数理解析研究所講究録 1228 巻 2001 年 1-12

Modular invariants of the modular data of finite groups

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Abstract

It is well known that for each finite group G there is associated a fusion algebra (in conformal field theory of mathematical physics), which is a commutative and associative finite dimensional algebra over the complex number field, and whose basis (primary fields) consists of all pairs of representatives a of the conjugacy classes of G and the irreducible characters of $C_G(a)$. Then, to this fusion algebra, the matrices S and T (called the modular data) are naturally associated. The size of these matrices is the number of primary fields. Also, S and T give a representation of the modular group $SL(2, Z)$. The modular invariants of G are the matrices M with nonnegative integral entries and $M_{00} = 1$ which commute with S and T. (Cf. A. Coste, T. Gannon and P. Ruelle: Finite group modular data, Nuclear Phys. B581 (2000), 679–717.)

In this paper, we will treat the following topics, which were motivated by the above mentioned paper of Coste, Gannon and Ruelle.

We determine the modular data S and T, and the modular invariants M for several finite groups G. Here are some examples of our results; there are 48 solutions of the modular invariants for the dihedral group of order 6 (symmetric group of degree 3), there are 65 solutions of the modular invariants for the dihedral group of order 10, there are 8719 solutions of the modular invariants for the alternating group of degree 5, etc.

We also discuss the question whether there exist two nonisomorphic groups whose modular data are identical.

1 Introduction

This paper is a slightly extended version of a talk by the first author (Eiichi Bannai) during the Nebres conference in February 22–24, 2001 which was held at De La Salle University, Manila, Philippines. The first author thanks the organizers of the conference for the invitation to join the celebration of the 60th birthday of Fr. Nebres.

The work presented in this paper is a preliminary report of the ongoing research of the first author and his collaborators, which was motivated by the influence of Terry Gannon; first

from his talk during the " Aigebraic Combinatorics, Vertex Operator Algebras, and Monster " meeting at the University of California at Santa Cruz in July 2000, and then from the papers [4], [3], [9]. We thank Professor Gannon for providing us with many useful informations. Although the main purpose of this paper is to present some new results which can be regarded as a kind of supplement to the papers of Coste-Gannon-Ruelle [4] and Gannon [9], we also hope that this paper may be used as a short introductory survey on this subject (from a combinatorial view point), especially as a handy introduction to the survey paper of Gannon [9] for those readers who are not familiar with this area.

It is known in mathematical physics (cf. [4], [9]) that for each conformal field theory (CFT), there is associated a finite dimensional commutative associative algebra which is called a fusion algebra. Here, we give a set of axioms of fusion algebra following [6], [4], [9].

Axioms of fusion algebra (See [6], [4], [9])

$\mathfrak{A} = \langle x_0, x_1, \ldots x_d \rangle$ is a finite dimensional vector space over the complex number field $\mathbb{C}$ with the basis $x_0, x_1, \ldots x_d$ satisfying the following conditions:

$$x_i x_j = \sum_{k=0}^{d} N_{i,j}^k x_k$$

(i) $N_{i,j}^k \in \mathbf{N} = \{0, 1, 2, \ldots\}$[1] for $\forall i,\ j,\ k \in \Phi$.
(ii) There exists an involutive map $\hat{}\ : \Phi \longrightarrow \Phi$ satisfying

(a) $N_{i,j}^k = N_{i,\hat{j}}^{\hat{k}}$,

(b) $N_{i,j}^0 = \delta_{i,j}$.

The indices in $\Phi = \{0, 1, 2 \ldots, d\}$ are called primary fields of the CFT, that is, the basis of the fusion algebra $\mathfrak{A}$ is indexed by the primary fields $\Phi = \{0, 1, 2 \ldots, d\}$. Note that these axioms imply that if we put $N_{i,j,k} = N_{i,j}^{\hat{k}}$, then $N_{i,j,k}$ is symmetric in $i,\ j,\ k$.

It should be noted that the definition of CFT as well as the definition of fusion algebra is not fixed and varies from the viewpoints of researchers, and there is room for debate on what are the right axioms.

Also, it is known in mathematical physics that there are associated two matrices S and T whose rows and columns are indexed by the primary fields Φ, which are called the modular data (or fusion data) for the fusion algebra (or the CFT) and are axiomized as follows.

