

# A bound for the number of columns $\ell_{(c,a,b)}$ in the intersection array of a distance-regular graph

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## Abstract

In this paper we give a bound for the number  $\ell_{(c,a,b)}$  of columns  $(c,a,b)^T$  in the intersection array of a distance-regular graph. We also show that this bound is intimately related to the Bannai-Ito Conjecture.

## 1 Introduction

Suppose that  $\Gamma$  is a finite connected graph with vertex set  $V\Gamma$ . As usual, we define the distance between any two vertices  $u$  and  $v$  of  $\Gamma$  to be the length of any shortest path in  $\Gamma$  between  $u$  and  $v$ , and the diameter  $d$  of  $\Gamma$  to be the largest distance between any pair of vertices in  $V\Gamma$ . For  $u \in V\Gamma$  and  $i$  any non-negative integer not exceeding  $d$ , let  $\Gamma_i(u)$  denote the set of vertices in  $V\Gamma$  that are at distance  $i$  from  $u$  and put  $\Gamma_{-1}(v) = \Gamma_{d+1}(v) := \emptyset$ . The graph  $\Gamma$  is called *distance-regular* if there are integers  $b_i, c_i$ ,  $0 \leq i \leq d$ , so that for any two vertices  $u$  and  $v$  in  $V\Gamma$  at distance  $i$ , there are precisely  $c_i$  neighbors of  $v$  in  $\Gamma_{i-1}(u)$  and  $b_i$  neighbors of  $v$  in  $\Gamma_{i+1}(u)$ . Clearly such a graph is regular with valency  $k := b_0$ . The numbers  $c_i, b_i$ , and  $a_i$ , where

$$a_i := k - b_i - c_i \quad (i = 0, \dots, d)$$

\*The authors thank the Com2MaC-KOSEF for its support.

†The author thanks the Swedish Research Council (VR) for its support.

is the number of neighbors of  $v$  in  $\Gamma_i(u)$  for  $u, v \in V\Gamma$  at distance  $i$ , are called the *intersection numbers* of  $\Gamma$ , and

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_j & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & \cdots & a_j & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_j & \cdots & b_{d-1} & b_d \end{bmatrix}$$

the *intersection array* of  $\Gamma$ .

Now, suppose that  $\Gamma$  is a distance-regular graph with valency  $k \geq 2$ , diameter  $d \geq 2$  and intersection numbers  $c_i, a_i, b_i$ ,  $0 \leq i \leq d$ . Given integers  $a \geq 0$  and  $b, c \geq 1$  with  $a + b + c = k$ , we define

$$\ell_{(c,a,b)} = \ell_{(c,a,b)}(\Gamma) := |\{i \mid 1 \leq i \leq d-1 \text{ and } (c_i, a_i, b_i) = (c, a, b)\}|,$$

that is, the number of columns  $(c, a, b)^T$  in the intersection array of  $\Gamma$ , and put

$$h = h(\Gamma) := \ell_{(1,a_1,b_1)}.$$

Note that since  $d \geq 2$  and  $c_1 = 1$  we have  $h \geq 1$ .

Finding good bounds for  $\ell_{(c,a,b)}$  is a powerful technique for understanding distance-regular graphs. For example, in [1] Bannai and Ito showed that, for a distance-regular graph with valency  $k \geq 3$ , if  $c$  is an integer with  $0 \leq 2c \leq k$  then  $\ell_{(c,k-2c,c)} \leq 10k2^k$ , from which they deduced that there are finitely many distance-regular graphs with valency 3. Also, in [4] Biggs et al. used circuit chasing to considerably improve this bound, which enabled them to classify the distance-regular graphs with valency 3.

In this paper we prove the following theorem.

