

New Forms for $GU(2, 2)$

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1 Introduction

In 2007 [5], Roberts and Schmidt established local New form theory for irreducible admissible generic representation of $\mathrm{PGSp}(4)$ ($\simeq \mathrm{SO}(3, 2)$) over nonarchimedean field. Following their result, we establish local New form theory for irreducible admissible generic *supercuspidal* representation of $\mathrm{PGU}(2, 2)$ ($\simeq \mathrm{SO}(4, 2)$) over nonarchimedean local field. To begin with, we recall Roberts-Schmidt's theory for $\mathrm{PGSp}(4)$. Let F be a nonarchimedean local field with ring of integers $\mathfrak{o}$. Let

$$\mathrm{GSp}_4(F) = \{g \in \mathrm{GL}_4(F) \mid {}^t g J g = aJ, \text{ for some } a \in F^\times\},$$

where

$$J = \begin{bmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

The mapping $g \rightarrow a$ defines a homomorphism to $F^\times$, called similitude character, and we will write $a = \mu(g)$. Let ψ be a nontrivial additive character on F . The Whittaker functions W on $\mathrm{GSp}_4(F)$ with respect to ψ are smooth functions such that

$$W\left(\begin{bmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{bmatrix} g\right) = \psi(x + y)W(g).$$

We denote by $\mathscr{W}_\psi$ the space of these W . Let $(\pi, V) \subset \mathscr{W}_\psi$ be an irreducible admissible representation of $\mathrm{GSp}_4(F)$ with trivial central character. For $W \in V$,

define the Novodvorsky zeta integral

$$Z_N(s, W) = \int_{F^\times} \int_F W \left(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} \right) dx d^\times a$$

with $s \in \mathbb{C}$, where dx and $d^\times a$ are Haar measures such that $\text{vol}(\mathfrak{o}) = 1$ and $\text{vol}(\mathfrak{o}^\times)$, respectively. Let $\mathfrak{p}$ denote the prime ideal of $\mathfrak{o}$ and $q = |\mathfrak{o}/\mathfrak{p}|$. Let $I(\pi)$ be the $\mathbb{C}$ -subspace of $\mathbb{C}[X, X^{-1}]$ spanned by these $Z_N(s, W)$ with $X = q^{-s}$. Since $\mathbb{C}[X, X^{-1}]$ is a principal ideal domain, $I(\pi)$ admits a generator $P(X)$ such that $P(1) = 1$. Denote by $L(s, \pi)$ the generator. Then, there exists $\varepsilon(s, \pi, \psi) = \varepsilon q^{-N_\pi(s - \frac{1}{2})}$ with $N_\pi \in \mathbb{Z}$, $\varepsilon \in \{\pm 1\}$ such that for arbitrary $W \in V$, it holds that

$$\frac{Z_N(1-s, \pi(u)W)}{L(1-s, \pi)} = \varepsilon(s, \pi, \psi) \frac{Z_N(s, W)}{L(s, \pi)}, \quad (1.1)$$

where

$$u = \begin{bmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{bmatrix}.$$

(1.1) is the functional equation of π . The monomial $\varepsilon(s, \pi, \psi)$ of X is called ε -factor of π , and varies according to ψ . For a positive integer n , define the compact subgroup of $\text{GSp}_4(\mathbb{F})$ by

$$K(n) = \left\{ g \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix} \mid \mu(g) \in \mathfrak{o}^\times \right\},$$

and call the paramodular group of level n . Now, suppose that $\text{Ker}(\psi) = \mathfrak{o}$. Then, N_π is positive. Roberts and Schmidt's *generic main theory* says that there exists uniquely $W \in V$ such that

- W is $K(N_\pi)$ -fixed.
- $W(1) = 1$ and $Z(s, W) = (1 - q^{-1})L(s, \pi)$.

Inspired by their theory, we obtained a similar result for $\text{GU}(2, 2)$.