Axioms of modular data (or fusion data) S and T (See [6], [4], [9])

[1] In what follows, we always denote the set of nonnegative integers by N

$$(*)\ \begin{cases} \text{(i) The matrix } S \text{ is symmetric and unitary (that is, } {}^tS = S \text{ and } {}^t\bar{S}\,S = I). \\ \text{(ii) The matrix } T \text{ is diagonal, unitary and of finite order.} \\ \text{(iii) } S \text{ and } T \text{ satisfy the relation } S^2 = (S\,T)^3 \text{ with } S^2 = C \text{ being a permutation} \\ \quad \text{matrix with } C^2 = I \text{ (namely, } S \text{ and } T \text{ give a representation of } SL(2,\mathbb{Z})). \\ \text{(iv) } S_{0,i} \geq 0 \text{ for any } i \in \Phi = \{0,1,\ldots,d\}. \end{cases}$$

The permutation matrix C in condition (iii) is called the charge conjugation.
Note that the matrix S satisfies the following conditions:

$$(**)\ \begin{cases} (1)\ N_{i,j}^0 = (S^2)_{i,j} \\ (2)\ N_{i,j}^k = \sum_{m=0}^{d} \dfrac{S_{i,m}S_{j,m}\overline{S_{k,m}}}{S_{0,m}} \end{cases}$$

The last condition (2) is called Verlinde's formula.

We should emphasize that the existence of these modular data S and T are based on physical considerations, and can not be obtained purely mathematically just from the axioms of fusion algebra. Again, there is room for debate on what is the right axiomization of modular data, although the one given here is pretty much standard.

Since $SL(2,\mathbb{Z})$ has the presentation

$$SL(2,\mathbb{Z}) = \langle\ x,\ y \mid x^2 = (xy)^3,\ x^4 = 1\ \rangle,$$

where $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have the representation ρ :

$$\rho\ :\ SL(2,\mathbb{Z}) \longrightarrow GL(d+1,\mathbb{C})$$

with $\rho(x) = S$ and $\rho(y) = T$. The following is a famous conjecture (cf. [6], [4], [8], [9]) .

Conjecture The kernel $\ker(\rho)$ is a congruence subgroup of $SL(2,\mathbb{Z})$.

Note that a subgroup Γ of $SL(2,\mathbb{Z})$ is called a congruence subgroup if Γ contains the principal congruence subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid a \equiv d \equiv 1,\ b \equiv c \equiv 0 \pmod{N} \right\}$$

for some $N \in \mathbb{N}$. A more precise form of the conjecture is that if $T^N = 1$, then $\Gamma(N) \subseteq \ker(\rho)$.

Assuming the above conjecture, Eholzer [6] gave a classification of modular fusion algebras with smaller dimensions. We believe it is interesting to attack this problem without assuming the conjecture, and so we propose the following research problem from a purely combinatorial view point.

Research Problem : Find the pair of matrices S and T (of small sizes) satisfying conditions $(*)$ and $(**)$ with all the $N_{i,j}^k \in \mathbb{N}$.
(We expect there should exist some noncongruence normal subgroups Γ in which the above

representation ρ of $SL(2,\mathbb{Z})$ is given by a representation of the finite group $SL(2,\mathbb{Z})/\Gamma$.) Also we believe that this problem is manageable and that group theory will be useful to attack this problem.

If we weaken the condition $N_{i,j}^k \in \mathbb{N}$ of the fusion algebra $\mathfrak{A}$ to the condition $N_{i,j}^k \in \mathbb{R}$ (or $N_{i,j}^k \geq 0$), fusion algebra $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ is closely related to the so called character algebra in the sense of the paper in 1942 of Kawada [11] (see also [2],§2.5]). (The character algebras are, in turn, a generalization of Bose-Mesner algebras of commutative association schemes.) This relation is fully explained in [1] (see also [9]). In fact, the work of Kawada [11] is a very pioneering work in algebraic combinatorics.

Here, let us allow the first author to mention a very personal comment on Yukiyosi Kawada (1916–1993).