**Theorem 1.1** *There exists a function  $k : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \rightarrow \mathbb{N}^+$  so that for all positive integers  $b, c, C$ ,  $k(b, c, C) \geq \max\{b + c, 3\}$  and for all distance-regular graphs  $\Gamma$  with valency  $k \geq k(b, c, C)$ , diameter  $d \geq 2$  and  $h \geq 2$ ,*

$$\ell_{(c,k-b-c,b)} \leq C.$$

As might be expected from the previously mentioned results for valency 3 distance-regular graphs, this theorem is closely related to the so-called Bannai-Ito Conjecture. Bannai and Ito conjectured that given an integer  $k \geq 3$  there are finitely many distance-regular graphs with valency  $k$ . In a series of papers [1, 2, 3] they showed that their conjecture was true for valency 3 and 4 and also that, for  $k \geq 3$  an integer, there are finitely many bipartite distance-regular graphs with valency  $k$  [2]. In addition, it was recently shown that the Bannai-Ito Conjecture is true for valencies 5, 6 and 7 [12] and also that there are finitely many triangle-free (i.e. containing no 3-cycles) distance-regular graphs with valency 8, 9 or 10 [13].

Using Theorem 1.1, we now prove that the Bannai-Ito Conjecture is basically equivalent to bounding  $\ell_{(c,k-b-c,b)}$  by a function of  $b$  and  $c$ .

**Theorem 1.2** *The following statements are equivalent:*

- (1) *For each integer  $k \geq 3$ , there are finitely many distance-regular graphs with valency  $k$ .*

(2) There exists a function  $f : \mathbb{N}^+ \times \mathbb{N}^+ \longrightarrow \mathbb{N}^+$  such that for all  $k, b, c \in \mathbb{N}^+$  and for all distance-regular graphs  $\Gamma$  with valency  $k \geq \max\{b + c, 3\}$ , diameter  $d \geq 2$  and  $h \geq 2$

$$\ell_{(c, k-b-c, b)} \leq f(b, c).$$

*Proof:* (1)  $\Rightarrow$  (2): By (1) there is a function  $g : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$  such that, for all distance-regular graphs  $\Gamma$  with valency  $k \geq 3$ , and diameter  $d \geq 2$ ,

$$d \leq g(k).$$

For  $b, c \in \mathbb{N}^+$  put

$$f(b, c) := \max\{g(k) \mid \max\{b + c, 3\} \leq k < k(b, c, 1)\},$$

where  $k : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \longrightarrow \mathbb{N}^+$  is a function with the properties given in Theorem 1.1.

Now suppose  $b, c \in \mathbb{N}^+$  and that  $\Gamma$  is a distance-regular graph with valency  $k \geq \max\{b + c, 3\}$ , diameter  $d \geq 2$ , and  $h \geq 2$ . Then

$$\ell_{(c, k-b-c, b)}(\Gamma) \leq d \leq g(k)$$

and, by Theorem 1.1 applied with  $C = 1$ , if  $k \geq k(b, c, 1)$  then

$$\ell_{(c, k-b-c, b)}(\Gamma) \leq 1.$$

Hence  $\ell_{(c, k-b-c, b)}(\Gamma) \leq f(b, c)$  and so (2) holds.

(2)  $\Rightarrow$  (1): Put

$$F(k) := \max\{f(b, 1) \mid 1 \leq b \leq k - 1\}.$$

Suppose that  $\Gamma$  is a distance-regular graph with valency  $k \geq 3$  and diameter  $d \geq 2$ . Note that  $k \geq 1 + b_1$  since otherwise  $k < b_1 + 1 = k - a_1$  which is a contradiction. By (2)

$$h = \ell_{(1, k-b_1-1, b_1)} \leq F(k)$$

and so, since  $d < \frac{1}{2}k^3h$  [10, Theorem 1.1],

$$d < \frac{1}{2}k^3F(k).$$

It is now straight-forward to check that (1) holds. ■

In view of results and examples contained in [6] and [8], it is plausible, for a distance-regular graph with  $h = 1$  and diameter  $d \geq 4$ , that  $c_4 \geq 2$ . If this were indeed the case, then Theorem 1.1 would also hold for  $h = 1$  and so the condition  $h \geq 2$  in Theorem 1.2 (2) could be removed. Bearing this in mind, we make the following conjecture.

