

L^2 -Solutions for Nonlinear Schrödinger Equations and Nonlinear Groups

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§1. Introduction and main results.

We consider the unique global existence of solutions in a weaker class than the energy space, i.e., $H^1(\mathbb{R}^n)$ for the Cauchy problem of the nonlinear Schrödinger equation:

$$(1.1) \quad i \frac{\partial u}{\partial t} = -\Delta u + \lambda |u|^{p-1} u, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

$$(1.2) \quad u(t_0, x) = u_0(x), \quad x \in \mathbb{R}^n,$$

where $t_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. By $\alpha(n)$ we denote ∞ if $n = 1$ or $n = 2$ and $(n+2)/(n-2)$ if $n \geq 3$. There are many papers concerning the global existence of solutions for Problem (1.1)-(1.2) (see, e.g., [1]-[2], [4]-[7], [9]-[10] and [13]-[14]). In [1] Baillon, Cazenave and Figueira show that if $1 \leq n \leq 3$, $1 < p < \alpha(n)$ and $\lambda > 0$, Problem (1.1)-(1.2) has a unique global strong solution $u(t) \in C(\mathbb{R}; H^2(\mathbb{R}^n)) \cap C^1(\mathbb{R}; L^2(\mathbb{R}^n))$ for any $u_0 \in H^2(\mathbb{R}^n)$. In [2] Ginibre and Velo show that if $1 < p < \alpha(n)$ and $\lambda > 0$ or if $1 < p < 1 + \frac{4}{n}$ and $\lambda < 0$, Problem (1.1)-(1.2) has a unique global weak solution $u(t) \in C(\mathbb{R}; H^1(\mathbb{R}^n))$ for any $u_0 \in H^1(\mathbb{R}^n)$.

In [6] Strauss shows that if $\lambda > 0$ and $p > 1$, Problem (1.1)-(1.2) has at least one global weak solution $u(t)$ in $L^\infty(\mathbb{R}; H^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n))$ for any $u_0 \in H^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$ (see also [5]). In [10] M. Tsutsumi and N. Hayashi discuss the unique global existence of classical solutions for (1.1)-(1.2) (see also Pecher and von Wahl [4]). In [9] M. Tsutsumi discusses the unique global solution in $\mathcal{S}'(\mathbb{R}^n)$ or in the weighted Sobolev space for (1.1)-(1.2). Recently in [13, 14] N. Hayashi, K. Nakamitsu and M. Tsutsumi have shown that the solution of (1.1)-(1.2) has the smoothing property in some sense. In [13] they also discuss the global existence of solutions of (1.1)-(1.2) for the initial data $u_0 \in L^2(\mathbb{R}^n)$ with $xu_0(x) \in L^2(\mathbb{R}^n)$, when $n = 1$. In almost all of previous papers the solution of (1.1)-(1.2) has been constructed in a space not larger than the energy space, that is, $H^1(\mathbb{R}^n)$, because the proofs in almost all of previous papers are based on the energy inequality. However, in [7] Strauss constructs the wave operators from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ for the equation (1.1) with $p = 1 + \frac{4}{n}$ (see [7, Theorem 5]). His results are almost equivalent to the construction in $L^2(\mathbb{R}^n)$ of unique local solutions for (1.1)-(1.2) with $p = 1 + \frac{4}{n}$. In this paper we prove that when $1 < p < 1 + \frac{4}{n}$, we can construct the unique global solution of (1.1)-(1.2) for any u_0 in $L^2(\mathbb{R}^n)$ (but possibly not in $H^1(\mathbb{R}^n)$). Such a solution is called an " L^2 -solution". Furthermore, we show that when $1 < p < 1 + \frac{4}{n}$, the solution operator of the evolution equation (1.1) constitutes a strongly continuous

nonlinear operator group in $L^2(\mathbb{R}^n)$. Our proof is based on the L^2 -norm conservation law and the dispersive effect of solutions (see, e.g., Lemma 2.2).

We put $U(t) = e^{i\Delta t}$ and $f(z) = \lambda|z|^{p-1}z$ ($z \in \mathbb{C}$). Our main theorem in this paper is the following.