As it is well known and as it was explained repeatedly during the Nebres conference, the initial mathematical exchange between Philippines and Japan was substantiated as the exchange program of mathematicians under the agreement between the Japan Society for the Promotion of Sciences (JSPS) and the Philippines' Department of Science and Technology (DOST), which was initiated by Professor Fr. Nebres and Professor Kawada with the support of Professor Shokichi Iyanaga. When the first author was an undergraduate student at Tokyo University, Professors Kawada and Iyanaga were faculty members as professors there, and he took advanced undergraduate mathematics courses from them. At that time Professor Kawada was his advisor. When the first author got a job as an Instructor at Tokyo University, Professor Kawada was a senior professor in that same department. The first author spent many years at Ohio State University and returned to Japan in 1989. Shortly after that, he was asked by Professor Kenichi Shinoda to participate in the DOST-JSPS exchange program. The first author was pleased to join the program since he was absolutely confident with the judgements of Professors Kawada and Iyanaga and thought that the program should be right if it was initiated by Professors Kawada and Iyanaga. It has been a great pleasure for the first author to be able to witness the success of such program as well as the developments of Filipino mathematics since then. The success of this exchange program is evident from the success of the present Nebres Conference.

When the first author had the chance to meet Professor Kawada more than 10 years ago, he told Professor Kawada that his paper [11] of 1942, which was a kind of forgotten work, had been highly regarded by combinatorialists as a pioneering work on algebraic combinatorics. At that time it seemed that Professor Kawada was not impressed much about it and explained to the first author that it was not his main work. However, some years later, the first author felt happy when he heard from someone that Professor Kawada was actually pleased to have been told that his old work was paid attention by combinatorialists.

It is described in Gannon [9] that there are many sources to get modular data S and T. For example, they are obtained from Euclidean lattices, affine Lie algebras, finite groups, Rational Conformal Theory (RCFT) and Vertex Operator Algebras (VOA), subfactors, Type III subfactors, number fields, etc., etc., ... We will not go into the details, but we concentrate ourselves on the examples obtained by finite groups.

Let us comment on the definition of (non unitary) fusion algebras and modular data. For the algebra $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ satisfying condition (*) of the fusion algebra, there is always associated a matrix S which diagonalizes the matrices $N_i = (N_{i,j}^k)_{j,\ k\in\Phi}$, $(\forall i \in \Phi)$ at the same time. In [1], we have normalized $S = (S_{i,j})$ in such a way that $S_{0,i} = S_{i,0}$ holds for $\forall i \in \Phi$. Therefore, the matrix S is a unitary matrix only when $k_i = m_i$, $\forall i \in \Phi$. (We concluded in [1] that Verlinde's formula is valid when S is symmetric, i.e., the corresponding character algebra is self-dual and that it is not valid in general. In particular, when S is not a unitary matrix.) As it is remarked in [9], it is possible to choose the matrix S (which diagonalizes the matrices N_i, $i \in \Phi$ at the same time and satisfies $S_{0,1} \in \mathbb{R}, \forall i \in \Phi$) as a unitary matrix from the beginning using a different normalization. (Hence, the original condition $S_{0,i} = S_{i,0}, \forall i \in \Phi$ is not necessarily true in general if S is not symmetric.) Even so, Verlinde's formula in the original expression (i.e., the formula (**)) is not valid in general if S is not symmetric. Therefore, in order to get Verlinde's formula in the original expression, some self-duality condition (such as ${}^tS = S$ or some slight weakening of it, cf. [9]) is in fact needed.

2 Modular data (fusion data) for finite groups and their modular invariants

The concept of fusion algebras naturally associated with finite groups is due to G. Lusztig [12], [13] and Dijkgraaf-Vafa-Verlinde-Verlinde [5] independently. Lusztig used this algebra to obtain the exact values of irreducible characters of certain exceptional finite Chevalley groups. In this paper, we follow the notation of [4], [8], [9] so that the notation of modular data would correspond to the notation given in the previous section. (The axioms of modular data used in this paper is slightly different from those of Lusztig's.)