**Conjecture 1.3** There exists a function  $f : \mathbb{N}^+ \times \mathbb{N}^+ \longrightarrow \mathbb{N}^+$  such that for all  $b, c \in \mathbb{N}^+$  satisfying  $b + c \leq k$  and for all distance-regular graph  $\Gamma$  with valency  $k \geq \max\{b + c, 3\}$

$$\ell_{(c, k-b-c, b)} \leq f(b, c).$$

In [7] Hiraki proved  $\ell_{(1,k-2,1)} \leq 20$  for every distance-regular graph with valency  $k \geq 3$ , and hence this conjecture is true in case  $b = c = 1$ . Using Theorem 1.1 we now prove a theorem that generalizes Hiraki's result in case  $h \neq 1$ .

**Theorem 1.4** *There exists a function  $f : \mathbb{N}^+ \rightarrow \mathbb{N}^+$  such that for all  $c \in \mathbb{N}^+$  and all distance-regular graphs  $\Gamma$  with valency  $k \geq \max\{2c, 3\}$ , diameter  $d \geq 2$  and  $h \geq 2$ ,*

$$\ell_{(c,k-2c,c)}(\Gamma) \leq f(c).$$

*Proof:* Suppose that  $k : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \rightarrow \mathbb{N}^+$  is a function with the properties given in Theorem 1.1. Given  $c \in \mathbb{N}^+$ , put  $k_c := k(c, c, 1) - 1$  and define

$$f(c) := 10 k_c 2^{k_c}.$$

Note that if  $k \geq \max\{2c, 3\}$ , then  $k(c, c, 1) \geq \max\{2c, 3\}$ , and hence  $f(c) > 1$ .

Now suppose that  $\Gamma$  is a distance-regular graph with valency  $k \geq \max\{2c, 3\}$  and  $h \geq 2$ . In view of Bannai and Ito's bound,  $\ell_{(c,k-2c,c)} \leq 10 k 2^k$ , mentioned above and since  $10 k 2^k$  is an increasing function on  $[\max\{2c, 3\}, \infty)$ , for all  $k$  with  $\max\{2c, 3\} \leq k \leq k(c, c, 1)$ ,

$$\ell_{(c,k-2c,c)} \leq 10 k 2^k \leq f(c).$$

The theorem now follows since by Theorem 1.1, for  $k \geq k(c, c, 1)$ ,

$$\ell_{(c,k-2c,c)} \leq 1 < f(c).$$

■

This rest of this paper is organized as follows. In Section 2 we present some definitions and results concerning distance-regular graphs. We also present a partial solution to a problem posed on [5, p.189] that is of independent interest and follows from Theorem 1.1. In Section 3 we derive some bounds for terms in the standard sequence associated to an eigenvalue of a distance-regular graph. Finally, in Section 4 we use these bounds to prove Theorem 1.1.

## 2 Distance-Regular Graphs

We begin this section by presenting some basic facts concerning distance-regular graphs (for more details see [5]). Suppose that  $\Gamma$  is a distance-regular graph with valency  $k \geq 2$ , diameter  $d \geq 2$  and intersection numbers  $c_i, a_i, b_i$ ,  $0 \leq i \leq d$ . Clearly,  $b_d = c_0 = a_0 = 0$  and  $c_1 = 1$ . In [5, Section 4.1], it is shown that  $\Gamma_i(u)$  contains  $k_i$  elements, where

$$k_0 := 1, \quad k_1 := k, \quad k_{i+1} := k_i b_i / c_{i+1}, \quad i = 0, \dots, d-1, \quad (1)$$

and in [5, Proposition 4.1.6] that

$$k = b_0 > b_1 \geq b_2 \geq \dots \geq b_{d-1} > b_d = 0 \text{ and } 1 = c_1 \leq c_2 \leq \dots \leq c_d \leq k. \quad (2)$$

Recall that the eigenvalues of  $\Gamma$  are the eigenvalues of the adjacency matrix of  $\Gamma$ . In particular, if  $\theta$  is an eigenvalue of  $\Gamma$  then  $\theta \in [-k, k]$ . We now state a result concerning the second largest eigenvalue of a distance regular graph.