2 Functional equation

Let F be a field and $E = F(\sqrt{d})$ be a quadratic extension of F with $d \in \mathfrak{o}^\times \setminus (\mathfrak{o}^\times)^2$. Let ψ be a nontrivial additive character on F and define the additive character ψ_E on E by $\psi_E(a + b\sqrt{d}) = \psi(b)$. Let $\text{Gal}(E/F) = \{1, c\}$, and

$$G = \text{GU}_{2,2}(E) = \{g \in \text{GL}_4(E) \mid {}^t g^c J g = \mu(g)J, \mu(g) \in F^\times\}.$$

We say a smooth function W on G is a Whittaker functions with respect to ψ , if

$$W\left(\begin{bmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x^c \\ & & & 1 \end{bmatrix} g\right) = \psi_E(x)\psi(y)W(g),$$

and denote by $\mathscr{W}_\psi$ the space of such functions. First of all, we should give a functional equation of irreducible admissible generic representation of G over a local nonarchimedean field. However, before that, let us consider the global situation. Let F be a global number field and E be a quadratic extension of F . Furusawa and Morimoto [3] defined a zeta integral for global Whittaker function W with respect to ψ_E (here ψ_E is an additive character on $\mathbb{A}_E$), and Schwartz-Bruhat function ϕ of $\mathbb{A}_F^2$ by

$$Z(s, W, \phi) = \int_{GL_2(\mathbb{A}_F)} \int_{\mathbb{A}_F} W\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} i(h)\right) \phi([0, 1]h) |\det(h)|^s dx dh$$

where i is the embedding such that

$$i\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b & & \\ & a & b & \\ c & & d & \\ & c & & d \end{bmatrix}.$$

When the global Whittaker function W is obtained by taking an integral of an automorphic cusp form, the zeta integral is written by this cusp form and the Eisenstein series $E(h, \phi, s)$ (defined in the standard way). Therefore, by using the well-known formula $E(h, \phi, s) = E({}^t h^{-1}, \phi^\#, 1-s)$, we obtain an incomplete global functional equation

$$Z(s, W, \phi) = Z(1-s, W', \phi^\#) \quad (2.1)$$

where $\phi^\#$ is the Fourier transformation of ϕ with respect to ψ , and

$$W'(g) = W\left(\begin{bmatrix} & & 1 \\ & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} {}^t g^{-1} \begin{bmatrix} & 1 \\ & 1 \\ & 1 \\ 1 & \end{bmatrix}\right).$$

Now, let F be a local nonarchimedean field. Let (π, V) be an irreducible, admissible generic representation of G , i.e., $V \subset \mathscr{W}_\psi$. For $x, y \in E$, and $z \in F$, let

$$n(x, y, z) = \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x^c \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & y^c \\ & & 1 \\ & & & 1 \end{bmatrix}.$$

This is an element of G . Let U denote the subgroup consisting of these elements. Let $U_F = \{n(x, y, z) \mid x, y, z \in F\}$. We define 'Klingen type' parabolic?? subgroup

$$Q' = E^\times \left\{ \begin{bmatrix} \det(g) & & \\ & g & \\ & & 1 \end{bmatrix} \mid g \in GL_2(F) \right\} U \subset G.$$

Then U_F is a normal subgroup of Q' . The following sets U_3, P_3 are subgroups of $GL_3(F)$:

$$U_3 = \left\{ \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix} \in GL_3(F) \right\} \subset P_3 = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ & & 1 \end{bmatrix} \in GL_3(F) \right\}.$$

Then, we have

$$Q'/U_F \simeq E^\times \times P_3(F)$$

via the homomorphism $i : Q' \rightarrow P_3$ defined by

$$i \left(t \begin{bmatrix} \det(g) & & \\ & g & \\ & & 1 \end{bmatrix} n(x, y, z) \right) = \left(t, \begin{bmatrix} g & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{y-y'}{2\sqrt{d}} \\ & \frac{x-x'}{2\sqrt{d}} \\ & & 1 \end{bmatrix} \right). \quad (2.2)$$

