \dots, i_p\} \cup \{j_1, \dots, j_q\}$ of $\{1, \dots, n\}$ such that $\frac{D(f_{h,1}, \dots, f_{h,q})}{D(z_{j_1}, \dots, z_{j_q})} \neq 0$ near some point of the regular part of V_h : then, near this point, $(z_{i_1}, \dots, z_{i_p}, f_{h,1}, \dots, f_{h,q})$ is a new system of local coordinates denoted by $(x_1, \dots, x_p, y_1, \dots, y_q)$, the local trivialization of ν associated to f_h becoming $\pi(\frac{\partial}{\partial y_\lambda})$, ($1 \leq \lambda \leq q$). Hence if X_0 writes locally $\sum_{i=1}^p P_i \frac{\partial}{\partial x_i} + \sum_{\mu=1}^q Q_\mu \frac{\partial}{\partial y_\mu}$, then $X_0.f_{h,\mu} = X_0.y_\mu = Q_\mu = \sum_{\lambda=1}^q y_\lambda \tilde{C}_{h,\lambda}^\mu$: hence, $C_{h,\lambda}^\mu = \frac{\partial Q_\mu}{\partial y_\lambda}|_{y=0}$. On the other hand, $\pi[X_0, \frac{\partial}{\partial y_\lambda}] = - \sum_{\mu=1}^q \left(\frac{\partial Q_\mu}{\partial y_\lambda}|_{y=0} \right) \pi(\frac{\partial}{\partial y_\mu})$: this proves part (iii) of the lemma.

Denote by Σ (resp. $(\Sigma_\alpha)_\alpha$) the singular set $\Sigma = [\text{Sing}(X_0) \cap V] \cup \text{Sing}(V)$ (resp. its connected components). (Recall that a singular point of X_0 is either a point where X_0 is not defined, or a point where it vanishes).

Assume Σ_α to be compact, and denote by U_α an open neighborhood of Σ_α in W , and $U_0 = U - \Sigma$. Let $V_\alpha = V \cap U_\alpha$. We shall assume furthermore that $U_\alpha \cap U_\beta = \emptyset$, for $\alpha \neq \beta$. (In particular, $V_\alpha - \Sigma_\alpha$ is in the regular part of V).

Denote by $\tilde{T}_\alpha$ a compact real manifold with boundary, of real dimension $2n$, included in U_α , such that Σ_α be inside the interior of $\tilde{T}_\alpha$, and the boundary $\partial\tilde{T}_\alpha$ of which being transverse to $V - \Sigma$. Put: $\mathcal{T}_\alpha = \tilde{T}_\alpha \cap V$, $\partial\mathcal{T}_\alpha = \partial\tilde{T}_\alpha \cap (V - \Sigma)$.

Assume:

- (i) U_α is included in the domain of a local holomorphic chart $(z_1, \dots, z_n)$ of W ,
- (ii) U_α is one of the U_h 's above, the index α being one of the indices h . (Write f_α and C_α the corresponding terms).

Let:

$$X_0|_{U_\alpha} = \sum_{i=1}^n A_i(z_1, \dots, z_n) \frac{\partial}{\partial z_i}.$$

Denote by $\mathcal{V}_i$ ($1 \leq i \leq n$) the open set of points m in $\partial\mathcal{T}_\alpha$ such that $A_i(m) \neq 0$. These open sets $\mathcal{V}_i$ constitute an open covering $\mathcal{V}$ of $\partial\mathcal{T}_\alpha$. Let $\mathcal{U}$ be any subcovering of $\mathcal{V}$. (Such a $\mathcal{U}$ always exists: take for instance $\mathcal{V}$ itself; see also the particular cases 2 and 3 below). We will denote by (R_i) , ($1 \leq i \leq n$) any system of “honey-cells” adapted to this covering $\mathcal{U}$ (see the definition in [L], section 1, under the name of “système d'alvéoles”). For instance, if the real hypersurfaces $|A_i| = |A_j|$ ($i \neq j$) in U_α are in general position, we may take for R_i the cell defined by: $|A_i| \geq |A_j|$ for all $j, j \neq i, \mathcal{V}_j \in \mathcal{U}$.

Denote by $\mathcal{M}$ the set of multiindices $u = (u_1, u_2, \dots, u_p)$ such that $1 \leq u_1 < u_2 < \dots < u_p \leq n$, and by $\mathcal{M}(\mathcal{U})$ the subset of those such that $\mathcal{V}_{u_j} \in \mathcal{U}$ and $\cap_{j=1}^p \mathcal{V}_{u_j}$ be not empty (that is the set of p simplices in the “nerve” of $\mathcal{U}$). For any $u \in \mathcal{M}(\mathcal{U})$, define $R_u = R_{u_1 u_2 \dots u_p} = \cap_{j=1}^p R_{u_j}$, oriented as in section 1 of [L].

Let $\varphi \in (\mathbb{Z}[c_1, \dots, c_q])^{2p}$ be a Chern polynomial having integral coefficients with respect to the Chern classes, and defining a characteristic class of dimension $2p$.

Theorem 1

Assume V to be *SLCI*. Define:

$$I_\alpha(\mathcal{F}, V, \varphi, \nu) = (-1)^{[\frac{p}{2}]} \sum_{u \in \mathcal{M}(\mathcal{U})} \int_{R_u} \frac{\varphi(-C_\alpha) dz_{u_1} \wedge dz_{u_2} \wedge \dots \wedge dz_{u_p}}{\prod_{j=1}^p A_{u_j}}.$$

- (i) $I_\alpha(\mathcal{F}, V, \varphi, \nu)$ does not depend on the various choices of $(z_1, \dots, z_n), \mathcal{U}, \tilde{T}_\alpha, f_\alpha, \tilde{C}_\alpha, R_i$, and depends only on the foliation $\mathcal{F}$ defined by X_0 , but not on X_0 itself.
- (ii) Assume furthermore V to be compact: $\sum_\alpha I_\alpha(\mathcal{F}, V, \varphi, \nu)$ is then an integer.
- (iii) This integer depends only on V and φ , but not on $\mathcal{F}$: it is equal to the evaluation $\langle \varphi(\nu), V \rangle$ of $\varphi(\nu)$ on the fundamental class $[V]$ of V .

Remark:

The index above depends obviously only on $\mathcal{F}$ and not on X_0 : if we take uX_0 instead of X_0 (u denoting some holomorphic non vanishing function on U), each A_i is multiplied by $u|_V$, the matrix C_α also, and the term under integration does not change.

In fact, we could write the theorem for a foliation $\mathcal{F}$ with singularities, defined only locally by an holomorphic vector field but non necessarily globally.

Particular cases:

1) For $p = q = 1$, $I_\alpha(\mathcal{F}, V, c_1, \nu)$ coincides with the index defined in [LN₁] by A.Lins Neto, if V_α is a (locally) irreducible curve. For a possibly reducible V_α , it coincides with the one in [Su] (notice that the sum of the indices of Lins Neto over the irreducible components is different from the above index: see [Su] (1.3) Remarks 1° and (1.4) Proposition). In fact, in this case, the 1-forms $\frac{dz_1}{A_1}$ and $\frac{dz_2}{A_2}$ coincide over $\mathcal{V}_1 \cap \mathcal{V}_2$ and glue therefore together, defining a 1-form η_α on ∂T_α , while $X_0 \cdot f_\alpha$ may be written $g_\alpha f_\alpha$ for some holomorphic function g_α . The formula of theorem 1 becomes now:

$$I_\alpha(\mathcal{F}, V, c_1, \nu) = \frac{-1}{2i\pi} \left[\int_{R_1} (-g_\alpha) \eta_\alpha + \int_{R_2} (-g_\alpha) \eta_\alpha \right] = \frac{1}{2i\pi} \int_{\partial T_\alpha} g_\alpha \eta_\alpha.$$

On the other hand, when f is irreducible, if $k\omega = \bar{h}.df + f\bar{\alpha}$ according to the notations of [LN] p.198 (up to the bars for avoiding confusions with our notations), his index is then equal to $\frac{-1}{2i\pi} \int_{\partial T_\alpha} \frac{\bar{\alpha}}{\bar{h}}$. But $\frac{\bar{\alpha}}{\bar{h}}$ and $g_\alpha \eta_\alpha$ are equal on ∂T_α , because they both take the same value g_α when applied to the restriction of X_0 , Q.E.D. See (1.1) Lemma and (1.2) in [Su], when f is possibly reducible. This coincidence is also obvious from the theorem 2 and the remark below. Thus the above theorem 1 may be seen as a generalization of the theorems A and C of [LN₁] and the theorem (2.1) of [Su]. In particular, since the sum of our indices is the self-intersection number of the curve V , the integer $3dg(S) - \chi(S) + \sum_B \mu(B)$, lying in the theorem A of [LN₁], is equal to $dg(S)^2$, if the curve S is locally irreducible at each of its singular points. In general, the integer is different from $dg(S)^2$ (see the theorems (2.1) and (2.5) in [Su], in fact, $dg(S)^2$ is equal to $3dg(S) - \chi(S) + \sum_p c(S, p)$ by the adjunction formula).