Theorem 1.1. Assume that $1 < p < 1 + \frac{4}{n}$. Then, for any $u_0 \in L^2(\mathbb{R}^n)$ and any $t_0 \in \mathbb{R}$ there exists a unique global solution $u(t)$ of (1.1)-(1.2) such that

$$(1.3) \quad u(t) \in C(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L_{loc}^r(\mathbb{R}; L^{p+1}(\mathbb{R}^n)),$$

$$(1.4) \quad u(t) = U(t-t_0) - i \int_{t_0}^t U(t-\tau) f(u(\tau)) d\tau, \quad t \in \mathbb{R},$$

$$(1.5) \quad \|u(t)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}, \quad t \in \mathbb{R},$$

where $r = \frac{4(p+1)}{n(p-1)}$ and the integral in (1.4) is the Bochner integral in $H^{-1}(\mathbb{R}^n)$. Furthermore, let u_{0j} , $j = 1, 2, \dots$, and u_0 be such that $u_{0j}, u_0 \in L^2(\mathbb{R}^n)$ and $u_{0j} \rightarrow u_0$ in $L^2(\mathbb{R}^n)$ ($j \rightarrow \infty$). Let $u_j(t)$ and $u(t)$ be the solutions of (1.1) with $u_j(t_0) = u_{0j}$ and $u(t_0) = u_0$, respectively. Then, for each $T > 0$

$$(1.6) \quad u_j(t) \rightarrow u(t) \text{ in } C([t_0-T, t_0+T]; L^2(\mathbb{R}^n)) \quad (j \rightarrow \infty).$$

Remark 1.1. Theorem 1.1 is almost the same as Theorem 1.1 in [15] except that (1.6) is stronger than (1.6) in [15]. Theorem 1.1 implies the well-posedness in $L^2(\mathbb{R}^n)$ of the Cauchy

problem of the nonlinear Schrödinger equation (1.1) with

$$1 < p < 1 + \frac{4}{n}.$$

By Theorem 1.1 we can define the solution operator of the evolution equation (1.1) as a mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, when $1 < p < 1 + \frac{4}{n}$. We denote it by $S(t)$. The following result is an immediate consequence of Theorem 1.1.

Corollary 1.2. Assume that $1 < p < 1 + \frac{4}{n}$. Then, $\{ S(t) ; -\infty < t < +\infty \}$ is a strongly continuous nonlinear operator group in $L^2(\mathbb{R}^n)$. That is, $S(t)$ is a homeomorphism from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ for each $t \in \mathbb{R}$, and

$$(1.7) \quad S(t+s) = S(t)S(s), \quad t, s \in \mathbb{R},$$

$$(1.8) \quad S(0) = I,$$

$$(1.9) \quad S(h)v \rightarrow v \text{ in } L^2(\mathbb{R}^n) \text{ (} h \rightarrow 0 \text{), } v \in L^2(\mathbb{R}^n),$$

where I is the identity operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Our plan in this paper is as follows. In Section 2 we summarize several lemmas needed for the proof of Theorem 1.1. In Section 3 we give a sketch of proof of Theorem 1.1.

We conclude this section with several notations given. We abbreviate $L^p(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n)$ to L^p and H^m , respectively. $(\cdot, \cdot)$ denotes the scalar product in L^2 . For a closed interval I in $\mathbb{R}$ and a Hilbert space H we denote the set of all weakly

continuous functions from I to H by $C_w(I;H)$. Let $h(x)$ be an even and positive function in $C_0^\infty(\mathbb{R}^n)$ with $\|h\|_{L^1} = 1$. We put $h_j(x) = j^n h(jx)$ for each positive integer j . $\ast$ denotes the convolution with respect to spatial variables. In the course of calculations below various constants will be simply denoted by C . In particular, $C = C(*, \dots, *)$ will denote a constant depending only on the quantities appearing in parentheses.

§2. Lemmas.

In this section we summarize several results needed for the proof of Theorem 1.1.

For $U(t)$ we have the following two lemmas.

Lemma 2.1. Let q and r be positive numbers such that $1/q + 1/r = 1$ and $2 \leq q \leq \infty$. For any $t \neq 0$, $U(t)$ is a bounded operator from L^r to L^q satisfying

$$(2.1) \quad \|U(t)v\|_{L^q} \leq (4\pi|t|)^{\frac{n}{q} - \frac{n}{2}} \|v\|_{L^r}, \quad v \in L^r, \quad t \neq 0,$$

and for any $t \neq 0$, the map $t \rightarrow U(t)$ is strongly continuous.

For $q = 2$, $U(t)$ is unitary and strongly continuous for all $t \in \mathbb{R}$.