Put

$$\bar{V} = V / \langle \pi(u)v - v \mid u \in U_F \rangle.$$

Via (2.2), $\bar{V}$ is a $E^\times \times P_3$ -module. Denote by $\bar{\pi}$ the action of $E^\times \times P_3$ and by $\bar{v}$ vectors of $\bar{V}$. For $t \in E^\times$,

$$\bar{\pi}(t)\bar{v} = \omega_\pi(t)\bar{v}.$$

According to the P_n -theory of Bernstein-Zelevinskii [1], $\bar{V}$ has the following filtration of P_3 -subspaces:

$$V_2 \subset V_1 \subset \bar{V}. \quad (2.3)$$

Define the character ψ_{U_3} on U_3 by

$$\psi_{U_3} \left(\begin{bmatrix} 1 & x_1 & x_2 \\ & 1 & x_3 \\ & & 1 \end{bmatrix} \right) = \psi(x_1 + x_3).$$

Put

$$\bar{V}_{\psi_{U_3}} = \bar{V} / \langle \psi_{U_3}(u')\bar{v} - \bar{\pi}(u')\bar{v} \mid u' \in U_3 \rangle.$$

Then, V_2 is a direct sum of $\dim_{\mathbb{C}} \bar{V}_{\psi_{U_3}}$ copies of an irreducible representation (c.f. p. 49 of loc. cite., this irreducible representation is denoted by τ_p^0). However,

$\dim_{\mathbb{C}} \bar{V}_{\psi_{U_3}} = 1$ by the uniqueness of Whittaker models of V (c.f. Shalika [6]). Thus, V_2 is irreducible.

Now, we are going to give a functional equation of π . Set

$$\begin{aligned} \mathbf{S} &= \{g = \begin{bmatrix} h & \\ & h' \end{bmatrix} \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \in \mathbf{G} \mid h \in \mathrm{GL}_2(\mathbf{F})\}, \\ \mathbf{T} &= \{g \in \mathbf{S} \mid h \in P_2\}, \end{aligned} \quad (2.4)$$

where

$$h' = \det(h) \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} {}^t h^{-1} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}.$$

These sets are subgroup of $\mathbf{G}$, and $\mathbf{S}$ is called Shalika subgroup. For $s \in \mathbb{C}$, define the 1-dimensional representation $\nu_s \psi$ of $\mathbf{S}$ by

$$\nu_s \psi \left(\begin{bmatrix} h & \\ & h' \end{bmatrix} \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \right) = |\det(h)|^{2s} \psi_{\mathbf{E}} \left(\frac{1}{2} \mathrm{tr} \left(\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} X \right) \right).$$

Let $\mathbf{S}$ act on $\phi \in \mathcal{S}(\mathbf{F}^2)$ by

$$\begin{bmatrix} h & x \\ & h' \end{bmatrix} \cdot \phi(z) = \phi(zh).$$

Lemma 2.1. *Except for finitely many $s \in \mathbb{C}$,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbf{S}}(\pi \otimes_{\mathbb{C}} \mathcal{S}_0(\mathbf{F}^2), \nu_s \psi) \leq \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbf{T}}(\pi, \nu_{s-1} \psi).$$

Proof. It suffices to construct a homomorphism $J : \mathrm{Hom}_{\mathbf{S}}(\pi \otimes_{\mathbb{C}} \mathcal{S}(\mathbf{F}^2), \nu_s \psi) \rightarrow \mathrm{Hom}_{\mathbf{T}}(\pi, \nu_{s-1} \psi)$ so that J is injective except for finitely many $s \in \mathbb{C}$. Take $\sigma \in \mathrm{Hom}_{\mathbf{S}}(\pi, \nu_s \psi)$. By the definition of $\nu_s \psi$,