More generally, for $p = 1$ and any q , there exists a 1-form η_α on ∂T_α , the restriction of which to each $\mathcal{V}_i$ being equal to $\frac{dz_i}{A_i}$. Then, still defining g_α by the same formula $X_0 \cdot f_\alpha = g_\alpha f_\alpha$, the formula of theorem 1 becomes:

$$I_\alpha(\mathcal{F}, V, c_1, \nu) = \frac{1}{2i\pi} \int_{\partial T_\alpha} g_\alpha \eta_\alpha.$$

2) When Σ_α is in the regular part of V , we may take for local chart:

$$(z_1, \dots, z_n) = (x_1, \dots, x_p, y_1, \dots, y_q)$$

such that $f_\lambda = y_\lambda$ for any $\lambda = 1, \dots, q$. Then $A_{p+\lambda}$ vanishes on V_α , in such a way that all open sets $\mathcal{V}_{p+\lambda}$ are empty, and that we may take $\mathcal{U} = \mathcal{V}_1, \dots, \mathcal{V}_p$: Then, $u = \{1, \dots, p\}$ is the unique element of $\mathcal{M}(\mathcal{U})$. On the other hand, c_λ^μ and $\frac{\partial A_{p+\mu}}{\partial y_\lambda}$ are equal on V_α . We recover therefore the formula of theorem 1 in [L], writing $I_\alpha(\mathcal{F}, V, \varphi, \nu)$ as a Grothendieck residue. Note that there are some sign errors in [L]. In the third line of p.237, the factor $(-1)^{[\frac{p}{2}]}$ should be omitted, in Théorème 1 of p.217, the integral giving the residue should be multiplied by $(-1)^{p+[\frac{p}{2}]} = (-1)^{[\frac{p+1}{2}]}$ instead of $(-1)^p$ and in Théorème 1' of p.233, the integral should be multiplied by $(-1)^{[\frac{p}{2}]}$.

3) Assume that Σ_α is a point m_α isolated in V , and that X_0 is meromorphic near m_α (thus X_0 has a zero, a pole or both at m_α). Then, we have the following

Theorem 2

There exists a local holomorphic chart $(z_1, \dots, z_n)$ near m_α in W , such that $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_p$ cover ∂T_α ($p = \dim_{\mathbb{C}} V$).

For this covering $\mathcal{U}$, $\mathcal{M}(\mathcal{U})$ has a unique element $u_0 = \{1, \dots, p\}$. Writing R instead of R_{u_0} , the formula of theorem 1 becomes now:

$$I_\alpha(\mathcal{F}, V, \varphi, \nu) = (-1)^{[\frac{p}{2}]} \int_R \frac{\varphi(-C_\alpha) dz_1 \wedge dz_2 \wedge \dots \wedge dz_p}{\prod_{i=1}^p A_i}.$$

Proof:

Let us write $X_0 = \sum_{i=1}^n A_i \frac{\partial}{\partial z_i}$, $A_i = \frac{P_i}{Q_i}$ with P_i and Q_i holomorphic near m_α . We think of P_i and Q_i as being in the ring $\mathcal{O}_n$ of germs of holomorphic functions at the origin O in $\mathbb{C}^n$ and assume that they are relatively prime for each i . Let Q be the least common multiple of the Q_i 's. Then QX_0 is a holomorphic vector field leaving V invariant.

Lemma 2

The holomorphic vector field QX_0 has an isolated zero at m_α on V .

In fact suppose QX_0 had a non-isolated zero at m_α on V and let V' be a positive dimensional irreducible subvariety of V containing m_α and contained in the zero set of QX_0 . For each i , we write $Q = Q_i Q'_i$, where $Q'_1, \dots, Q'_n$ have no common factors. Since $QX_0 = \sum_{i=1}^n P_i Q'_i \frac{\partial}{\partial z_i}$, the functions $P_i Q'_i$ are all in the defining ideal $I(V')$ of V' . Hence, since $I(V')$ is prime and X_0 is non-zero away from m_α , there exists i_0 such that $Q'_{i_0} \in I(V')$. Thus there is a prime factor P of Q'_{i_0} such that $P \in I(V')$.

Now, since $Q_i Q'_i = Q = Q_{i_0} Q'_{i_0}$, P is a factor of $Q_i Q'_i$ for any i . On the other hand, since the pole of X_0 is the union of the zero sets of the Q_i 's, we have $Q_i \notin I(V')$, by the assumption that the pole of X_0 is at most isolated on V . Therefore, P must be a factor of Q'_i for all i . This contradicts the fact that the Q'_i 's have no common factors. This proves the lemma.

In the above situation, since the zero set of $P_i Q'_i$ is not smaller than that of P_i , it suffices to prove the proposition for vector fields holomorphic near m_α . Note that the index of X_0 at m_α is equal to that of $Q X_0$. Note also that if X_0 has an isolated pole on V , then V is in fact 1-dimensional, since the pole of X_0 has codimension 1 in the ambient space and in V .

In what follows, for an ideal I in the ring $\mathcal{O}_n$, we denote by $\text{ht } I$ its height and by $V(I)$ the (germ of) the analytic set defined by I . Thus $\text{ht } I = \text{codim } V(I)$. Also, for germs $a_1, \dots, a_r$ in $\mathcal{O}_n$, we denote by $(a_1, \dots, a_r)$ the ideal generated by them.

Lemma 3

Let $A_1, \dots, A_n, f_1, \dots, f_q$ be germs in $\mathcal{O}_n$, $n = p + q$, with $\text{ht}(f_1, \dots, f_q) = q$ and $\text{ht}(A_1, \dots, A_n, f_1, \dots, f_q) = n$. Then there exist germs $A'_1, \dots, A'_p$ in $\mathcal{O}_n$ such that

- (i) $A'_1, \dots, A'_p$ are linear combinations of $A_1, \dots, A_n$ with $\mathbb{C}$ coefficients,
- (ii) $\text{ht}(A'_1, \dots, A'_p, f_1, \dots, f_q) = n$.

Since $\text{ht}(f_1, \dots, f_q) = q$, it suffices to show the following for $r = 1, \dots, p$:

(*) if $A'_1, \dots, A'_{r-1}$ are linear combinations of $A_1, \dots, A_n$ (with $\mathbb{C}$ coefficients) with $\text{ht}(A'_1, \dots, A'_{r-1}, f_1, \dots, f_q) = r - 1 + q$, then there exists A'_r which is a linear combination of $A_1, \dots, A_n$ (with $\mathbb{C}$ coefficients) with $\text{ht}(A'_1, \dots, A'_r, f_1, \dots, f_q) = r + q$.

To show this, let $V(A'_1, \dots, A'_{r-1}, f_1, \dots, f_q) = V_1 \cup \dots \cup V_s$ be the irreducible decomposition of $V(A'_1, \dots, A'_{r-1}, f_1, \dots, f_q)$. Since $\text{ht}(A_1, \dots, A_n, f_1, \dots, f_q) = n$, for any point x in $V(A'_1, \dots, A'_{r-1}, f_1, \dots, f_q)$ near O but different from O , there exists A_i with $A_i(x) \neq 0$. Hence we see that there exists A'_r which is a linear combination of $A_1, \dots, A_n$ with $V_k \not\subset V(A'_r)$ for $k = 1, \dots, s$. We have

$$V(A'_1, \dots, A'_r, f_1, \dots, f_q) = (V_1 \cap V(A'_r)) \cup \dots \cup (V_s \cap V(A'_r)).$$

Since each V_k is irreducible and $V_k \not\subset V(A'_r)$, we have $\dim(V_k \cap V(A'_r)) < \dim V_k$. Therefore, we get $\text{ht}(A'_1, \dots, A'_r, f_1, \dots, f_q) = r + q$, hence the lemma.

Note that the condition $\text{ht}(f_1, \dots, f_q) = q$ means that the variety V defined by $f_1 = \dots = f_q = 0$ is a complete intersection and the condition $\text{ht}(A_1, \dots, A_n, f_1, \dots, f_q) = n$ means that the singularity of the holomorphic vector field $X = \sum_{i=1}^n A_i \frac{\partial}{\partial z_i}$ is isolated in V .

In the above situation, if we choose a suitable coordinate system $(z_1, \dots, z_n)$ in $\mathbf{C}^n$, then we may suppose that $\text{ht}(A_1, \dots, A_p, f_1, \dots, f_q) = n$. The theorem 2 follows.