**Lemma 2.1** [12, Theorem 6.2] Suppose  $b, c \in \mathbb{N}^+$  and  $k \geq \max\{b+c, 3\}$  is a positive integer. Let  $\Gamma$  be a distance-regular graph with valency  $k$  and put  $\ell := \ell_{(c, k-b-c, b)}$ . The second largest eigenvalue  $\theta_1$  of  $\Gamma$  satisfies

$$\theta_1 \geq k - b - c + 2\sqrt{bc} \cos\left(\frac{2\pi}{\ell+1}\right).$$

The standard sequence  $(u_i = u_i(\theta) \mid 0 \leq i \leq d)$  associated to an eigenvalue  $\theta$  of  $\Gamma$  is defined recursively by the equations

$$u_0 = 1, \quad u_1 = \theta/k, \quad b_i u_{i+1} - (\theta - a_i) u_i + c_i u_{i-1} = 0 \quad \text{for } i = 1, 2, \dots, d-1.$$

As is well-known, see e.g. [5, Theorem 4.1.4], if  $v := |V\Gamma|$ , then the multiplicity  $m(\theta)$  of  $\theta$  is given by

$$m(\theta) = \frac{v}{M(\theta)}, \quad (3)$$

where

$$M(\theta) = \sum_{i=0}^d k_i u_i(\theta)^2.$$

Now given a positive integer  $c$ , define

$$\begin{aligned} \xi_c &:= \min\{i \mid 1 \leq i \leq d \text{ and } c_i = c\}, \quad \text{and} \\ \eta_c &:= |\{i \mid 1 \leq i \leq d \text{ and } c_i = c\}|. \end{aligned}$$

To prove the next lemma we will use the following relationships between these numbers that were given in [10] (Lemma 2.1 and Proposition 3.2, respectively). If  $c > 1$  is an integer, then

$$\eta_c \leq 2\xi_c - 3, \quad (4)$$

and if  $c$  is a positive integer and  $\eta_c \neq 0$ , then

$$\xi_c \leq \frac{c^2}{4} \eta_1 + 1. \quad (5)$$

Put

$$e := \max\{i \mid 1 \leq i \leq d-1 \text{ and } c_i \leq b_i\}.$$

**Lemma 2.2** Suppose that  $\Gamma$  is a distance-regular graph with valency  $k \geq 3$  and diameter  $d \geq 2$ , and that  $b, c$  are positive integers with  $k \geq b+c$ . If  $\ell_{(c, k-b-c, b)} \geq 1$ , then

$$d < \begin{cases} 2(\eta_1 + 1) & \text{if } c_e = 1, \\ \frac{3}{2} \max\{b, c\}^2 \eta_1 & \text{if } c_e \geq 2. \end{cases}$$

**Proof:** Since  $c_{e+1} > b_{e+1}$ , by [5, Proposition 4.1.6 (ii)]

$$d < 2(e+1). \quad (6)$$

Thus, if  $c_e = 1$ , then since  $e \leq \eta_1$  it follows that  $d \leq 2\eta_1 + 1$  holds.

Now suppose  $c_e \geq 2$ . Since  $\{i \mid c_i = c_e\} = \{\xi_{c_e}, \xi_{c_e} + 1, \dots, \xi_{c_e} + \eta_{c_e} - 1\}$ ,

$$e \leq \xi_{c_e} + \eta_{c_e} - 1.$$

By applying (4) and then (5) to the righthand side of this inequality, we have

$$e \leq \frac{3}{4}c_e^2\eta_1 - 1. \quad (7)$$

But  $c_e \leq \max\{b, c\}$ , since  $1 \leq \ell_{(c, k-b-c, b)}$ . Thus, in view of (6) and (7) we have  $d < \frac{3}{2}\max\{b, c\}^2\eta_1$ . This completes the proof.  $\blacksquare$

### 3 Bounding Terms of the Standard Sequence

In this section we derive some bounds for terms in the standard sequence associated to an eigenvalue of a distance-regular graph that we use in the proof of Theorem 1.1. We begin with some definitions.