$$\sigma(\pi(t)v) = \omega_{\pi}(t)\sigma(v) = |t|^{4s}\sigma(v)$$

for $t \in \mathbf{F}^{\times}$ and $v \in V$. Hence, except for finitely many $s \in \mathbb{C}$,

$$\mathrm{Hom}_{\mathbf{S}}(\pi, \nu_s \psi) = \{0\}. \quad (2.5)$$

By restriction of $\mathcal{S}(\mathbf{F}^2)$ to its $\mathbf{S}$ -subspace $\mathcal{S}_0(\mathbf{F}^2) = \{\phi \in \mathcal{S}(\mathbf{F}^2) \mid \phi([0,0]) = 0\}$, we have a homomorphism

$$J' : \mathrm{Hom}_{\mathbf{S}}(\pi \otimes_{\mathbb{C}} \mathcal{S}(\mathbf{F}^2), \nu_s \psi) \rightarrow \mathrm{Hom}_{\mathbf{S}}(\pi \otimes_{\mathbb{C}} \mathcal{S}_0(\mathbf{F}^2), \nu_s \psi).$$

However, the kernel of J' is embedded into $\mathrm{Hom}_{\mathbf{S}}(\pi, \nu_s \psi)$ since

$$\pi \otimes_{\mathbb{C}} \mathcal{S}(\mathbf{F}^2) / \pi \otimes_{\mathbb{C}} \mathcal{S}_0(\mathbf{F}^2) \simeq \pi$$

as $\mathbf{S}$ -modules via the mapping: $v \otimes \phi \rightarrow \phi([0,0])v \in \pi$. Therefore, by (2.5), except for finitely many s , the kernel of J' is $\{0\}$ and J' is injective. The quotient $P_2 \backslash \mathrm{GL}_2(\mathbf{F})$ acts on $\mathbf{F}^2 \setminus \{[0,0]\}$ faithfully, and $\mathbf{T} \backslash \mathbf{S} \simeq P_2 \backslash \mathrm{GL}_2(\mathbf{F})$. So, the

$\mathcal{S}$ -space $\mathcal{S}_0(\mathbb{F}^2)$ is isomorphic to $\text{ind}_{\mathbf{T}}^{\mathcal{S}}(1)$, the compactly supported induced representation from the trivial representation 1 of $\mathbf{T}$, via the correspondence: $\phi \mapsto \phi([0, 1]h) \in \text{ind}_{\mathbf{T}}^{\mathcal{S}}(1)$ where we write an element of $\mathcal{S}$ as in (2.4). So,

$$\text{Hom}_{\mathcal{S}}(\pi \otimes_{\mathbb{C}} \mathcal{S}_0(\mathbb{F}^2), \nu_s \psi) \simeq \text{Hom}_{\mathcal{S}}(\pi \otimes_{\mathbb{C}} \text{ind}_{\mathbf{T}}^{\mathcal{S}}(1), \nu_s \psi).$$

However, in general, for representations τ, ξ of a group G , and one-dimensional representation χ , it holds that

$$\begin{aligned} \text{Hom}_G(\tau \otimes_{\mathbb{C}} \xi, \chi) &\simeq \text{Hom}_G(\tau, \text{Hom}_{\mathbb{C}}(\xi, \chi)) \\ &\simeq \text{Hom}_G(\tau, \text{Hom}_{\mathbb{C}}(\chi^{-1} \xi, \mathbb{C})), \end{aligned}$$

where $g \in G$ acts on $f \in \text{Hom}_{\mathbb{C}}(\xi, \chi)$ by $g \cdot f(t) = \chi(g)f(\xi(g^{-1})t)$. Therefore,

$$\begin{aligned} \text{Hom}_{\mathcal{S}}(\pi \otimes_{\mathbb{C}} \text{ind}_{\mathbf{T}}^{\mathcal{S}}(1), \nu_s \psi) &\simeq \text{Hom}_{\mathcal{S}}(\pi, \text{Hom}_{\mathbb{C}}(\text{ind}_{\mathbf{T}}^{\mathcal{S}}((\nu_s \psi)^{-1}), \mathbb{C})) \\ &\simeq \text{Hom}_{\mathcal{S}}(\pi, \text{Ind}_{\mathbf{T}}^{\mathcal{S}}(\nu_{s-1} \psi)) \\ &\simeq \text{Hom}_{\mathbf{T}}(\pi, \nu_{s-1} \psi). \end{aligned}$$