Remark:

Let V_α be defined by $f_\lambda = 0$, $\lambda = 1, \dots, q$. Suppose that V_α is invariant by a holomorphic vector field X_0 and that Σ_α is an isolated point m_α in V_α . Then as is shown above, there exists a holomorphic chart $(z_1, \dots, z_n)$ near m_α such that, when we write $X_0 = \sum_{i=1}^n A_i \frac{\partial}{\partial z_i}$, $\text{ht}(A_1, \dots, A_p, f_1, \dots, f_q) = n$, i.e., $A_1, \dots, A_p, f_1, \dots, f_q$ form a regular sequence. We may set

$$\tilde{T}_\alpha = \{z = (z_1, \dots, z_n) \mid |A_i(z)| \leq \varepsilon, |f_\lambda(z)| \leq \varepsilon, i = 1, \dots, p, \lambda = 1, \dots, q\}.$$

Thus we have $T_\alpha = \{z \mid |A_i(z)| \leq \varepsilon, f_\lambda(z) = 0\}$ and we may also set

$$R_i = \{z \in \partial T_\alpha \mid |A_i(z)| \geq |A_j(z)| \text{ for } j \neq i\}.$$

Then we have

$$R = R_{12\dots p} = \{z \mid |A_i(z)| = \varepsilon, f_\lambda(z) = 0, i = 1, \dots, p, \lambda = 1, \dots, q\},$$

which is a smooth closed submanifold of real dimension p in ∂T_α , the link of the singularity V_α . If we set $\theta_i = \arg A_i(z)$, R is oriented so that the form $(-1)^{[\frac{p}{2}]} d\theta_1 \wedge \dots \wedge d\theta_p$ is positive. Thus if we set $R' = (-1)^{[\frac{p}{2}]} R$ so that $d\theta_1 \wedge \dots \wedge d\theta_p$ is positive on R' , we get

$$I_\alpha(\mathcal{F}, V, \varphi, \nu) = \int_{R'} \frac{\varphi(-C_\alpha) dz_1 \wedge dz_2 \wedge \dots \wedge dz_p}{\prod_{i=1}^p A_i}.$$

More generally, let $E \rightarrow V$ be a continuous complex vector bundle of rank $r \geq 1$, the restriction of which to the regular part of V being holomorphic, and such that there exists a C^∞ extension $\tilde{E} \rightarrow U$ of E to some neighborhood U of V in W . We shall assume also that there exists a $\mathbf{C}$ action of X_0 over $E|_{V-\Sigma}$ in the sense of Bott ([B₂]): a $\mathbf{C}$ -linear operator θ_{X_0} from the space of C^∞ sections of $E|_{V-\Sigma}$ into itself is given, such that:

$\theta_{X_0}(\sigma)$ is holomorphic whenever σ is holomorphic,

$\theta_{X_0}(u\sigma) = (X_0.u)\sigma + u\theta_{X_0}(\sigma)$ for any C^∞ function u and any section σ .

Let $\varphi \in (\mathbb{Z}[c_1, \dots, c_r])^{2p}$. We have the following generalization of theorem 1:

Let $(\sigma_1, \dots, \sigma_r)$ be a trivialization of $E|_{U_\alpha}$ (assumed to be trivial), and M_α be the $r \times r$ matrix with holomorphic coefficients $(M_\alpha)_a^b: V_\alpha - \Sigma_\alpha \rightarrow \mathbb{C}$ such that $\theta_{X_0}(\sigma_a) = \sum_b (M_\alpha)_a^b \sigma_b$.

Theorem 1'

Define:

$$I_\alpha(\theta_{X_0}, V, \varphi, E) = (-1)^{[\frac{p}{2}]} \sum_{u \in \mathcal{M}(\mathcal{U})} \int_{R_u} \frac{\varphi(M_\alpha) dz_{u_1} \wedge dz_{u_2} \wedge \dots \wedge dz_{u_p}}{\prod_{j=1}^p A_{u_j}}.$$

(i) $I_\alpha(\theta_{X_0}, V, \varphi, E)$ does not depend on the various choices of $(z_1, \dots, z_n), \mathcal{U}, \tilde{T}_\alpha, (\sigma_1, \dots, \sigma_r), R_i$,

(ii) Assume V to be compact: $\sum_\alpha I_\alpha(\theta_{X_0}, V, \varphi, E)$ is then an integer.

(iii) This integer depends only on V, φ and E , but not on X_0 and θ_{X_0} . It is in fact equal to the evaluation $\langle \varphi(E), V \rangle$ of $\varphi(E)$ on the fundamental class $[V]$ of V .

Remarks:

1) For theorem 1', V does not need to be SLCI not even LCI; this assumption was only useful for being sure that ν and $\tilde{\nu}$ exist in the example 1 below. This is still true, even for theorem 1, if we have some other reason to know that ν and $\tilde{\nu}$ exist.

2) If V is smooth, we recover the theorem 1' of [L], some particular cases of which being also in Baum-Bott [when $E = T_{\mathbb{C}}(W)|_V$ ([BB₁)]], and in Bott ([B₂)] [when X_0 is non degenerate along Σ_α].

3) Let V_α be defined by $f_\lambda = 0, \lambda = 1, \dots, q$ and invariant by a holomorphic vector field X_0 . Suppose that Σ_α is an isolated point m_α in V_α , X_0 still being holomorphic near m_α . Then, as in the previous remark, there exists a holomorphic chart $(z_1, \dots, z_n)$ near m_α such that $A_1, \dots, A_p, f_1, \dots, f_q$ form a regular sequence. In this case, we have

$$I_\alpha(\theta_{X_0}, V, \varphi, E) = \int_{R'} \frac{\varphi(M_\alpha) dz_1 \wedge dz_2 \wedge \dots \wedge dz_p}{\prod_{i=1}^p A_i},$$

where

$$R' = \{z \mid |A_i(z)| = \varepsilon, f_\lambda(z) = 0, i = 1, \dots, p, \lambda = 1, \dots, q\},$$

which is oriented so that the form $d\theta_1 \wedge \dots \wedge d\theta_p$ is positive, $\theta_i = \arg A_i(z)$.

Example 1

Assume V to be SLCI. Take $E = \nu$, and θ_{X_0} defined such as in section 2 above, with $M_\alpha = -C_\alpha$. Then we get the theorem 1 above from the theorem 1'. We shall write in this case $I_\alpha(\mathcal{F}, V, \varphi, \nu)$ instead of $I_\alpha(\theta_{X_0}, V, \varphi, \nu)$.

Example 2

Take $E = T_{\mathbb{C}}(W)|_V$, and define $\theta_{X_0}(Y) = [X_0, \tilde{Y}]|_V$, depending only on the vector field Y tangent to W along V , and not on its extension $\tilde{Y}$ to some neighbourhood of V . Then, $M_\alpha = -\frac{D(A_1, \dots, A_n)}{D(z_1, \dots, z_n)}$. The index is now this one defined in section 8 of [L], theorem 1' giving a formula for computing it. In this case, we shall write $I_\alpha(X_0, V, \varphi, T_{\mathbb{C}}(W))$ instead of $I_\alpha(\theta_{X_0}, V, \varphi, T_{\mathbb{C}}(W)|_V)$. [Notice that if we replace here X_0 by uX_0 as in theorem 1, the index is now changing!]

3- Proof of theorem 1'

Let ω be a connexion on $\tilde{E}|_{U_0}$, defined by a derivation law ∇ satisfying:

$$\begin{cases} \nabla_{X_0} \tilde{\sigma}|_{V-\Sigma} = \theta_{X_0} \sigma \text{ for every section } \sigma \text{ of } E, \\ \nabla_Z \sigma = 0 \text{ for every section } Z \in T^{0,1}(V-\Sigma) \\ \text{and every holomorphic section } \sigma \text{ of } E \end{cases}$$

(We shall say that such an ω is *special* relatively to θ_{X_0} .)

Let us give also an arbitrary connection ω_α on $\tilde{E}|_{U_\alpha}$.

Let $\varphi \in (\mathbb{Z}[c_1, \dots, c_r])^{2p}$ be a Chern polynomial having integral coefficients with respect to the Chern classes $c_1, \dots, c_r$, and defining a characteristic class of dimension $2p$. We use the notations Δ_ω for the Chern-Weil homomorphism defined by a connection ω , and $\Delta_{\omega_0 \omega_1 \dots \omega_r}(\varphi)$ the Bott's operator for iterated differences ($[B_1]$), such that: $d \circ \Delta_{\omega_0 \omega_1 \dots \omega_r} = \sum_{j=0}^r (-1)^j \Delta_{\omega_0 \dots \omega_j \dots \omega_r}$.
(In particular: $d \circ \Delta_{\omega \omega'} = \Delta_{\omega'} - \Delta_\omega$).