Suppose that  $\Gamma$  is a distance-regular graph with valency  $k \geq 3$  and diameter  $d \geq 2$ , and that  $\theta$  is an eigenvalue of  $\Gamma$  with  $a_1 + 2\sqrt{b_1} < \theta < k$ . Let  $1 \leq p < d$  be the largest integer for which  $c_p \leq b_p$  and  $a_p + 2\sqrt{b_p c_p} < \theta$  both hold. Define

$$T := T(\theta) = \{i \mid 0 \leq i \leq p \text{ and } (c_i, a_i, b_i) \neq (c_{i+1}, a_{i+1}, b_{i+1})\}.$$

Put  $s := |T| - 1$  and let  $t_0, t_1, \dots, t_s$  be the ordering of  $T$  with  $0 = t_0 < t_1 < \dots < t_s = p$ .

Now, if  $(u_i = u_i(\theta) \mid 0 \leq i \leq d)$  is the standard sequence associated to  $\theta$  and, for  $1 \leq i \leq s$ , the largest and smallest roots of the equation

$$b_{t_i} u_{t_i+1} + (a_{t_i} - \theta) u_{t_i} + c_{t_i} u_{t_i-1} = 0$$

are  $\rho_i := \rho_i(\theta)$  and  $\sigma_i := \sigma_i(\theta)$ , respectively, then by the theory of three-term recurrences there are numbers  $\gamma_i$  and  $\delta_i$  with

$$u_j = \gamma_i \rho_i^{j-t_{i-1}} + \delta_i \sigma_i^{j-t_{i-1}} \quad (t_{i-1} \leq j \leq t_i + 1). \quad (8)$$

Note that since  $a_i + 2\sqrt{b_i c_i} < \theta < k$ , we have  $0 < \sigma_i < \rho_i < 1$ ,  $1 \leq i \leq s$ .

We now list some inequalities that will be used in the proof of Theorem 1.1.

**Proposition 3.1** *Suppose  $1 \leq i \leq s$  and  $u_i$ ,  $\gamma_i$  and  $\rho_i$  are as defined just above. Then the following inequalities hold*

$$(i) \quad \rho_{i+1} < \rho_i, \quad i \neq s,$$

$$(ii) \quad u_{t_{i-1}+1} > \rho_i u_{t_{i-1}},$$

$$(iii) \gamma_i > u_{t_{i-1}},$$

$$(iv) u_{t_i} > \prod_{j=1}^i \rho_j^{t_j - t_{j-1}}.$$

Proof: (i): For positive integers  $b, c$  satisfying  $b + c \leq k$ ,  $c \leq b$  and  $k - b - c + 2\sqrt{bc} < \theta$  we define

$$f_{b,c}(x) := bx^2 + (k - b - c - \theta)x + c.$$

Let  $\rho_{b,c}$  be the largest root of  $f_{b,c}(x) = 0$ . Since  $b \geq c$ ,

$$\theta > k - b - c + 2\sqrt{bc} > k - (b + 1) - c + 2\sqrt{(b + 1)c},$$

and hence both  $\rho_{b,c}$  and  $\rho_{b+1,c}$  are positive. Moreover,  $0 < \rho_{b,c} < 1$  since  $k - b - c + 2\sqrt{bc} < \theta < k$ . Hence

$$f_{b+1,c}(\rho_{b,c}) = \rho_{b,c}^2 - \rho_{b,c} = \rho_{b,c}(\rho_{b,c} - 1) < 0$$

and therefore  $\rho_{b,c} < \rho_{b+1,c}$ . It is straight-forward to show in a similar fashion that  $\rho_{b,c} < \rho_{b,c-1}$  holds. It now follows in view of (2) that (i) must hold.