Let G be a finite group. Let $C_0 = \{1\}$, $C_1, \ldots, C_d$ be all the conjugacy classes of G, where 1 is the identity element of G. Let R be a set of representatives of C_0, $C_1, \ldots, C_d$. Take

$$\Phi = \{(a, \chi) \mid a \in R,\ \chi \text{ is an irreducible character of } C_G(a)\}.$$

The matrices (modular data) S and T are defined as follows.

$$S_{(a,\chi),(b,\chi')} = \frac{1}{|C_G(a)|\ |C_G(b)|} \sum_{g\in G(a,b)} \overline{\chi(gbg^{-1})}\ \overline{\chi'(gag^{-1})}$$

where

$$G(a,b) = \{g \in G \mid a \cdot gbg^{-1} = gbg^{-1} \cdot a\}$$

and $\overline{\chi(*)}$ denote the complex conjugate of $\chi(*)$.

$$T_{(a,\chi),(b,\chi')} = \delta_{a,b}\delta_{\chi,\chi'}\frac{\chi(a)}{\chi(1)}.$$

Then it is shown that S and T satisfy the conditions (*) and (**) with $N_{i,j}^k \in \mathbb{N}$, $\forall i, j, k \in \Phi$.

Remark. It seems that it is expected from the viewpoint of physics that there is associated a RCFT or Rational VOA to this modular data (fusion data). However, it seems that it is still a conjecture from a rigorous mathematical viewpoint.

Definition ((Physical) modular invariant). The non-negative integer matrix $M = (M_{i,j})_{i\in\Phi, j\in\Phi}$ satisfying the conditions

$$MS = SM, \quad MT = TM, \quad \text{and } M_{00} = 1$$

is called a (physical) modular invariant (for the modular data S and T).

Proposition (See [3]). There are only finitely many modular invariants M for the modular data S and T of a finite group G.

Remark. The proof of this proposition given in [3] is based on the condition that

$$S_{0,0} \leq S_{0,i}, \quad \forall i \in \Phi,$$

which is certainly true for the modular data of a finite group. More explicitly, we get for example,

$$\sum_{a,b\in\Phi} M_{a,b} \leq |G|^2.$$

Before going into the details of our concrete determination of modular invariants for certain finite groups, let us mention why we are interested in modular invariants M (cf. [4], [9], etc.). It is known in mathematical physics that for each RCFT, there is associated for each $a \in \Phi$ a character ch_a defined on the upper half plane satisfying the conditions:

$$\mathrm{ch}_a(-\frac{1}{\tau}) = \sum_{b\in\Phi} S_{a,b}\ \mathrm{ch}_a(\tau),$$

$$\mathrm{ch}_a(\tau + 1) = \sum_{b\in\Phi} T_{a,b}\ \mathrm{ch}_a(\tau).$$

(Here $\mathrm{ch}_a(\tau) = q^{-\frac{c}{24}}\ \mathrm{Tr}_a\ q^{L_0}$ in the Rational VOA theory, and the variable τ is in the upper half plane. It is a well known conjecture that the ch_a, $(a \in \Phi)$ are modular functions for some subgroups in $SL(2,\mathbb{Z})$ of level N.)
Then the one-loop partition function $Z(\tau)$ of RCFT is given by

$$Z(\tau) = \sum_{a,b\in\Phi} M_{a,b}\ \mathrm{ch}_a(\tau)\ \overline{\mathrm{ch}_b(\tau)}.$$

So, we need to know the modular invariants in order to know $Z(\tau)$. However, it seems that $\mathrm{ch}_a(\tau)$ are known only for very special cases (cf. [4], [9], etc.), and it is generally not easy to know $\mathrm{ch}_a(\tau)$ explicitly.

3 Results

We first want to find the modular data S and T for some finite groups, and then want to determine the modular invariants M for each modular data.

It is known that this question can basically be answered for finite abelian groups, although it seems not easy in general to find the modular invariants explicitly (cf. [7], [9]). In what follows, we restrict our study to non-abelian groups.

First, let $G = D_{2n}$ denote the dihedral group of order $2n$. Then we obtain the following table for $|\Phi|$, and we can have S and T very explicitly ([3]). For example, we have

G	$\lvert\Phi\rvert$
D_{4m}	$2m^2+14$
D_{4m+2}	$2m^2+2m+4$

So, in particular we have

$$|\Phi(D_6)| = 8, \quad |\Phi(D_8)| = 22, \quad |\Phi(D_{10})| = 16, \quad |\Phi(D_{12})| = 32, \quad |\Phi(D_{14})| = 28.$$

Theorem 1. The numbers of solutions of modular invariants M for the groups D_6, D_{10} and D_{14} are given as follows:
$|M| = 48$ for D_6,
$|M| = 65$ for D_{10},
$|M| = 27$ for D_{14},