Proposition 2

Let: $J_\alpha(\theta_{X_0}, V, \varphi, E) = \int_{T_\alpha} \Delta_{\omega_\alpha}(\varphi) + \int_{\partial T_\alpha} \Delta_{\omega_\alpha \omega}(\varphi)$.

(i) $J_\alpha(\mathcal{F}, V, \varphi, E)$ does not depend on the choices of $\tilde{T}_\alpha$, ω , ω_α .

(ii) Assume V to be compact: $\sum_\alpha J_\alpha(\theta_{X_0}, V, \varphi, E)$ is then an integer.

(iii) This integer depends only on V and φ , but not on $\mathcal{F}$. (It is in fact nothing else but the evaluation $\langle \varphi(E), V \rangle$ of $\varphi(E)$ on the fundamental class $[V]$ of V).

[Notice that, in Proposition 2, we do not assume neither that U_α is included in the domain of a local chart, nor that $E|_{U_\alpha}$ is trivial].

The proof is exactly the same as the proof of the 3 first parts in theorem 8 of [L], just writing $\nabla_{x_0} \sigma = \theta_{x_0} \sigma$, instead of $\nabla_{x_0} Y = [X_0, Y]$.

The theorem 1' (hence the theorem 1) will follow immediately from Proposition 2 above, and from

Proposition 3

When U_α is included in the domain of a local chart, and when $\tilde{E}|_{U_\alpha}$ is trivial, then

$$I_\alpha(\theta_{x_0}, V, \varphi, E) = J_\alpha(\theta_{x_0}, V, \varphi, E).$$

In the formula of proposition 2, we may choose ω_α equal to the trivial connection ω_0 whose connection form with respect to the trivialization $(\sigma_1, \dots, \sigma_r)$ of $\tilde{E}|_{U_\alpha}$ is the matrix 0. Hence,

$$J_\alpha(\theta_{x_0}, V, \varphi, E) = \int_{\partial T_\alpha} \Delta_{\omega_0 \omega}(\varphi).$$

Remarks:

1) Notice that the integration of the same expression over only one of the connected components of $\partial T_\alpha \cap V$ would give the partial index corresponding to the corresponding "sheet" or "branch" through Σ_α .

2) If V is not LCI, we still can define $I_\alpha(\mathcal{F}, V, \varphi, \nu)$ and $J_\alpha(\mathcal{F}, V, \varphi, \nu)$ under the condition that the bundle $\nu|_{V_\alpha - \Sigma_\alpha}$ is trivializable, and conclusion of proposition 3 will still remain true. But this index will now depend on the choice of the homotopy class of the trivialization. Furthermore, if this is possible at any point of Σ , the sum of these indices has now no reason neither to be an integer nor to be independant on $\mathcal{F}$.

There are 3 steps in the proof of proposition 3:

- 1) We first study the properties of the holomorphic connections ω_i on $E|_{V_i}$, the connection form of which with respect to the given trivialization being $\frac{dz_i}{A_i} M_\alpha$.
- 2) Then, we prove that $\Delta_{\omega_0 \omega}(\varphi)$, which is a cocycle on ∂T_α , is cohomologous, when imbedded in the total Čech-de Rham complex $CDR^*(\mathcal{U})$, to the element μ in

$CDR^{2p-1}(\mathcal{U})$ defined by:

$$\begin{cases} \mu_u = \Delta_{\omega_0 \omega_{u_1} \omega_{u_2} \dots \omega_{u_p}}(\varphi) \text{ for } u \in \mathcal{M}(\mathcal{U}), \\ \mu_I = 0 \text{ for any simplex } I \text{ of dimension } \neq p-1 \text{ in the nerve of } \mathcal{U}. \end{cases}$$

3) Finally, we prove that

$$\mu_u = \frac{\varphi(M_\alpha) dz_{u_1} \wedge dz_{u_2} \wedge \dots \wedge dz_{u_p}}{\prod_{j=1}^p A_{u_j}}.$$

Using integration on $CDR^*(\mathcal{U})$ as recalled in lemma 6 below, this will achieve the proof of proposition 3.

First step:

Let Ω be an open set in $V_\alpha - \Sigma_\alpha$, Y a holomorphic non vanishing vector field tangent to Ω , and Γ a holomorphic map from Ω into the space of $r \times r$ matrices with complex coefficients. A connection $\bar{\omega}$ on $E|_\Omega$ will be said "adapted" to (Y, Γ) if its connection form relatively to the trivialization $(\sigma_1, \dots, \sigma_r)$ of $E|_\Omega$, still denoted $\bar{\omega}$, satisfies:

$$\begin{cases} \bar{\omega}(Y) = \Gamma, \\ \bar{\omega}(Z) = 0 \text{ for every section } Z \text{ of } T^{0,1}(V_\alpha - \Sigma_\alpha). \end{cases}$$

Hence the restriction to Ω of a "special" connection, such as defined for proposition 2, is adapted to (X_0, M_α) , while the restriction to Ω of the trivial connection ω_0 is adapted to any $(Y, \text{matrix } 0)$ for Y holomorphic tangent to Ω . From the usual vanishing theorem (Bott ([B₁]), Kamber-Tondeur ([KT])), we deduce the

Lemma 4

Let $\dim \varphi = 2p$.

$$\begin{cases} \text{If } \bar{\omega} \text{ is adapted to some } (Y, \Gamma), \Delta_{\bar{\omega}}(\varphi) = 0, \\ \text{If } \bar{\omega}_1, \dots, \bar{\omega}_k \text{ are adapted to the same } (Y, \Gamma), \Delta_{\bar{\omega}_1, \dots, \bar{\omega}_k}(\varphi) = 0. \end{cases}$$

For any q multiindex $I = (1 \leq i_1, i_2, \dots, i_q \leq n)$ (the i_j 's being all distinct), define

$$D_I = \det \frac{D(f_1, \dots, f_q)}{D(z_{i_1}, \dots, z_{i_q})}.$$

For any $u \in \mathcal{M}$, define the q multiindex $\bar{u} = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_q)$ so that

$$1 \leq \bar{u}_1 < \bar{u}_2 < \dots < \bar{u}_q \leq n, \text{ and } \{1, 2, \dots, n\} = \{u_1, u_2, \dots, u_p\} \cup \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_q\},$$

and by $\Omega_{\bar{u}}$ the open set of points in V_α where $D_{\bar{u}} \neq 0$: $\Omega_{\bar{u}}$ is a union of open sets where the restrictions of the functions $z_{u_1}, \dots, z_{u_p}$ constitute a system of local coordinates. For any $q+1$ multiindex $I = (1 \leq i_0, i_1, \dots, i_q \leq n)$, Y_I will denote the holomorphic vector field:

$$Y_I = \sum_{k=0}^q (-1)^k D_{I-i_k} \frac{\partial}{\partial z_{i_k}}.$$

Lemma 5

(i) Y_I is tangent to V .

(ii) For $m \in \mathcal{V}_i$ ($1 \leq i \leq n$), there exists $u \in \mathcal{M}$ containing i such that $D_{\bar{u}} \neq 0$ at the point m .

(iii) For any i ($1 \leq i \leq n$), the connection $\omega_i = \frac{dz_i}{A_i} M_\alpha$ on $E|_{\mathcal{V}_i}$ satisfies the following condition: for any $u \in \mathcal{M}$ containing i , the restriction of ω_i to $\Omega_{\bar{u}}$ is simultaneously adapted to (X_0, M_α) and to any $(Y_{u_j+\bar{u}}, \text{matrix } 0)$ such that $u_j \neq i$.

Let in fact I be some $q+1$ multi index such that $D_{I-i_k} \neq 0$ at some point m in V for some $i_k \in I$: it means that the restrictions $\tilde{z}_i$ to V of the functions z_i constitute, for i belonging to $\{1, 2, \dots, n\} - \{I - i_k\}$ (in particular for $i = i_k$), a system of local coordinates on V near m . But then, the restriction of Y_I to the domain of such a local chart is equal to $(-1)^k D_{I-i_k} \frac{\partial}{\partial \tilde{z}_{i_k}}$ and is therefore tangent to V , hence part (i) of the lemma.

The condition for X_0 to be tangent to V may be written:

$$\sum_{j=1}^n A_j (f_\lambda)'_{z_j} = 0 \text{ on } V_\alpha \text{ for all } \lambda = 1, \dots, q.$$

Hence, if $m \in \mathcal{V}_i$, the q dimensional vector $\left((f_\lambda)'_{z_i} \right)_{\lambda=1, \dots, q}$ is, on V_α , a linear combination of the others $\left((f_\lambda)'_{z_j} \right)_{\lambda=1, \dots, q}$ ($j \neq i$): D_J must be zero at m for any q multiindex J containing i . But, since $\mathcal{V}_i$ is in the regular part of V , one at least of the D_J must be $\neq 0$: the only possibility is therefore that $i \notin J$ for such an J , hence part (ii) of the lemma.