(ii) and (iii): We will prove that these hold using induction on  $i$ . Suppose  $i = 1$ . Then  $u_{t_0} = u_0 = 1$  and  $u_{t_0+1} = u_1 = \frac{\theta}{k}$ . Since  $a_1 + 2\sqrt{b_1} < \theta < k$  and  $\rho_1$  is the largest root of

$$b_1x^2 + (a_1 - \theta)x + 1 = 0,$$

we have

$$b_1\left(\frac{\theta}{k}\right)^2 + (a_1 - \theta)\frac{\theta}{k} + 1 = \left(1 - \frac{\theta}{k}\right)\left(1 + (a_1 + 1)\frac{\theta}{k}\right) > 0.$$

Hence  $\frac{\theta}{k} > \rho_1$ . Thus  $\gamma_1 > 1$  since  $\gamma_1\rho_1 + \delta_1\sigma_1 = u_1 = \frac{\theta}{k} > \rho_1$ ,  $\gamma_1 + \delta_1 = u_0 = 1$  and  $\rho_1 > \sigma_1 > 0$ . Therefore (ii) and (iii) hold for  $i = 1$ .

Now suppose  $2 \leq i < s$  and suppose  $u_{t_{i-1}+1} > \rho_i u_{t_{i-1}}$  and  $\gamma_i > u_{t_{i-1}}$  both hold. Then  $\delta_i < 0$  since  $\gamma_i + \delta_i = u_{t_{i-1}}$ . Thus, using equations

$$u_{t_i} = \gamma_i \rho_i^{t_i - t_{i-1}} + \delta_i \sigma_i^{t_i - t_{i-1}} \text{ and } u_{t_{i+1}} = \gamma_i \rho_i^{t_i - t_{i-1} + 1} + \delta_i \sigma_i^{t_i - t_{i-1} + 1},$$

we obtain

$$\rho_i u_{t_i} < u_{t_{i+1}}. \tag{9}$$

Hence  $\rho_{i+1} u_{t_i} < \rho_i u_{t_i} < u_{t_{i+1}}$  by (i) and (9) and so (ii) holds.

Now, in view of

$$u_{t_i} = \gamma_{i+1} + \delta_{i+1} \text{ and } u_{t_{i+1}} = \gamma_{i+1} \rho_{i+1} + \delta_{i+1} \sigma_{i+1},$$

it follows that

$$\gamma_{i+1} = \frac{u_{t_{i+1}} - \sigma_{i+1} u_{t_i}}{\rho_{i+1} - \sigma_{i+1}}$$

holds, and hence by (i) and (9)

$$\gamma_{i+1} > \frac{\rho_i - \sigma_{i+1}}{\rho_{i+1} - \sigma_{i+1}} u_{t_i} > u_{t_i}$$

holds. Thus (iii) holds.

(iv) We prove this by using induction on  $i$ . Suppose  $i = 1$ . Then by (8), (ii) and (iii) we have

$$\begin{aligned} u_{t_1} - \rho_1^{t_1} &= (\gamma_1 - 1)\rho_1^{t_1} + \delta_1\sigma_1^{t_1} \\ &= (\gamma_1 - 1)\rho_1^{t_1} + \sigma_1^{t_1-1}(u_1 - \gamma_1\rho_1) \\ &> \rho_1(\gamma_1 - 1)(\rho_1^{t_1-1} - \sigma_1^{t_1-1}) > 0. \end{aligned}$$

Therefore (iv) holds for  $i = 1$ .