Remark. (i) It is stated in [4] that $|M| = 32$ for D_6 $(= S_3)$. However, the number 48 seems correct. At the end of this paper, we will list all the 48 solutions of M.
(ii) To save the space, we will not list all the solutions for D_{10} and D_{14} in this paper, however the complete solutions will be available from the authors on request, (or see http://www.math.kyushu-u.ac.jp/AlgComb/). This is also true for other data presented in this paper. It seems unclear whether the numbers of solutions $|M(D_{2p})|$ increase with p, but it seems that the increase (if that is the case) is not so drastic. Thus, we believe that the complete solutions $M(D_{2p})$ may be manageable to obtain for odd prime numbers p or more generally for odd numbers p. On the other hand, the number of solutions $|M(D_{4m})|$ is generally expected to be larger than the case of $D_{2(2m+1)}$, and at the present time we do not know the explicit number $|M(D_8)|$ (possibly larger than 10^5).

Theorem 2.

$$|\Phi(A_4)| = 14, \text{ and } \quad |M(A_4)| = 38$$

($|\Phi(S_4)| = 21$, but it seems that $|M(S_4)|$ is very large, and is not explicitly determined at the time of this writing.

Theorem 3.

$$|\Phi(A_5)| = 22, \text{ and } \quad |M(A_5)| = 8719$$

Remark 8719 is a prime number. In [4], it is mentioned that a finite group G has generally many modular invariants if any of $|\mathrm{Out}(G)|$, $|G/G'|$, $|Z(G)|$, or $|\mathbb{Q}\ (S_{i,j}(i,j=0,\ldots,d))/\mathbb{Q}|$ is large. Neither of these numbers is large for $G = A_5$. It is remarkable that $|M(A_5)|$ is considerably larger than the numbers obtained by these standard constructions. We do not have any accurate feelings or guess on what will the numbers $|M(A_n)|$ be for $n \longrightarrow \infty$. At the end of this paper, we will list the modular data S and T and the solution M in which $M_{a,b}$ takes the largest value 25 among all the 8719 solutions $M(A_5)$.

We conclude this paper by answering a special case of the question of Coste-Gannon-Ruelle [4].

Problem: Do the modular data S and T determine the group G uniquely? Namely, let S_i and T_i, $(i=1,2)$ be the modular data for G_i, $(i=1,2)$ and assume that there is a permutation matrix P satisfying $P^{-1}S_1P = S_2$, and $P^{-1}T_1P = T_2$. Then are G_1 and G_2 isomorphic as abstract groups ?
(It was suggested in [4] that the groups of order 16 may be a place to look for counter examples.)

We prove the following:

Proposition The above problem is answered affirmatively for groups of order 16.

Proof. There exist 14 groups of order 16. We will use MAGMA's numbering of the groups. At first we classify their modular data S and T with $|\Phi|$. The result is as follows.

$\lvert\Phi\rvert$	Groups
256 (abelian)	(16,1), (16,2), (16,5), (16,10), (16,14)
88	(16,3), (16,4), (16,6), (16,11), (16,12), (16,13)
46	(16,7), (16,8), (16,9)