On $\Omega_{\bar{u}}$, $X_0 = \sum_{j=1}^p A_{u_j} \frac{\partial}{\partial \tilde{z}_{u_j}} = \frac{1}{D_{\bar{u}}} \sum_{j=1}^p A_{u_j} Y_{u_j+\bar{u}}$ and, on $\mathcal{V}_i \cap \Omega_{\bar{u}}$, the p holomorphic vector fields X_0 and $\left(Y_{u_j+\bar{u}} \right)_{u_j \neq i}$ are linearly independant. The part (iii) of the lemma becomes now obvious to check, since $\mathcal{V}_i$ is covered by the $\Omega_{\bar{u}}$ such that $i \in u$.

Second step:

For any k simplex $I = (i_0 \cdots i_k)$ in the nerve of $\mathcal{U}$, write:

$$\Delta_{\omega_0 \omega_I}(\varphi) = \Delta_{\omega_0 \omega_{i_0} \cdots \omega_{i_k}}(\varphi), \Delta_{\omega \omega_I}(\varphi) = \Delta_{\omega \omega_{i_0} \cdots \omega_{i_k}}(\varphi),$$

and $\Delta_{\omega_0 \omega_I}(\varphi) = \Delta_{\omega_0 \omega_{i_0} \cdots \omega_{i_k}}(\varphi)$.

Define $\gamma \in CDR^{2p-1}(\mathcal{U})$ as the family $(\gamma_I)_I$ given by:

$$\gamma_I = (-1)^{[\frac{k+1}{2}]} \Delta_{\omega_0 \omega_I}(\varphi), \text{ where } k \text{ denotes the dimension } |I| \text{ of } I.$$

Then, the total differential $D\gamma$ of γ in $CDR^*(\mathcal{U})$ is given by:

$$\begin{aligned} (D\gamma)_I &= (-1)^{[\frac{k+1}{2}]+k} \left(\Delta_{\omega \omega_I}(\varphi) - \Delta_{\omega_0 \omega_I}(\varphi) + \sum_{\alpha=0}^k (-1)^\alpha \Delta_{\omega_0 \omega_{I-i_\alpha}}(\varphi) \right) \\ &\quad + \sum_{\alpha=0}^k (-1)^{[\frac{k}{2}]+\alpha+1} \Delta_{\omega_0 \omega_{I-i_\alpha}}(\varphi) \\ &= (-1)^{[\frac{k+1}{2}]+k} \left(\Delta_{\omega \omega_I}(\varphi) - \Delta_{\omega_0 \omega_I}(\varphi) \right)(\varphi), \text{ for } |I| > 0, \end{aligned}$$

$$\text{and } (D\gamma)_i = \Delta_{\omega \omega_i}(\varphi) - \Delta_{\omega_0 \omega_i}(\varphi) + \Delta_{\omega_0 \omega}(\varphi) \text{ for } |I| = 0.$$

But all terms $\Delta_{\omega \omega_I}(\varphi)$ vanish because the connections $\omega, \omega_{i_0}, \dots, \omega_{i_k}$ are all adapted to the same (X_0, M_α) , all terms $\Delta_{\omega_0 \omega_I}(\varphi)$ vanish for $|I| < p-1$ because the connections $\omega_0, \omega_{i_0}, \dots, \omega_{i_k}$ are all adapted to a same $(Y, \text{matrix } 0)$, and all terms of $(D\gamma)_I$ vanish for $|I| \geq p$ because $\Delta_{\omega_0 \cdots \omega_r}(\varphi)$ is always 0 for any family of $r+1$ connections when $r > p$. Therefore, it remains only:

$$(D\gamma)_i = \Delta_{\omega_0 \omega}(\varphi) \text{ for } I = \{i\} \text{ of dimension } 0,$$

$$(D\gamma)_u = -\mu_u \text{ for } u \in \mathcal{M}(\mathcal{U}) \text{ of dimension } p-1,$$

$$\text{all others } (D\gamma)_I \text{ 's being } 0. \text{ This proves: } D\gamma = \iota \left(\Delta_{\omega_0 \omega}(\varphi) \right) - \mu,$$

where ι denotes the natural imbedding of the de Rham complex $\Omega_{DR}^*(\partial\mathcal{T}_\alpha)$ into $CDR^*(\mathcal{U})$.

Third step:

The set $\mathcal{V}_u$ equal to $\cap_{j=1}^p \mathcal{V}_{u_j}$ is included into $\Omega_{\bar{u}}$. In fact, as already seen at lemma 5, if m belongs to $\mathcal{V}_i$, D_I must be zero when $i \in I$: so if $m \in \mathcal{V}_u$, u is the only possible element v in $\mathcal{M}(\mathcal{U})$ such that $D_v \neq 0$.

For computing $\Delta_{\omega_0 \omega_{u_1} \cdots \omega_{u_p}}$, we introduce (Bott [B₁]) the connection $\tilde{\omega}$ on $(\tilde{E}|_{\mathcal{V}_u}) \times \Delta^p \rightarrow \mathcal{V}_u \times \Delta^p$, (Δ^p denoting the p -simplex $0 \leq \sum_{i=1}^p t_i \leq 1, 0 \leq t_i \leq 1$, in $\mathbf{R}^p$), defined by $\tilde{\omega} = \sum_{i=1}^p t_i \omega_i + \left[1 - (\sum_{i=1}^p t_i) \right] \omega_0 = \left(\sum_{j=1}^p \frac{t_j}{A_{u_j}} dz_{u_j} \right) M_\alpha$.

The curvature $\tilde{\Omega}$ of this connection is then equal to

$$\tilde{\Omega} = \left(\sum_{j=1}^p dt_j \wedge \frac{1}{A_{u_j}} dz_{u_j} \right) M_\alpha + \text{ (terms without any } dt_k \text{)}.$$

Therefore, for every polynomial φ in $Chern^{2p}[c_1 \dots c_n]$,

$$\Delta_{\omega}(\varphi) = p!(-1)^{[\frac{p}{2}]} dt_1 \wedge dt_2 \wedge \dots \wedge dt_p \wedge \frac{\varphi(M_{\alpha}) dz_{u_1} \wedge \dots \wedge dz_{u_p}}{\prod_{j=1}^p A_{u_j}} \\ + (\text{terms of degree} < p \text{ in } dt_j)$$

By integration over Δ^p , and using the equality $\int_{\Delta^p} dt_1 \wedge \dots \wedge dt_p = \frac{1}{p!}$, we get ([B₁] p.64):

$$\Delta_{\omega_0 \omega_1 \dots \omega_p}(\varphi) = \frac{\varphi(M_{\alpha}) dz_{u_1} \wedge dz_{u_2} \wedge \dots \wedge dz_{u_p}}{\prod_{j=1}^p A_{u_j}}.$$

This achieves the proof of proposition 3, hence of theorems 1' and 1, once using:

Lemma 6

There exists a linear map $L : CDR^{2p-1}(\mathcal{U}) \rightarrow \mathbb{C}$ with the following properties:

- i) L vanishes on the total coboundaries $D\left(CDR^{2p-2}(\mathcal{U})\right)$,*
 - ii) L extends simultaneously the integration $\int_{\partial T_{\alpha}} : \Omega_{DR}^{2p-1}(\partial T_{\alpha}) \rightarrow \mathbb{C}$,*
- and the map: $(-1)^{[\frac{p}{2}]} \sum_{u \in \mathcal{M}(\mathcal{U})} \int_{R_u} : C^{p-1}(\mathcal{U}, \Omega_{DR}^p) \rightarrow \mathbb{C}$.*

Proof: See section 6 of [L].