Now, suppose  $2 \leq i < s$  and assume

$$u_{t_i} > \prod_{j=1}^i \rho_j^{t_j - t_{j-1}}. \quad (10)$$

Then using (iii),  $u_{t_i} = \gamma_{i+1} + \delta_{i+1}$  and  $u_{t_{i+1}} = \gamma_{i+1}\rho_{i+1}^{t_{i+1}-t_i} + \delta_{i+1}\sigma_{i+1}^{t_{i+1}-t_i}$ , we obtain

$$u_{t_{i+1}} - u_{t_i}\rho_{i+1}^{t_{i+1}-t_i} = \delta_{i+1}(\sigma_{i+1}^{t_{i+1}-t_i} - \rho_{i+1}^{t_{i+1}-t_i}) > 0.$$

But by (10) it then follows that

$$u_{t_{i+1}} > u_{t_i}\rho_{i+1}^{t_{i+1}-t_i} > \prod_{j=1}^i \rho_j^{t_j - t_{j-1}} \rho_{i+1}^{t_{i+1}-t_i} = \prod_{j=1}^{i+1} \rho_j^{t_j - t_{j-1}}$$

holds. This completes the proof of (iv) ■

## 4 Proof of Theorem 1.1

Before proving the theorem, we first present some definitions. Suppose that  $b, c$  and  $C$  are arbitrary positive integers. Put

$$\begin{aligned} \phi &= \phi_{b,c} := -b - c - 2\sqrt{bc} \quad \text{and} \\ \phi' &= \phi'_{b,c,C} := -b - c + 2\sqrt{bc} \cos\left(\frac{2\pi}{C+2}\right). \end{aligned}$$

Note

$$\phi < -b - c - \sqrt{bc} \leq \phi'.$$

For each  $c'$  with  $1 \leq c' \leq c$ , let  $\beta_{c'}$  be the smallest positive integer satisfying both  $\beta_{c'} \geq c'$  and  $\phi \geq -\beta_{c'} - c' + 2\sqrt{\beta_{c'}c'}$ .

Now, for  $l, m$  any positive integers and for any real number  $\lambda \geq -l - m - 2\sqrt{lm}$ , let  $\eta_m(\lambda)$  denote the largest root of the equation

$$lx^2 - (l + m + \lambda)x + m = 0.$$



Note that since  $2\sqrt{\beta_{c'}c'} \leq \phi + \beta_{c'} + c' < \phi' + \beta_{c'} + c'$ , it follows that

$$0 < \sqrt{\frac{c'}{\beta_{c'}}} < \tau_{\beta_{c'},c'}(\phi') < 1. \quad (11)$$

Define

$$\begin{aligned} \rho = \rho_{b,c,C} &:= \min\{\tau_{\beta_{c'},c'}(\phi') \mid 1 \leq c' \leq c\} \text{ and} \\ \alpha &:= \max\left\{\frac{\beta_{c'}}{c'} \mid 1 \leq c' \leq c\right\}. \end{aligned}$$

By (11) and  $\beta_1 \geq 9$ , we have

$$\rho < 1 \quad \text{and} \quad 9 \leq \alpha. \quad (12)$$

*Proof of Theorem 1.1:* We define a function  $k$  and prove that it has the required properties. For  $b, c$  and  $C$  arbitrary positive integers, put

$$k(b, c, C) := \max\left\{\frac{\alpha^{20}}{\rho^{12}}, 2\left(\frac{\alpha^{2\max\{b,c\}^2}}{\rho c^2}\right)^9, b + c, 3\right\}.$$

Now suppose that  $\Gamma$  is a distance-regular graph with  $h(\Gamma) \geq 2$ , valency  $k \geq \max\{b + c, 3\}$ , diameter  $d \geq 2$  and

$$\ell_{(c,k-b-c,b)} > C.$$

We prove

$$k < \begin{cases} \frac{\alpha^{20}}{\rho^{12}} & \text{if } c = 1, \\ 2\left(\frac{\alpha^{2\max\{b,c\}^2}}{\rho c^2}\right)^9 & \text{if } c \geq 2, \end{cases}$$

from which the theorem immediately follows.

Let  $w$  be the largest non-negative integer so that  $t := t_w$  is the largest element of  $T(\theta_1)$  with

$$k - b_t - c_t + 2\sqrt{b_t c_t} < k - b - c + 2\sqrt{bc}. \quad (13)$$

Note that this last equation implies  $c_t \leq c$ .