The second isomorphism is by 2.25 c) of [1], and the last is by the Frobenius reciprocity law 2.29 of loc. cite.. This completes the proof. $\square$

Let

$$\pi^{\vee}(g) = \pi(\mu(g)^t g^{-1}).$$

Proposition 2.2. π is equivalent to $\pi^{\vee}$.

Proof. Consider the action γ of G on G defined by $\gamma(g)g_1 = g^{-1}g_1g$, and the homeomorphism $\sigma : g \rightarrow \mu(g)^t g^{-1}$ in G . Then, all the conditions of the second Theorem in p.91 of [4] are satisfied. Indeed, $\gamma(g)\sigma = \sigma\gamma(g^{-1})$, σ^2 is identity, and $\sigma(g) = J^{-1}gJ$. Therefore, the character $\text{tr}(\pi)$ of π (see 2.17 of [1] for the definition of $\text{tr}(\pi)$) coincides with $\text{tr}(\pi^{\vee})$. By Corollary 2.20 of loc. cite, π is equivalent to $\pi^{\vee}$. $\square$

For a local Whittaker function $W \in \mathcal{W}_{\psi}$, and $\phi \in \mathcal{S}(\mathbb{F}^2)$, we define the zeta integral by

$$Z(s, W, \phi) = \int_{N_2 \backslash \text{GL}_2(\mathbb{F})} \int_{\mathbb{F}} W \left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} i(h) \right) \phi([0, 1]h) |\det(h)|^s dx dh.$$

A common argument shows this integral is absolutely convergent when $\Re(s)$ is sufficiently large. Define $L(s, \pi)$ in the same way as in the $\text{GSp}(4)$ case. Let $\mathbf{S}'$ (resp. $\mathbf{T}'$) be the conjugation of $\mathbf{S}$ (resp. $\mathbf{T}$) by

$$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & 1 & & \\ & & & 1 \end{bmatrix}.$$

Theorem 2.3. *There exists a function $\varepsilon(s, \pi, \psi)$ such that*

$$\frac{Z(1-s, W', \phi^\#)}{L(1-s, \pi)} = \varepsilon(s, \pi, \psi) \frac{Z(s, W, \phi)}{L(s, \pi)} \quad (2.6)$$

for arbitrary $W \in V$ and $\phi \in \mathcal{S}(\mathbf{F}^2)$. $\varepsilon(s, \pi, \psi)$ is in a form of $\varepsilon q^{-N_\pi(s-1/2)}$ with $\varepsilon \in \{\pm 1\}, N_\pi \in \mathbb{Z}$.

Proof. Consider the functionals on $\pi \otimes_{\mathbb{C}} \mathcal{S}(\mathbf{F}^2)$ sending $W \otimes \phi$ to $\frac{Z(s, W, \phi)}{L(s, \pi)}$ and to $\frac{Z(1-s, W', \phi^\#)}{L(1-s, \pi)}$. Both these functionals belong to $\text{Hom}_{S'}(\pi \otimes_{\mathbb{C}} \mathcal{S}(\mathbf{F}^2), \nu_s \psi)$. First, we will show that, except for finitely many s ,

$$\dim_{\mathbb{C}} \text{Hom}_{S'}(\pi \otimes_{\mathbb{C}} \mathcal{S}(\mathbf{F}^2), \nu_s \psi) \leq 1. \quad (2.7)$$