4- Examples

Let W be the 3-dimensional complex projective space $\mathbf{CP}^3$, of points $[X, Y, Z, T]$ with homogeneous coordinates X, Y, Z, T . Take for V the cone V_l of equation

$$X^l + Y^l + Z^l = 0 \quad (l \text{ being any integer } \geq 1),$$

which has a single isolated singular point $O = [0, 0, 0, 1]$. Denote by U_T, U_Z and U_Y the affine spaces $T \neq 0, Z \neq 0$ and $Y \neq 0$ with respective coordinates $(x = \frac{X}{T}, y = \frac{Y}{T}, z = \frac{Z}{T})$, $(x' = \frac{X}{Z}, y' = \frac{Y}{Z}, t' = \frac{T}{Z})$ and $(x'' = \frac{X}{Y}, z'' = \frac{Z}{Y}, t'' = \frac{T}{Y})$. The 3 open sets U_T, U_Z, U_Y cover V_l since the point $[1, 0, 0, 0]$ does not belong to V_l . The corresponding equations of V_l may be written respectively: $f_T = 0, f_Z = 0, f_Y = 0$, with:

$$f_T(x, y, z) = x^l + y^l + z^l,$$

$$f_Z(x', y', t') = x'^l + y'^l + 1, \text{ and } f_Y(x'', z'', t'') = x''^l + z''^l + 1.$$

The bundle $\tilde{\nu}$ is defined by the cocycle

$$(g_{TZ} = z^l = \frac{1}{t'^l}, \quad g_{TY} = y^l = \frac{1}{t''^l}, \quad g_{ZY} = y''^l = \frac{1}{z'^l}).$$

In general, for a hypersurface V_l of degree l in $\mathbf{CP}^n$ ($\dim_{\mathbf{C}} V_l = p = n - 1$), we have (see Example 2 in section 2)

$$\langle (c_1)^p(\nu), V_l \rangle = l^{n-1} \int_{V_l} c^{n-1} = l^n.$$

Also, from $T_{\mathbf{C}}(\mathbf{CP}^n) \oplus 1 = (n+1)\check{L}$, we have:

$$1 + c_1(T_{\mathbf{C}}) + c_2(T_{\mathbf{C}}) + \dots = (1 + c)^{n+1},$$

hence:

$$c_1(T_{\mathbf{C}}(\mathbf{CP}^n)) = (n+1)c, \quad c_2(T_{\mathbf{C}}(\mathbf{CP}^n)) = \frac{(n+1)n}{2}c^2, \dots$$

In particular, for $p = 2$, $q = 1$, we get:

$$\begin{aligned} \langle (c_1)^2(T_{\mathbf{C}}(\mathbf{CP}^3)), V_l \rangle &= (3+1)^2 \int_{V_l} c^2 = 16l, \\ \langle c_2(T_{\mathbf{C}}(\mathbf{CP}^3)), V_l \rangle &= \frac{4 \cdot 3}{2} \int_{V_l} c^2 = 6l. \end{aligned}$$

Example 1:

Take for X_0 the extension H to the whole $\mathbf{CP}^3$ of the vector field of infinitesimal homotheties $x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$ in U_T . (In U_Z and U_Y , H is equal respectively to $-t' \frac{\partial}{\partial t'}$ and $-t'' \frac{\partial}{\partial t''}$). This vector field has for singular set the union of $\{O\}$ and of the hyperplane $T = 0$, and Σ has 2 connected components: Σ_1 is the isolated point $\{O\}$, and Σ_2 the curve $(X' + Y' + Z' = 0, T = 0)$. Notice however that Σ_2 does not contain any singularity for the foliation $\mathcal{F}$ generated by H , so that we can already assert:

$$I_2(\mathcal{F}, V_l, (c_1)^2, \nu) = 0.$$

1) Computation of $I_1(\mathcal{F}, V_l, (c_1)^2, \nu)$ and $I_1(H, V_l, \varphi, T_{\mathbf{C}}(W))$ ($\varphi = (c_1)^2$ or c_2):

For $E = \nu$, $H.f_T = lf_T$ and $M_0 = -C_0$ is the 1×1 constant matrix $(-l)$. For $E = T_{\mathbf{C}}(W)|_V$, $M_0 = -\frac{D(x,y,z)}{D(x,y,z)}$ is equal to the opposite of the 3×3 identity matrix, in such a way that for $E = \nu$, $(c_1)^2(M_0)$ is a constant equal to $\frac{-l^2}{4\pi^2}$,

while for $E = T_{\mathbf{C}}(W)|_V$, $\varphi(M_0)$ is also a constant equal to $\begin{cases} \frac{-9}{4\pi^2} & \text{if } \varphi = (c_1)^2, \\ \frac{-3}{4\pi^2} & \text{if } \varphi = c_2. \end{cases}$

(Recall that, c_k applied to some matrix is equal to $(\frac{i}{2\pi})^k$ times the k th elementary symmetric function of the eigenvalues).

We compute the indices in two ways; first directly by the definition in theorem 1 or 1' and then applying theorem 2.

(i) Take for $\tilde{T}$ the ball $\text{Sup}(|x|, |y|, |z|) \leq \varepsilon$ for some positive constant ε . Let R_z be the region in the boundary ∂T defined by $|z| \geq |x|, |z| \geq |y|$, and define R_x and R_y similarly. The index $I_1(\theta_H, V_l, \varphi, E)$ at the origin O is equal in both cases to

$$-\varphi(M_0) \left(\int_{R_{xy}} \frac{dx}{x} \wedge \frac{dy}{y} + \int_{R_{yz}} \frac{dy}{y} \wedge \frac{dz}{z} + \int_{R_{xz}} \frac{dx}{x} \wedge \frac{dz}{z} \right).$$

On R_{xy} , we may write: $x = \varepsilon e^{i\theta}$, $y = \varepsilon e^{i\sigma}$, and $\frac{dx}{x} \wedge \frac{dy}{y} = -d\theta \wedge d\sigma$, which is positive on R_{xy} . [In fact, remember ([L]) the convention about the orientation of R_{xy} by the normal from R_x to R_y : let us write $x = re^{i\theta}$ and $y = se^{i\sigma}$ on T ; then $dr \wedge d\theta \wedge ds \wedge d\sigma$ is positive on T with r increasing when approaching $\partial T \cap R_x$, $r = \varepsilon$ and $d\theta \wedge ds \wedge d\sigma$ is positive on R_x with s increasing when approaching the boundary near R_{xy} , in such a way that $-d\theta \wedge d\sigma$ is positive on R_{xy}]. But there, we have $z^l = -(x^l + y^l) = -2\varepsilon^l \cos \frac{l(\sigma - \theta)}{2} e^{i \frac{l(\sigma + \theta)}{2}}$, so that R_{xy} is an l -fold covering of the set of (θ, σ) such that $2\varepsilon^l |\cos(\sigma - \theta)| \leq \varepsilon^l$ (because $|z| \leq \varepsilon$ on R_{xy}). It is easy to check that the set of (θ, σ) in the square $[0, 2\pi]^2$ where the previous condition holds is made of l strips, the area of each one being $\frac{2\pi}{3} \times 2\pi = \frac{4\pi^2}{3}$. Then, because of the l sheets of the covering, we get: $\int_{R_{xy}} \frac{dx}{x} \wedge \frac{dy}{y} = \frac{4l\pi^2}{3}$. The computation is the same for the two others integrals, so that

$$\int_{R_{xy}} \frac{dx}{x} \wedge \frac{dy}{y} + \int_{R_{yz}} \frac{dy}{y} \wedge \frac{dz}{z} + \int_{R_{xz}} \frac{dx}{x} \wedge \frac{dz}{z} = 4l\pi^2.$$

(ii) We observe that, in this case, x , y and f_T form a regular sequence (see the Remark after Theorem 2 and Remark 3) after Theorem 1'), and we may take for $\tilde{T}$ the ball $\text{Sup}(|x|, |y|, |f_T|) \leq \varepsilon$. The index $I_1(\theta_H, V_l, \varphi, E)$ at the origin O is equal to

$$\varphi(M_0) \int_{R'} \frac{dx}{x} \wedge \frac{dy}{y},$$

where R' is the 2-submanifold in the boundary ∂T given by

$$R' = \{(x, y, z) \mid |x| = |y| = \varepsilon, x^l + y^l + z^l = 0\}.$$

On R' , we may write: $x = \varepsilon e^{i\theta}$, $y = \varepsilon e^{i\sigma}$, and $\frac{dx}{x} \wedge \frac{dy}{y} = -d\theta \wedge d\sigma$, which is negative on R' . But there, we have $z^l = -(x^l + y^l)$, so that R' is an l -fold covering of the set of (θ, σ) in the square $[0, 2\pi]^2$. Thus we get

$$\int_{R'} \frac{dx}{x} \wedge \frac{dy}{y} = -4l\pi^2.$$

In either way we get:

$$I_1(\mathcal{F}, V_l, (c_1)^2, \nu) = l^3, \text{ and}$$

$$I_1(H, V, \varphi, T_{\mathbf{C}}(W)) = \begin{cases} 9l & \text{if } \varphi = (c_1)^2, \\ 3l & \text{if } \varphi = c_2, \end{cases}$$

2) Computation of $I_2(H, V_l, \varphi, T_{\mathbf{C}}(W))$:

Since Σ_2 is a smooth compact holomorphic manifold in the regular part of V_l , we may use the Bott's theorem ([B₂] p.314) for computing the index, under the condition that the infinitesimal action of H on the bundle N normal to Σ_2 in V_l be non degenerate. Since V_l is compact, this action will be of constant type along Σ_2 , and the same thing is true for the action $\theta_H|_{\Sigma_2}$ of H . So, it is enough to calculate them for instance along $\Sigma_2 \cap U_Z$. Since $\frac{\partial f_Z}{\partial x'} = lx'^{l-1}$, and $\frac{\partial f_Z}{\partial y'} = ly'^{l-1}$, and because both coordinates x' and y' may not vanish simultaneously over $\Sigma_2 \cap U_Z$, we may assume for instance $x' \neq 0$. Near such a point in $\Sigma_2 \cap U_Z$, we may replace the coordinates (x', y', t') by $(u = f_Z(x', y', t'), v = y', w = t')$, so that V_l has now for local equation $u = 0$, while Σ_2 is now locally defined by $u = 0, w = 0$. The bundle N is generated by $\frac{\partial}{\partial w}$, $H = -w \frac{\partial}{\partial w}$, and $[H, \frac{\partial}{\partial w}] = \frac{\partial}{\partial w}$: therefore this action, represented by the constant 1×1 matrix $(+1)$, is effectively non degenerate. On the other hand, ν is generated by $\frac{\partial}{\partial u}$, so that $[H, \frac{\partial}{\partial u}] = 0$, while the third bracket $[H, \frac{\partial}{\partial v}]$ being also 0, the action $\theta_H|_{\Sigma_2}$ on $T_{\mathbf{C}}(W)$ will be represented by the constant matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Denote a, b, c the formal classes such that the k th Chern class of W is equal to the k th elementary symmetric function of a, b, c . After Bott, we have:

$$I_2(H, V_l, \varphi, T_{\mathbf{C}}(W)) = \left\langle \frac{\hat{\varphi} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c+1 \end{pmatrix}}{1 + c_1(N)}, \Sigma_2 \right\rangle,$$

where $\hat{\varphi}$ denotes $(a + b + c + 1)^2$ for $\varphi = (c_1)^2$, and $ab + (a + b)(c + 1)$ for $\varphi = c_2$. Hence, we get:

$$I_2(H, V, \varphi, T_{\mathbf{C}}(W)) = \begin{cases} \langle 2c_1(T_{\mathbf{C}}(W)) - c_1(N), \Sigma_2 \rangle, & \text{for } (c_1)^2, \\ \text{and } \langle a + b, \Sigma_2 \rangle & \text{for } c_2. \end{cases}$$

Notice that N coincides with the restriction to Σ_2 of the hyperplane bundle $\tilde{L} \rightarrow \mathbf{CP}^2$ after identification of $\mathbf{CP}^2$ with the hyperplane $T = 0$ in $\mathbf{CP}^3$, while $T_{\mathbf{C}}(W)$ is stably equivalent to $4\tilde{L}$, and $(a+b)|_{\mathbf{CP}^2} = c_1(\mathbf{CP}^2) = 3c_1(\tilde{L})$. We get therefore $7 < c_1(\tilde{L}), \Sigma_2 > = 7l$ for $(c_1)^2$, and $3 < c_1(\tilde{L}), \Sigma_2 > = 3l$ for c_2 .

Finally, we recover:

$$< (c_1)^2(\nu), V_l > = l^3 + 0 = l^3,$$

$$< (c_1)^2(T_{\mathbf{C}}(W)), V_l > = 9l + 7l = 16l, \text{ and } < c_2(T_{\mathbf{C}}(W)), V_l > = 3l + 3l = 6l.$$

In particular, for $l = 2$, we get:

$$< (c_1)^2(\nu), V_2 > = 8, \quad \text{and}$$

$$< (c_1)^2(T_{\mathbf{C}}(W)), V_2 > = 32, \quad < c_2(T_{\mathbf{C}}(W)), V_2 > = 12.$$

Example 2:

Take $l = 2$, with now for X_0 the extension $\mathcal{R}$ to the whole $\mathbf{CP}^3$ of the vector field of infinitesimal "complex rotations" $y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$ in U_T .

In U_Z (resp. in U_Y), $\mathcal{R}$ writes $y' \frac{\partial}{\partial x'} - x' \frac{\partial}{\partial y'}$ (resp. $(x''^2 + 1) \frac{\partial}{\partial x''} + x'' z'' \frac{\partial}{\partial z''} + x'' t'' \frac{\partial}{\partial t''}$). Now Σ is made of 3 isolated points: $m_1 = [0, 0, 0, 1]$, $m_2 = [i, 1, 0, 0]$ and $m_3 = [-i, 1, 0, 0]$. Notice that V_2 is regular at m_2 and m_3 . We have :

$\mathcal{R}.f_T = 0$, $\mathcal{R}.f_Z = 0$, and $\mathcal{R}.f_Y = 2x''f_Y$, this proves that $\mathcal{R}$ still preserves V , and that $I_1(\mathcal{R}, V, (c_1)^2, \nu) = 0$ since $m_1 \in U_T$.

1) Computation of $I_1(\mathcal{R}, V_2, \varphi, T_{\mathbf{C}}(W))$:

In this case, y , $-x$ and f_T form a regular sequence and we may take for $\tilde{T}$ the ball

$\text{Sup}(|x|, |y|, |f_T|) \leq \varepsilon$ for some positive constant ε . The index $I_1(\theta_{X_0}, V, \varphi, E)$ at the origin O is then equal to

$$\int_{R'} \varphi(M_1) \frac{dx \wedge dy}{-xy},$$

where R' is the 2-submanifold in the boundary ∂T given by

$$R' = \{ (x, y, z) \mid |y| = |-x| = \varepsilon, x^2 + y^2 + z^2 = 0 \}.$$

If we write: $x = \varepsilon e^{i\theta}$, $y = \varepsilon e^{i\sigma}$ on R' , $d\sigma \wedge d\theta$ is positive on R' . Hence we have

$$\int_{R'} \frac{dx \wedge dy}{-xy} = -8\pi^2. \quad \text{When } E = T_{\mathbf{C}}(W)|_V, M_1 \text{ is now the matrix } \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}:$$

$\varphi(M_1)$ is still a constant, now equal to 0 for $\varphi = (c_1)^2$, and to $\frac{-1}{4\pi^2}$ for $\varphi = c_2$. Then we have,

$$I_1(\mathcal{F}, V_2, (c_1)^2, \nu) = I_1(X_0, V_2, (c_1)^2, T_{\mathbf{C}}(W)) = 0, \text{ and } I_0(X_0, V_2, c_2, T_{\mathbf{C}}(W)) = 2.$$

2) Computation of indices at points m_2 and m_3 :

Observe that $\frac{\partial f_Y}{\partial x''} = 2x'' \neq 0$ near these points. Then we may use $(u = f_Y, v = z'', w = t'')$ instead of (x'', z'', t'') as local coordinates, with $\mathcal{R} = x(2u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w})$. The tangent space to V is generated by $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial w}$. Since the restriction $x(v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w})$ is nondegenerate at m_2 and m_3 , with eigenvalues $(\varepsilon i, \varepsilon i)$ with $\varepsilon = 1$ (resp. -1) at m_2 (resp. m_3), we may use the Bott's formula. The normal bundle ν is generated by $\frac{\partial}{\partial u}$, and the action of R on ν at points m_2 and m_3 is given by the 1×1 matrix $(-2\varepsilon i)$, and :

$$I_2(\mathcal{F}, V, (c_1)^2, \nu) = I_3(\mathcal{F}, V, (c_1)^2, \nu) = 4.$$

The action of $\mathcal{R}$ on $T_{\mathbf{C}}(W)$ is given by the matrix $-\varepsilon i \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and

$$I_2(\mathcal{R}, V_2, (c_1)^2, T_{\mathbf{C}}(W)) = I_3(\mathcal{R}, V_2, (c_1)^2, T_{\mathbf{C}}(W)) = 16,$$

$$I_2(\mathcal{R}, V_2, c_2, T_{\mathbf{C}}(W)) = I_3(\mathcal{R}, V_2, c_2, T_{\mathbf{C}}(W)) = 5.$$

We may notice that we still have, as in example 1:

$$\langle (c_1)^2(\nu), V_2 \rangle = 0 + 4 + 4 = 8,$$

$$\langle (c_1)^2(T_{\mathbf{C}}(W)), V_2 \rangle = 0 + 16 + 16 = 32,$$

$$\text{and } \langle c_2(T_{\mathbf{C}}(W)), V_2 \rangle = 2 + 5 + 5 = 12.$$

Example 3:

Take still $l = 2$, with now for X_0 the linear combination $X_\omega = aH + b\mathcal{R}$ of examples 1 and 2, where $\omega \in [0, \frac{\pi}{2}]$, $a = \cos \omega$, $b = \sin \omega$, ($a \neq 0$). In U_T , $X_\omega = a \left[x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right] + b \left[y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right]$ has only for singular point the origin m_1 . In U_Z , $X_\omega = b(y' \frac{\partial}{\partial x'} - x' \frac{\partial}{\partial y'}) - at' \frac{\partial}{\partial t'}$, has no singular point on V_2 . In U_Y , $X_\omega = b(x''^2 + 1) \frac{\partial}{\partial x''} + bx''z'' \frac{\partial}{\partial z''} + t''(bx'' - a) \frac{\partial}{\partial t''}$ has the same singular points m_2 and m_3 as in example 2.