Lemma 2.2. Let q and r be positive numbers such that $1 \leq q - 1 < \alpha(n)$ and $(\frac{n}{2} - \frac{n}{q})r = 2$. Then,

$$(2.2) \quad \|U(\cdot)v\|_{L^r(\mathbb{R}; L^q)} \leq C \|v\|_{L^2},$$

where $C = C(n, q)$.

Lemma 2.1 is well known (see, e.g., [2, Lemma 1.2]). For Lemma 2.2, see Strichartz [8, Corollary 1 in §3] and Ginibre and Velo [3, Proposition 7].

Furthermore, we need the following two lemmas.

Lemma 2.3. Let I be an open interval in $\mathbb{R}$. Let $1 < q, r < \infty$ and $a, b > 0$. We put

$$M = \{ v(t) \in L^\infty(I; L^2) \cap L^r(I; L^q); \\ \|v\|_{L^\infty(I; L^2)} \leq a, \quad \|v\|_{L^r(I; L^q)} \leq b \}.$$

Then M is a closed subset in $L^r(I; L^q)$.

Lemma 2.4. Let T_1 and T_2 be constants with $T_1 < T_2$. Assume that $v(t) \in C([T_1, T_2]; H^{-1})$ and for some $K > 0$

$$(2.3) \quad \|v(t)\|_{L^2}^2 \leq K, \quad \text{a.e. } t \in [T_1, T_2].$$

Then, $v(t) \in C_w([T_1, T_2]; L^2)$ and (2.3) holds for all $t \in [T_1, T_2]$.

Lemmas 2.3 and 2.4 are identical to Lemmas 2.3 and 2.4 in [15], respectively. For the proofs of Lemmas 2.3 and 2.4, see [15, §2].

We conclude this section by giving the following lemma concerning the mollifier $h_j(x)$.

Lemma 2.5. Let I be a bounded closed interval in $\mathbb{R}$. Let $f(t) \in C(I; L^2)$. We put $f_j(t) = (h_j * f)(t)$. Then,

$$(2.4) \quad f_j(t) \in \bigcap_{k=1}^{\infty} C(I; H^k), \quad j = 1, 2, \dots,$$

$$(2.5) \quad \|f_j(t)\|_{H^m} \leq C_{jm} \|f(t)\|_{L^2}, \quad t \in I, j = 1, 2, \dots,$$

for each positive integer m ,

$$(2.6) \quad f_j(t) \rightarrow f(t) \quad \text{in } C(I; L^2) \quad (j \rightarrow \infty),$$

where $C_{jm} = C(j, m)$.

Proof. (2.4) and (2.5) are clear. We prove only (2.6).

We note that $f(t)$ is uniformly continuous on I . Since

$$\|f_j(t) - f_j(s)\|_{L^2} \leq \|f(t) - f(s)\|_{L^2}, \quad t, s \in I,$$

we conclude that $f_j(t)$, $j = 1, 2, \dots$, are equi-continuous on I . On the other hand, $f_j(t) \rightarrow f(t)$ in L^2 ($j \rightarrow \infty$) for each $t \in I$. Therefore, we can prove (2.6) by using the same argument as in the proof of the Ascoli-Arzelà theorem.

(Q. E. D.)

§3. Sketch of the Proof of Theorem 1.1.

In this section we give a sketch of the proof of Theorem 1.1. By I_t and $\bar{I}_t$ we denote an open interval $(t_0 - t, t_0 + t)$ and a closed interval $[t_0 - t, t_0 + t]$, respectively, for $t \geq 0$. Let $r = \frac{4(p+1)}{n(p-1)}$ throughout this section.

We have the following result concerning the unique local existence of L^2 -solutions for (1.1)-(1.2).

Lemma 3.1. Assume that $1 < p < 1 + \frac{4}{n}$. Then, for any $t_0 \in \mathbb{R}$ and any $\rho > 0$ there exists a $T = T(p, n, \lambda, \rho) > 0$ such that for any $u_0 \in L^2$ with $\|u_0\|_{L^2} \leq \rho$ Problem (1.1)-(1.2) has a unique local solution $u(t)$:

$$(3.1) \quad u(t) \in C(\bar{I}_T; L^2) \cap L^r(I_T; L^{p+1}),$$

$$(3.2) \quad u(t) = U(t-t_0)u_0 - i \int_{t_0}^t U(t-\tau)f(u(\tau)) d\tau, \quad t \in \bar{I}_T,$$

where the integral in (3.2) is the Bochner integral in H^{-1} .