By Lemma 2.1, it suffices to show $\dim_{\mathbb{C}} \text{Hom}_{T'}(\pi, \nu_s \psi) \leq 1$, except for finitely many s . Take a $\Lambda \in \text{Hom}_{T'}(\pi, \nu_s \psi)$. For $t \in \mathbf{F}^\times, x \in \mathbf{E}, z \in \mathbf{F}$,

$$\begin{aligned} \Lambda(\pi(\mathbf{n}(x, y, z))v) &= \psi_E(x)\Lambda(v), \\ \Lambda\left(\pi\left(\begin{bmatrix} t & & & \\ & t & & \\ & * & 1 & \\ & & & 1 \end{bmatrix}\right)v\right) &= \nu_s(t)\Lambda(v). \end{aligned}$$

Via (2.2), Λ corresponds to the functional λ on the P_3 -space $\overline{V}$, such that

$$\begin{aligned} \lambda\left(\begin{bmatrix} 1 & & \\ & 1 & z \\ & & 1 \end{bmatrix}v\right) &= \psi(z)\lambda(v), \\ \lambda\left(\begin{bmatrix} t & & \\ * & 1 & \\ & & 1 \end{bmatrix}v\right) &= \nu_s(t)\lambda(v). \end{aligned}$$

The proof for Proposition 2.5.7. of [5] says that, except for finitely many s , the space $\{\lambda\}$ is at most one-dimensional. (see also Lemma 2.5.4., 2.5.5., and 2.5.6. of loc.cite.) Thus, (2.7) is showed. Therefore, there is a function $\varepsilon(s, \pi, \psi)$ depending only on s, π, ψ such that (2.6) holds except for finitely many s . By definition, there is a finite set of pairs $\{(W_i, \phi_i)\}$ such that $L(s, \pi) = \sum_i Z(s, W_i, \phi_i)$. By (2.6),

$$\varepsilon(s, \pi, \psi) = \frac{\sum_i Z(1-s, W'_i, \phi_i^\#)}{L(1-s, \pi)} \in \mathbb{C}[q^{-s}, q^s].$$

(Each W'_i belongs to V .) Therefore, we may write $\varepsilon(s, \pi, \psi) = R(q^{-s})q^{-Ms}$ by some $R[X] \in \mathbb{C}[X]$ and $M \in \mathbb{Z}$. From (2.6) and the fact that $(W')' = W, (\phi^\#)^\#(z) = \phi(-z)$, it follows that

$$\varepsilon(s, \pi, \psi)\varepsilon(1-s, \pi, \psi) = 1.$$

Therefore, R has no zeros, and is a monomial. Now the assertion follows immediately. $\square$

3 D-paramodular vectors

For a nonnegative integer m , set

$$\mathfrak{D}_m = \mathfrak{o} + \mathfrak{p}^m \sqrt{d}$$

and

$$K_d(2m) = \{g \in G \cap \begin{bmatrix} \mathfrak{D}_m & \mathfrak{p}^{-m}\mathfrak{D}_m & \mathfrak{p}^{-m}\mathfrak{D}_m & \mathfrak{p}^{-2m}\mathfrak{D}_m \\ \mathfrak{p}^m\mathfrak{D}_m & \mathfrak{D}_m & \mathfrak{D}_m & \mathfrak{p}^{-m}\mathfrak{D}_m \\ \mathfrak{p}^m\mathfrak{D}_m & \mathfrak{D}_m & \mathfrak{D}_m & \mathfrak{p}^{-m}\mathfrak{D}_m \\ \mathfrak{p}^{2m}\mathfrak{D}_m & \mathfrak{p}^m\mathfrak{D}_m & \mathfrak{p}^m\mathfrak{D}_m & \mathfrak{D}_m \end{bmatrix} \mid \mu(g) \in \mathfrak{o}^\times\},$$