Now, since  $\ell_{(c,k-b-c,b)} \geq C + 1 \geq 2$ , by Lemma 2.1 the second largest eigenvalue  $\theta_1$  of  $\Gamma$  satisfies

$$\theta_1 \geq k + \phi'.$$

Hence, in view of the definitions of  $\rho_i$  and  $\rho$ ,

$$\rho_w(\theta_1) \geq \rho_w(k + \phi') = \tau_{b_t c_t}(\phi') \geq \rho.$$

Therefore, since  $\rho_i(\theta_1) \geq \rho$  for  $1 \leq i \leq w$ , it follows by Proposition 3.1 (i) and (iv) that

$$u_t > \rho^t. \quad (14)$$

Thus, by (3) and (14) we have

$$m(\theta_1) < \frac{v}{k_t u_t^2} < \frac{v}{k_t \rho^{2t}}. \quad (15)$$

Moreover, since  $b_1 \geq \frac{1}{2}k$  and  $h \geq 2$ , the Terwilliger Tree bound [11, Proposition 3.3] implies

$$2\left(\frac{k}{2}\right)^{\frac{1}{2}h} \leq 2(b_1)^{\frac{1}{2}h} \leq m(\theta_1). \quad (16)$$

In addition, by (1) and (2) we have

$$\begin{aligned} k_i &\leq k_t \leq \alpha^i k_t & 0 \leq i \leq t-1, \\ k_{t+i} &\leq \alpha^i k_t \leq \alpha^{t+i} k_t & 0 \leq i \leq d-t, \end{aligned}$$

and so, as  $d \geq 2$  and  $\alpha \geq 2$ ,

$$v \leq k_t \sum_{j=0}^d \alpha^j = k_t \left[ \frac{\alpha^{d+1} - 1}{\alpha - 1} \right] < k_t \alpha^{\frac{3}{2}d}. \quad (17)$$

Thus, by (12), (15), (16), (17) and  $h \geq 2$ ,

$$k < 2 \left( \frac{\alpha^{\frac{3}{2}d}}{2\rho^{2t}} \right)^{\frac{2}{h}}. \quad (18)$$

Now, suppose  $c = 1$ . Since  $c_t \leq c = 1$  we have  $t \leq \eta_1$ . Hiraki [9, Theorem 2] has shown that if  $h = h(\Gamma) \geq 2$ , then

$$\eta_1 \leq 2(h+1). \quad (19)$$

Thus Lemma 2.2 implies  $d \leq 2\eta_1 + 1 \leq 4h + 5$  and so

$$\frac{\alpha^{\frac{3}{2}d}}{2\rho^{2t}} < \frac{\alpha^{6h+8}}{2\rho^{4h+4}}.$$

So, by (18) and  $h \geq 2$ , we obtain

$$k < \frac{2\alpha^{12}}{\rho^8} \left( \frac{\alpha^{16}}{4\rho^8} \right)^{\frac{1}{h}} \leq \frac{\alpha^{20}}{\rho^{12}}.$$

Now, to complete the proof, suppose  $c \geq 2$ . Since  $c_t \leq c$ , by (4), (5) and (19), we have

$$t < \xi_c + \eta_c \leq \frac{3}{2}c^2(h+1).$$

Also, by Lemma 2.2 and (19),

$$d < \frac{3}{2} \max\{b, c\}^2 \eta_1 \leq 3 \max\{b, c\}^2 (h+1).$$

Thus by (18),  $h \geq 2$  and the last two bounds on  $t$  and  $d$ ,

$$k < 2 \left( \frac{\alpha^{\frac{3}{2} \max\{b, c\}^2 (h+1)}}{2\rho^{3c^2(h+1)}} \right)^{\frac{2}{h}} = 2^{1-\frac{2}{h}} \left( \frac{\alpha^{\frac{3}{2} \max\{b, c\}^2}}{\rho^{c^2}} \right)^{\frac{6(h+1)}{h}} < 2 \left( \frac{\alpha^{2 \max\{b, c\}^2}}{\rho^{c^2}} \right)^9.$$

This completes the proof. ■

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