Furthermore, the solution $u(t)$ satisfies

$$(3.3) \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad t \in \bar{I}_T.$$

Proof. We only give the outline of the proof of Lemma 3.1. For the details, see [15, §3].

We consider the following integral equation:

$$(3.4) \quad u_j(t) = U(t-t_0)h_j * u_0 - i \int_{t_0}^t U(t-\tau)f(u_j(\tau)) d\tau, \\ j = 1, 2, \dots.$$

From the result of Ginibre and Velo [2, Theorem 3.1] we already know that for each j there exists a unique global solution $u_j(t)$ of (3.4) in $C(\mathbb{R}; H^1)$ such that

$$(3.5) \quad \|u_j(t)\|_{L^2}^2 = \|h_j * u_0\|_{L^2}^2 \leq \|u_0\|_{L^2}^2, \quad t \in \mathbb{R},$$

$$j = 1, 2, \dots.$$

Let ρ be a positive constant with $\|u_0\|_{L^2}^2 \leq \rho$. By δ we denote the constant appearing in (2.2) with $q = p+1$ and $r = \frac{4(p+1)}{n(p-1)}$. We note that δ depends only on n and p . We put

$$(3.6) \quad M = \{ v(t) \in L^\infty(I_T; L^2) \cap L^r(I_T; L^{p+1});$$

$$\|v\|_{L^\infty(I_T; L^2)} \leq \rho, \quad \|v\|_{L^r(I_T; L^{p+1})} \leq 2\delta\rho \},$$

where T is a small positive constant to be determined later. We note that by Lemma 2.3 M is closed in $L^r(I_T; L^{p+1})$.

We first show that if T is sufficiently small, then

$$(3.7) \quad u_j(t) \in M \quad \text{for all } j.$$

For $0 \leq s \leq T$ we take the $L^r(I_s; L^{p+1})$ norm of (3.4) and use (2.1), (2.2) and the generalized Young inequality to obtain

$$(3.8) \quad \|u_j\|_{L^r(I_s; L^{p+1})} \leq \delta\rho + C_0 T^{p/q_1} \|u_j\|_{L^r(I_s; L^{p+1})}^p,$$

$$0 \leq s \leq T, \quad j = 1, 2, \dots,$$

where $q_1 = \frac{4p}{n+4-np}$ and $C_0 = C_0(n, p, \lambda)$. Now we choose $T > 0$ so small that there exists a positive number y satisfying $C_0 T^{p/q_1} y^p + \delta\rho - y < 0$ and $0 < y \leq 2\delta\rho$. For that purpose, it is sufficient to choose $T > 0$ so that

$$(3.9) \quad T < (2C_0(2\delta\rho)^{p-1})^{-q_1/p}.$$

Then we put

$$(3.10) \quad y_0 = \min \{ 2\delta\rho \geq y > 0; c_0 T^{p/q_1} y^p + \delta\rho - y = 0 \}.$$

If T is chosen so small that (3.9) holds, then by (3.8) and (3.10) we obtain

$$(3.11) \quad \|u_j\|_{L^r(I_T; L^{p+1})} \leq y_0 \leq 2\delta\rho, \quad j = 1, 2, \dots.$$

(3.5) and (3.11) give us (3.7), if T is chosen so small that (3.9) holds.

We next consider the estimate of the difference between u_j and u_k for any j and k with $j \neq k$. For $u_j, u_k \in M$ we have

$$(3.12) \quad \|u_j - u_k\|_{L^r(I_T; L^{p+1})} \leq \delta K(j, k) + \bar{c}_0 T^{p/q_1} \cdot 2(2\delta\rho)^{p-1} \|u_j - u_k\|_{L^r(I_T; L^{p+1})},$$

where $K(j, k) = \|h_j * u_0 - h_k * u_0\|_{L^2}^2$, $q_1 = \frac{4p}{n+4-np}$ and $\bar{c}_0 = \bar{c}_0(n, p, \lambda)$. If we choose T so small in (3.12) that

$$(3.13) \quad \bar{c}_0 T^{p/q_1} \cdot 2(2\delta\rho)^{p-1} \leq \frac{1}{2},$$

then we have by (3.12)

$$(3.14) \quad \|u_j - u_k\|_{L^r(I_T; L^{p+1})} \leq 2\delta K(j, k)$$

for all j and k . Since $K(j, k) \rightarrow 0$ ($j, k \rightarrow \infty$), we obtain by (3.14)