$$K_d(2m+1) = \{g \in G \cap \begin{bmatrix} \mathfrak{D}_m & \mathfrak{p}^{-m}\mathfrak{D}_m & \mathfrak{p}^{-m}\mathfrak{D}_m & \mathfrak{p}^{-2m-1}\mathfrak{D}_m \\ \mathfrak{p}^{m+1}\mathfrak{D}_m & \mathfrak{D}_m & \mathfrak{D}_m & \mathfrak{p}^{-m}\mathfrak{D}_m \\ \mathfrak{p}^{m+1}\mathfrak{D}_m & \mathfrak{D}_m & \mathfrak{D}_m & \mathfrak{p}^{-m}\mathfrak{D}_m \\ \mathfrak{p}^{2m+1}\mathfrak{D}_m & \mathfrak{p}^{m+1}\mathfrak{D}_m & \mathfrak{p}^{m+1}\mathfrak{D}_m & \mathfrak{D}_m \end{bmatrix} \mid \mu(g) \in \mathfrak{o}^\times\}.$$

$\mathfrak{D}_m$ is a subring of the ring of integers of E , and $K_d(n)$ is a compact subgroup of G . We call $K_d(n)$ the D-paramodular group of level n . Let ω be a generator of $\mathfrak{p}$. The followings are elements of G :

$$t_n = \begin{bmatrix} & & -\omega^{-n} \\ & 1 & \\ & & 1 \\ \omega^n & & \end{bmatrix},$$

$$u_{2m} = \begin{bmatrix} & -\omega^{-m} & & \\ \omega^m & & & \\ & & \omega^{-m} & \\ & & -\omega^m & \end{bmatrix}, v_{2m} = \begin{bmatrix} & -\omega^{-m} & & \\ \omega^m & & & \\ & & \omega^m & \\ & & -\omega^{-m} & \end{bmatrix},$$

$$u_{2m+1} = \begin{bmatrix} & -\omega^{-m} & & \\ \omega^{m+1} & & & \\ & & \omega^{-m} & \\ & & -\omega^{m+1} & \end{bmatrix}, v_{2m+1} = \begin{bmatrix} & -\omega^{-m} & & \\ \omega^{m+1} & & & \\ & & \omega^{m+1} & \\ & & -\omega^{-m} & \end{bmatrix}.$$

Note that $t_{2m}, u_{2m}, v_{2m} \in K_d(n)$. Note that $t_{2m+1} \in K_d(2m+1)$, $u_{2m+1}, v_{2m+1} \notin K_d(2m+1)$, however, u_{2m+1}, v_{2m+1} are normalizers of $K_d(2m+1)$. We call $v \in V$ a D-paramodular vector if v is invariant under some $K_d(n)$, and denote by $V(n)$ the subspace consisting of $K_d(n)$ -invariant vectors. Similar to the argument of [5], we may find a $W \in V$ fixed by a 'Klingen-type' compact subgroup such that $W(1) \neq 0$, and construct a D-paramodular form $W' = \int_{K_d(n)} \pi(k)Wdk$ for some n , such that $W'(1) \neq 0$. So,

$$V(n) \neq \{0\}$$

for some n . When $n = 2m+1$, the subspace $V(2m+1)$ is decomposed to the direct sum $V(2m+1)_+ + V(2m+1)_-$ with

$$V(2m+1)_\varepsilon = \{v \in V(2m+1) \mid \pi(u_{2m+1})v = \varepsilon v\}.$$

Note that $\pi(v_{2m+1})v = \pi(u_{2m+1})v$ since $t_{2m+1}^{-1}u_{2m+1}t_{2m+1} = v_{2m+1}$. Set

$$\eta = \begin{bmatrix} \omega^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \omega \end{bmatrix}.$$

It is easy to see that

$$\pi(\eta) : V(n) \rightarrow V(n+2), V(2m+1)_\varepsilon \rightarrow V(2m+3)_\varepsilon.$$

Define

$$Z_d(s, W) = \int_{\mathbb{F}^\times} W(i \begin{bmatrix} a & \\ & 1 \end{bmatrix}) |a|^{s-2} d^\times a.$$