The modular data S and T of abelian groups were classified already. So we will check pairs {(16,3), (16,4), (16,6), (16,11), (16,12), (16,13)} and {(16,7), (16,8), (16,9)}. There are only two pairs {(16,3),(16,13)}and {(16,4),(16,12)} for which the diagonal entries of T are identical (including multiplicities) as set. However, the entries (including multiplicities) of S for (16,4) and (16,12) are different. Therefore, {(16,3), (16,13)} is the only possibility. Actually, their matrices S have identical (including multiplicities) entries as set, and they survived in many further nonisomorphism tests. So, we first thought that these two groups might give the identical matrices S and T. Now, let S_3 and T_3 be the modular data of (16,3), S_{13} and T_{13} be the modular data of (16,13). Since T_3 and T_{13} have the same set of diagonal entries, there exist permutation matrices Q_3 and Q_{13} such that $T := Q_3^{-1}T_3Q_3 = Q_{13}^{-1}T_{13}Q_{13} = E_1(1,44) \oplus E_2(-1,28) \oplus E_3(\xi_4,8) \oplus E_4(-\xi_4,8)$,where $E_i(\lambda,p)$ is a scalar matrix of λ and of degree p. We define $S_3' := Q_3^{-1}S_3Q_3$, $S_{13}' := Q_{13}^{-1}S_{13}Q_{13}$. $\exists P$ such that $P^{-1}S_3P = S_{13}$ and $P^{-1}T_3P = T_{13}$ if and only if $\exists P'$ such that $P'^{-1}S_3'P' = S_{13}'$ and $P'^{-1}TP' = T$ where $P' = P_1' \oplus P_2' \oplus P_3' \oplus P_4'$ such that $E_iP_i' = P_i'E_i$. Thus, we let $N_1 := \{1,\ldots,44\}, N_2 := \{45,\ldots,72\}, N_3 := \{73,\ldots,80\}, N_4 := \{81,\ldots,88\}$, $\exists P$ such that $P^{-1}S_3P = S_{13}$ and $P^{-1}T_3P = T_{13}$ if and only if $[S_3']_{N_i,N_i}P_i' = P_i'[S_{13}']_{N_i,N_i}, \forall i \in \{1,\cdots,4\}$ and $[S_3']_{N_i,N_j}P_j' = P_i'[S_{13}']_{N_i,N_j}, \forall i,j \in \{1,\cdots,4\}$ where $[S_k']_{N_i,N_i}$ is the $|N_i| \times |N_j|$ matrix such that $([S_k']_{N_i,N_j})_{x,y} = (S_k')_{x,y}, (x \in N_i, y \in N_j)$.

However, there is no permutation matrix which satisfies the above equations for $i := 3$ and $j := 4$. ∎

Remark The problem above is yet unanswered for groups of order 32. We have calculated the modular data S and T for each group of order 32 (and 64) using GAP. There are several possible candidates of pairs of groups G_1 and G_2 (of order 32) which might give a counter example. However, in all of these cases it is proved that the character tables of G_1 and G_2 are not equivalent. Hence, in view of the fact that Coste-Gannon-Ruelle [4] expected that the modular data determines the character tables of the group, it may be unlikely that a counter example is obtained from the groups of order 32.

Appendix 1

Modular data S and T of dihedral group of order 6

$$6S = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & 3 & -3 \\ 3 & -3 & 0 & 0 & 0 & 0 & -3 & 3 \end{bmatrix}, T = \text{diagonal}[1, 1, 1, 1, w, w^2, 1, -1].$$

Modular invariants of dihedral group of order 6

$$MI_1 = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, MI_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, MI_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$MI_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, MI_5 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, MI_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$MI_7 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, MI_8 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, MI_9 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$MI_{10} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, MI_{11} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, MI_{12} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$MI_{13} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, MI_{14} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, MI_{15} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$MI_{16} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, MI_{17} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, MI_{18} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$MI_{19} = \begin{bmatrix} 1&1&0&1&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&3&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, MI_{20} = \begin{bmatrix} 1&1&0&2&0&0&0&0\\ 1&1&0&2&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 2&2&0&4&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, MI_{21} = \begin{bmatrix} 1&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$MI_{22} = \begin{bmatrix} 1&1&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, MI_{23} = \begin{bmatrix} 1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, MI_{24} = \begin{bmatrix} 1&1&0&2&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&0&2&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&0&2&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$MI_{25} = \begin{bmatrix} 1&1&2&0&0&0&0&0\\ 1&1&2&0&0&0&0&0\\ 1&1&2&0&0&0&0&0\\ 1&1&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, MI_{26} = \begin{bmatrix} 1&1&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&1&3&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, MI_{27} = \begin{bmatrix} 1&0&1&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$MI_{28} = \begin{bmatrix} 1&1&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, MI_{29} = \begin{bmatrix} 1&1&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&2&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, MI_{30} = \begin{bmatrix} 1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$MI_{31} = \begin{bmatrix} 1&1&0&2&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&0&2&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&0&2&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, MI_{32} = \begin{bmatrix} 1&1&2&0&0&0&0&0\\ 1&1&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 2&2&4&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, MI_{33} = \begin{bmatrix} 1&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 1&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$MI_{34} = \begin{bmatrix} 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 2&2&2&2&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix},$$

$MI_{35} = {}^tMI_{21}$, $MI_{36} = {}^tMI_{22}$, $MI_{37} = {}^tMI_{23}$, $MI_{38} = {}^tMI_{24}$, $MI_{39} = {}^tMI_{25}$, $MI_{40} = {}^tMI_{26}$, $MI_{41} = {}^tMI_{27}$, $MI_{42} = {}^tMI_{28}$, $MI_{43} = {}^tMI_{29}$, $MI_{44} = {}^tMI_{30}$, $MI_{45} = {}^tMI_{31}$, $MI_{46} = {}^tMI_{32}$, $MI_{47} = {}^tMI_{33}$, $MI_{48} = {}^tMI_{34}$.