$$(3.15) \quad \|u_j - u_k\|_{L^r(I_T; L^{p+1})} \rightarrow 0 \quad (j, k \rightarrow \infty),$$

if T is chosen so small that (3.13) holds. In addition we have by (3.15)

$$\begin{aligned}
(3.16) \quad |(u_j(t) - u_k(t), \psi)| &\leq K(j, k) \|\psi\|_{L^2} \\
&+ CT^{q_2} \|\psi\|_{H^1} \cdot 2(2\delta\rho)^{p-1} \|u_j - u_k\|_{L^r(I_T; L^{p+1})} \\
&\rightarrow 0 \quad (j, k \rightarrow \infty) \quad \text{uniformly on } \bar{I}_T,
\end{aligned}$$

for $\psi \in H^1$, where $q_2 = \frac{4+(n+4)p-np^2}{4(p+1)} > 0$. (3.16) implies that $\{u_j(t)\}_{j=1}^\infty$ is the Cauchy sequence in $C(\bar{I}_T; H^{-1})$.

Therefore, by (3.7), (3.15), (3.16) and Lemma 2.3 we obtain the solution $u(t)$ of (1.1)-(1.2) such that

$$(3.17) \quad u(t) \in L(I_T; L^2) \cap L^r(I_T; L^{p+1}) \cap C(\bar{I}_T; H^{-1}),$$

$$(3.18) \quad u(t) = U(t-t_0)u_0 - i \int_{t_0}^t U(t-\tau)f(u(\tau)) d\tau, \quad t \in \bar{I}_T,$$

$$(3.19) \quad \|u(t)\|_{L^2} \leq \|u_0\|_{L^2}, \quad \text{a.e. } t \in I_T,$$

$$(3.20) \quad u_j(t) \rightarrow u(t) \text{ in } L^r(I_T; L^{p+1}) \text{ and in } C(\bar{I}_T; H^{-1}) \quad (j \rightarrow \infty),$$

where T is a positive constant determined by (3.9) and (3.13) and the integral in (3.18) is the Bochner integral in H^{-1} .

(3.17), (3.19) and Lemma 2.4 imply that

$$(3.21) \quad u(t) \in C_w(\bar{I}_T; L^2)$$

and that for all $t \in \bar{I}_T$ (3.19) holds. The uniqueness of solutions satisfying (3.17-18) follows from the estimate of the type (3.14) and the standard argument.

Thus, for any $s \in \bar{I}_T$ we can uniquely solve (1.1)-(1.2) in the time interval $[s-T, s+T]$ with the initial time t_0 and the

initial datum u_0 replaced by s and $u(s)$, respectively, where T is the same as in the case of the initial time t_0 and the initial datum u_0 . Therefore, reversing the roles of 0 and t , we obtain the reverse inequality to (3.19) for all $t \in \bar{I}_T$, which implies (3.3). (3.3) and (3.21) give us

$$(3.22) \quad u(t) \in C(\bar{I}_T; L^2).$$

This completes the proof of Lemma 3.1.

(Q. E. D.)

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. The unique global existence of L^2 -solutions for (1.1)-(1.2) follows directly from Lemma 3.1, which shows the unique local solvability in L^2 of (1.1)-(1.2) and the a priori bound of the L^2 -norm of L^2 -solutions.

It remains only to prove the continuous dependence of L^2 -solutions on the initial data. Let u_{0j} , $j = 1, 2, \dots$, and u_0 be such that $u_{0j}, u_0 \in L^2$ and $u_{0j} \rightarrow u_0$ in L^2 ($j \rightarrow \infty$). Let $u_j(t)$ and $u(t)$ be the global L^2 -solutions of (1.1) with $u_j(t_0) = u_{0j}$ and $u(t_0) = u_0$, respectively. We put $\rho = \sup \{ \|u_0\|_{L^2}, \|u_{0j}\|_{L^2}, j = 1, 2, \dots \}$. For this ρ , let $T > 0$ be defined as in (3.9) and (3.13). Then, by using the same argument as in the proof of Lemma 3.1 we have

$$(3.23) \quad u_j(t) \rightarrow u(t) \quad \text{in } L^r(I_T; L^{p+1}) \quad (j \rightarrow \infty),$$

$$\begin{aligned}
(3.24) \quad |(u_j(t) - u(t), g(t))| &\leq K \sup_{t \in \bar{I}_T} \|g(t)\|_{H^1} \\
&\times (\|u_{0j} - u_0\|_{L^2}^2 + \|u_j - u\|_{L^r(I_T; L^{p+1})}^2), \\
&t \in \bar{I}_T, \quad j = 1, 2, \dots,
\end{aligned}$$