Similar to the $\mathrm{GSp}(4)$ -case, we have

Proposition 3.1 (η -principle). *Let $m \geq 1$. Suppose that $W \in V(2m)$ (resp. $W \in V(2m+1)_\varepsilon$). If $Z_d(s, W) = 0$, then there exists a $W' \in V(2m-2)$ (resp. $W \in V(2m-1)_\varepsilon$) such that $\pi(\eta)W' = W$.*

4 New forms

Suppose that (π, V) is supercuspidal. The argument of [2] for L -function of a representation of $\mathrm{GL}(2) \times \mathrm{GL}(2)$ works for the $\mathrm{GU}(2, 2)$ case, and we conclude $L(s, \pi)$ is one of following forms.

- i) $L(s, \pi) = (1 - q^{-s})^{-1}$. We say π is distinguished.
- ii) $L(s, \pi) = (1 + q^{-s})^{-1}$. We say π is quasi-distinguished.
- iii) $L(s, \pi) = (1 - q^{-2s})^{-1}$. We say π is dual-distinguished.
- iv) $L(s, \pi) = 1$. We say π is nondistinguished.

Define two Hecke operators for $V(n)$ by

$$T_{1,0} = \pi(K_d(n)\eta^{-1}K_d(n)), \quad T_{0,1} = \pi(K_d(n)i \begin{bmatrix} \omega & \\ & 1 \end{bmatrix} K_d(n)),$$

which are self-adjoint. But, note that $T_{1,0}$ doesnot act on $V(2m+1)_\varepsilon$. If $V(M_\pi) \neq \{0\}$ and $V(n) = \{0\}$ for all $n < M_\pi$, then we say M_π is the minimal level of π . For $W \in V(M_\pi)$, we obtain a recursion formula of $W(i \begin{bmatrix} \omega^n & \\ & 1 \end{bmatrix})$ by using the Hecke operators. The η -principle, and this recursion formula, give rough one-dimensionality of $V(M_\pi)$ (in case that M_π is odd, the one-dimensionality may lost as below). Let

$$\phi_m = \mathrm{Ch}(\mathfrak{p}^m \oplus \mathfrak{o}) \in \mathcal{S}(\mathbb{F}^2).$$

Observing the form of $L(s, \pi)$ and the functional equation, we obtain

Theorem 4.1. *Suppose that (π, V) is supercuspidal. Then M_π is odd and $N_\pi = M_\pi + 1$.*

- i) *When π is distinguished, the subspace $V(M_\pi)_+$ is one dimensional, and $V(M_\pi)_- = \{0\}$. For the unique $W \in V(M_\pi)_+$ such that $W(1) = 1$,*

$$Z(s, W, \phi_{\frac{N_\pi}{2}}) = \frac{q-1}{q+1} L(s, \pi).$$

- ii) *When π is quasi-distinguished, the subspace $V(M_\pi)_-$ is one dimensional, and $V(M_\pi)_+ = \{0\}$. For the unique $W \in V(M_\pi)_-$ such that $W(1) = 1$,*

$$Z(s, W, \phi_{\frac{N_\pi}{2}}) = \frac{q-1}{q+1} L(s, \pi).$$

- iii) *When π is dual-distinguished, both of the subspaces $V(M_\pi)_+, V(M_\pi)_-$ are one dimensional. For each unique $W_\pm \in V(M_\pi)_\pm$ such that $W_\pm(1) = 1$,*

$$Z(s, W_+ + W_-, \phi_{\frac{N_\pi}{2}}) = 2 \frac{q-1}{q+1} L(s, \pi).$$

- iv) *When π is nondistinguished, both of the subspaces $V(M_\pi)_+, V(M_\pi)_-$ are one dimensional. For each unique $W_\pm \in V(M_\pi)_\pm$ such that $W_\pm(1) = 1$,*

$$Z(s, W_\pm, \phi_{\frac{N_\pi}{2}}) = \frac{q-1}{q+1} L(s, \pi).$$

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