1) Computation of indexes at point m_1 :

Since, $X_\omega \cdot f_T = 2af_T$, the 1×1 matrix C_1 is constant equal to $((-2a))$, so that $(c_1)^2(C_1) = \frac{-a^2}{\pi^2}$.

Write: $A = ax + by$, $B = -bx + ay$ and $C = az$.

We have $\frac{D(A,B,C)}{D(x,y,z)} = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & a \end{pmatrix}$, and $\varphi(-\frac{D(A,B,C)}{D(x,y,z)})$ is still a constant equal to

$$\begin{cases} \frac{-9a^2}{4\pi^2} & \text{if } \varphi = (c_1)^2, \\ \frac{-(3a^2 + b^2)}{4\pi^2} & \text{if } \varphi = c_2. \end{cases}$$

In this case, A , B and f_T form a regular sequence and we may take for $\tilde{T}$ the ball $\text{Sup}(|A|, |B|, |f_T|) \leq \varepsilon$ for some positive constant ε . The index $I_1(\mathcal{F}, V_2, \varphi, E)$ at the origin O is equal to

$$\varphi(M_1) \int_{R'} \frac{dx \wedge dy}{AB},$$

where R' is the 2-submanifold in the boundary ∂T given by

$$R' = \{(x, y, z) \mid |A| = |B| = \varepsilon, x^2 + y^2 + z^2 = 0\}.$$

Since $dx \wedge dy = dA \wedge dB$, the integral is computed as in example 1 to get: $\int_{R'} \frac{dx \wedge dy}{AB} = -8\pi^2$. Thus we have

$$I_1(\mathcal{F}, V_2, \varphi, E) = \begin{cases} 8a^2 & \text{for } E = \nu \text{ and } \varphi = (c_1)^2, \\ 18a^2 & \text{for } E = T_{\mathbb{C}}W \text{ and } \varphi = (c_1)^2, \\ 2(3a^2 + b^2) & \text{for } E = T_{\mathbb{C}}W \text{ and } \varphi = c_2. \end{cases}$$

2) Computation of indices at points m_2 and m_3 :

We already observed that $\frac{\partial f_Y}{\partial x''} = 2x'' \neq 0$ near these points, so that we may use $(u = f_Y, v = z'', w = t'')$ instead of (x'', z'', t'') as local coordinates, with $X_\omega = bx''(2u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}) + (bx'' - a)w \frac{\partial}{\partial w}$. The tangent space to V_2 is generated by $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial w}$. The restriction $bx''v \frac{\partial}{\partial v} + (bx'' - a)w \frac{\partial}{\partial w}$ of X_ω to V_2 has for eigenvalues $(b\varepsilon i, b\varepsilon i - a)$ with $\varepsilon = 1$ (resp. -1) at m_2 (resp. m_3). It is therefore nondegenerate at these points, and we may use the Bott's formula.

The normal bundle ν is generated by $\frac{\partial}{\partial u}$, the action of X_ω on ν at points m_2 and m_3 is given by the 1×1 matrix $((-2b\varepsilon i))$, and :

$$I_2(\mathcal{F}, V, (c_1)^2, \nu) = -\frac{4b^2}{ib(ib-a)} = 4b(b-ai), \text{ while}$$

$$I_3(\mathcal{F}, V, (c_1)^2, \nu) = 4b(b+ai). \text{ We recover:}$$

$$\langle (c_1)^2(\nu), V_2 \rangle = 8a^2 + 4b(b-ai) + 4b(b+ai) = 8.$$

The action of X_ω on $T_{\mathbf{C}}(W)$ has for eigenvalues: $(-2b\epsilon i, -b\epsilon i, -(b\epsilon i - a))$.

$$I_2(X_\omega, V_2, (c_1)^2, T_{\mathbf{C}}(W)) = \frac{(4ib-a)^2}{ib(ib-a)} = (16b^2 + 7a^2) - i\frac{a(8b^2-a^2)}{b}, \text{ while}$$

$$I_3(X_\omega, V_2, (c_1)^2, T_{\mathbf{C}}(W)) = (16b^2 + 7a^2) + i\frac{a(8b^2-a^2)}{b}. \text{ We recover:}$$

$$\langle (c_1)^2(T_{\mathbf{C}}(W)), V_2 \rangle = 18a^2 + 2(16b^2 + 7a^2) = 32.$$

$$I_2(X_\omega, V_2, c_2, T_{\mathbf{C}}(W)) = \frac{2(bi)^2 + 2bi(bi-a) + bi(bi-a)}{ib(ib-a)} = 5b^2 + 3a^2 - 2iab, \text{ while}$$

$$I_3(X_\omega, V_2, c_2, T_{\mathbf{C}}(W)) = 5b^2 + 3a^2 + 2iab. \text{ We recover:}$$

$$\langle c_2(T_{\mathbf{C}}(W)), V_2 \rangle = 2(3a^2 + b^2) + 2(5b^2 + 3a^2) = 12.$$

We may notice, in accordance with the theory, that the indices themselves are not necessarily integers and depend on a, b , contrary to their sum. Notice also that we recover the values of example 1 ($l = 2$) for $\omega = 0$, and of example 2 for $\omega = \frac{\pi}{2}$. However the calculation for this last case had to be done separately, because we assumed explicitly $C \neq 0$ near m_0 in the calculation of example 3.

References

- [B₁] R. Bott: *Lectures on characteristic classes and foliations*, Lectures on Algebraic and Differential Topology, Lecture Notes in Mathematics 279, Springer-Verlag, New York, Heidelberg, Berlin, 1972, 1-94.
- [B₂] R. Bott: *A residue formula for holomorphic vector fields*, J. of Diff. Geom. 1 (1967) 311-330.
- [BB₁] P. Baum and R. Bott: *On the zeroes of holomorphic vector fields*, Essays on Topology and related topics (Mémoires dédiés à Georges de Rham), Springer (1970) 29-47.
- [BB₂] P. Baum and R. Bott: *Singularities of holomorphic foliations*, J. of Diff. Geom. 7 (1972) 279-342.
- [CS] C. Camacho and P. Sad: *Invariant varieties through singularities of holomorphic vector fields*, Ann. of Math. 115 (1982) 579-595.

[G] B. Gmira: *Une généralisation d'un théorème de C. Camacho et P. Sad relatif aux feuilletages holomorphes singuliers*. Thèse de III^{ème} cycle, Lille, 1984. See also: *Sur les feuilletages holomorphes singuliers de codimension 1*, Publ. Mathématiques 36 (1992), 229-240.

[KT] F. Kamber and P. Tondeur: *Foliated Bundles and Characteristic Classes*, Lecture Notes in Mathematics 493, Springer-Verlag, New York, Heidelberg, Berlin, 1975.

[L] D. Lehmann: *Résidus des sous variétés invariantes d'un feuilletage singulier*, Ann. Inst. Fourier 41 (1991) 211-258.

[LN₁] A. Lins Neto: *Algebraic solutions of polynomial differential equations and foliations in dimension two*, Holomorphic Dynamics, Mexico 1986, Lecture Notes in Mathematics 1345, Springer-Verlag, New York, Heidelberg, Berlin, 1988, 192-232.

[LN₂] A. Lins Neto: *Complex codimension one foliations leaving a compact submanifold invariant*, Dynamical Systems and Bifurcation Theory 1985, Pitman Research Notes in Mathematics Series 160, Longman Scientific and Technical, Harlow, New York, 1987, 295-317.

[So] M. Soares: *A note on algebraic solutions of foliations in dimension 2*, Dynamical Systems 1990, Pitman Research Notes in Mathematics Series 285, Longman Scientific and Technical, Harlow, New York, 1993, 250-254.

[Su] T. Suwa: *Indices of holomorphic vector fields relative to invariant curves*, preprint, 1993.

Daniel LEHMANN

GETODIM, CNRS, UA 1407. Université de Montpellier II, 34095 Montpellier cedex, France.

Email: LEHMANN@frmop22.cnusc.fr Fax: (33) 67 54 30 79

Tatsuo SUWA

Department of Mathematics, Hokkaido University
Sapporo 060, Japan

Email: SUWA@math.hokudai.ac.jp Fax : (81) 11 727 3705