Note that MI_2, MI_5, MI_7, MI_{12}, MI_{13}, MI_{17}, MI_{18}, MI_{19}, MI_{26}, MI_{27}, MI_{29}, MI_{33}, MI_{40}, MI_{41}, MI_{43}, MI_{47} are not listed in Coste-Gannon-Ruelle [4].

Appendix 2

Modular data S and T of A_5, the alternating group of order 60.

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ {}^tS_{12} & S_{22} & S_{23} & S_{24} \\ {}^tS_{13} & {}^tS_{23} & S_{33} & S_{34} \\ {}^tS_{14} & {}^tS_{24} & {}^tS_{34} & S_{44} \end{bmatrix} \text{ where } S_{11} = \frac{1}{60}\begin{bmatrix} 1&3&3&4&5\\ 3&9&9&12&15\\ 3&9&9&12&15\\ 4&12&12&16&20\\ 5&15&15&20&25 \end{bmatrix},$$

$$S_{12} = \frac{1}{60}\begin{bmatrix} 12&12&20&15&15\\ -12b&-12a&0&-15&-15\\ -12a&-12b&0&-15&-15\\ -12&-12&20&0&0\\ 0&0&-20&15&15 \end{bmatrix}, S_{13} = \frac{1}{5}\begin{bmatrix} 1&1&1&1&1&1&1&1\\ -b&-a&-b&-a&-b&-a&-b&-a\\ -a&-b&-a&-b&-a&-b&-a&-b\\ -1&-1&-1&-1&-1&-1&-1&-1\\ 0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$S_{14} = \frac{1}{12}\begin{bmatrix} 3&3&4&4\\ -3&-3&0&0\\ -3&-3&0&0\\ 0&0&4&4\\ 3&3&-4&-4 \end{bmatrix}, S_{22} = \frac{1}{60}\begin{bmatrix} 24&24&0&0&0\\ 24&24&0&0&0\\ 0&0&40&0&0\\ 0&0&0&45&-15\\ 0&0&0&-15&45 \end{bmatrix}, S_{23} = \frac{1}{5}\begin{bmatrix} b&a&a&b&a&b&b&a\\ a&b&b&a&b&a&a&b\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$S_{24} = \frac{1}{12}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ -3 & -3 & 0 & 0 \\ -3 & -3 & 0 & 0 \end{bmatrix}, \quad S_{33} = \frac{1}{5}\begin{bmatrix} 2 & b & b & b & a & a & a & 2 \\ b & 2 & b & b & a & a & 2 & a \\ b & b & 2 & a & b & 2 & a & a \\ b & b & a & 2 & 2 & b & a & a \\ a & a & b & 2 & 2 & a & b & b \\ a & a & 2 & b & a & 2 & b & b \\ a & 2 & a & a & b & b & 2 & b \\ 2 & a & a & a & b & b & b & 2 \end{bmatrix}, \quad S_{34} = [0], \quad S_{44} = \frac{1}{12}\begin{bmatrix} -3 & 9 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 0 & 0 & 8 & -4 \\ 0 & 0 & -4 & 8 \end{bmatrix},$$

with $a = \zeta + \zeta^4$, $b = \zeta^2 + \zeta^3$ and ζ is a primitive 5-th root of unity.
$T = \mathrm{diag}\,[1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \zeta,\ \zeta,\ \zeta^2,\ \zeta^2,\ \zeta^3,\ \zeta^3,\ \zeta^4,\ \zeta^4,\ -1,\ -1,\ \omega,\ \omega^2]$, with ω a primitive 3rd root of unity.

The maximum value of the entries of all the 8719 solutions M is 25. The solution M having the upper left corner $60S_{11}$ and all other entries are 0, with S_{11} given above, is the only one with this maximum value.

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