for $g(t) \in C(\bar{I}_T; H^1)$ (see, e.g., (3.15) and (3.16)), where $K = K(n, p, \lambda, \rho) > 0$. We evaluate

$$\begin{aligned}
(3.25) \quad \|u_j(t) - u(t)\|_{L^2}^2 &= (u_j(t) - u(t), u_j(t) - u(t)) \\
&\leq \left| \|u_j(t)\|_{L^2}^2 - (u(t), u_j(t)) \right| + |(u_j(t) - u(t), u(t))|, \\
&t \in \bar{I}_T, \quad j = 1, 2, \dots.
\end{aligned}$$

We first evaluate the second term at the right hand side of (3.25). Let ε be an arbitrary positive constant. We put $\tilde{u}_k(t) = (h_k * u)(t)$ for each positive integer k . By Lemma 2.5 we can choose k so large that

$$\begin{aligned}
(3.26) \quad |(u_j(t) - u(t), u(t) - \tilde{u}_k(t))| &\leq 2\rho \|u(t) - \tilde{u}_k(t)\|_{L^2} < \frac{1}{2}\varepsilon, \\
&t \in \bar{I}_T.
\end{aligned}$$

For such a k we have by (3.23), (3.24) and Lemma 2.5

$$\begin{aligned}
(3.27) \quad |(u_j(t) - u(t), \tilde{u}_k(t))| &\leq K \sup_{t \in \bar{I}_T} \|\tilde{u}_k(t)\|_{H^1} \\
&\times (\|u_{0j} - u_0\|_{L^2}^2 + \|u_j - u\|_{L^r(I_T; L^{p+1})}^2) < \frac{1}{2}\varepsilon, \\
&t \in \bar{I}_T,
\end{aligned}$$

if j is sufficiently large. Therefore, we obtain by (3.26) and (3.27)

$$\begin{aligned}
(3.28) \quad & |(u_j(t) - u(t), u(t))| \\
& \leq |(u_j(t) - u(t), \tilde{u}_k(t))| + |(u_j(t) - u(t), u(t) - \tilde{u}_k(t))| \\
& < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \quad t \in \bar{I}_T,
\end{aligned}$$

for sufficiently large j . (3.28) implies that

$$(3.29) \quad |(u_j(t) - u(t), u(t))| \rightarrow 0 \quad (j \rightarrow \infty) \quad \text{uniformly on } \bar{I}_T.$$

We next evaluate the first term at the right hand side of

(3.25). Since $\|u_j(t)\|_{L^2}^2 = \|u_{0j}\|_{L^2}^2$ and $\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2$ for $t \in \bar{I}_T$, we have by (3.29)

$$\begin{aligned}
(3.30) \quad & \left| \|u_j(t)\|_{L^2}^2 - (u(t), u_j(t)) \right| \\
& \leq \left| \|u_{0j}\|_{L^2}^2 - \|u_0\|_{L^2}^2 \right| + |(u(t), u_j(t) - u(t))| \\
& \rightarrow 0 \quad (j \rightarrow \infty) \quad \text{uniformly on } \bar{I}_T.
\end{aligned}$$

Combining (3.25), (3.29) and (3.30), we obtain

$$(3.31) \quad u_j(t) \rightarrow u(t) \quad \text{in } C(\bar{I}_T; L^2) \quad (j \rightarrow \infty).$$

On the other hand, the length of T is determined only by n, p, λ and ρ (see (3.9) and (3.13)). By the L^2 -norm conservation law we see that $\sup\{\|u(t)\|_{L^2}^2, \|u_j(t)\|_{L^2}^2, j = 1, 2, \dots\}$ is constant for $t \in \mathbb{R}$. Accordingly, we use the above argument with the initial time t_0 and the initial data $u_0, u_{0j}, j = 1, 2, \dots$, replaced by t_0+T and $u(t_0+T), u_j(t_0+T), j = 1, 2, \dots$, or by t_0-T and $u(t_0-T), u_j(t_0-T), j = 1, 2, \dots$, respectively, to obtain (3.31) with $\bar{I}_T$ replaced by $\bar{I}_{2T}$.

Repeating this procedure, we obtain (1.6). This completes the proof of Theorem 1.1.

(Q. E. D.)

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