

Rational curves on coindex 3 Fano varieties

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This poster is based on the paper [JO]. We describe the irreducible components of the moduli spaces of rational curves on smooth coindex 3 Fano varieties. This is a higher-dimensional analog of the study of smooth Fano 3-folds [BLRT]. In particular, we prove the moduli space of rational curves representing each numerical class is irreducible when the dimension is at least 5.

0 Settings

- X : Fano manifold/ $\mathbb{C}$ of coindex 3, dimension $n \geq 4$
- $\text{coindex} := n + 1 - \max\{r \in \mathbb{Z} \mid -K_X/r \in \text{Pic}(X)\}$
- $H := -K_X/(n-2)$: fundamental divisor
- Coindex 3 Fano manifolds are classified by [Muk]

1 Main Theorems

Theorem 1. Let $n \geq 5$ and $\alpha \in NE(X)$

$\implies \text{Mor}(\mathbb{P}^1, X, \alpha)$ irreducible.

Moreover, $\alpha \in \text{Nef}_1(X) \setminus \partial NE(X)$

$\implies \text{Mor}(\mathbb{P}^1, X, \alpha)$ gen. para. emb. free curves.

Theorem 2. Let $n = 4$ and $\alpha \in \text{Nef}_1(X) \setminus \partial NE(X)$

Assume that $X \not\cong \mathbb{P}^1 \times V$ (V : Fano 3-fold of even index)

- $H \cdot \alpha = 1$ and X is general
 $\implies \text{Mor}(\mathbb{P}^1, X, \alpha)$ smooth irreducible.
- $H \cdot \alpha \geq 2$
 $\implies \exists! M \subset \text{Mor}(\mathbb{P}^1, X, \alpha)$ gen. para. emb. free curves.
Any other component parametrizes either
 - (i) non-dominant family of curves, or
 - (ii) multiple covers of H -lines.

2 Strategy for the proof

- (A) Distinguish between *accumulating* and *Manin* components
- (B) Base case (spaces of H -lines)
- (C) Induction step (*Movable bend-and-break*)

3 Accumulating components

$M \subset \text{Mor}(\mathbb{P}^1, X, \alpha)$: irreducible component

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{s} & X: \text{evaluation morphism} \\ \pi \downarrow & & \\ M & & \end{array}$$

π : universal family

M is *accumulating*

“def” $\iff \begin{cases} \text{(i) } s \text{ is non-dominant, or} \\ \text{(ii) general fiber of } s \text{ is not irreducible} \end{cases}$

Accumulating comps can be detected by the a -inv.

Definition 3 (a -invariant). Y : proj var, L : nef, big divisor

$$a(Y, L) := \inf\{t \in \mathbb{R} \mid K_{\tilde{Y}} + t\phi^*L: \text{pseudo-effective on } \tilde{Y}\},$$

which is indep. of the choice of a resolution $\phi: \tilde{Y} \rightarrow Y$.

Then, [LT] tells that:

- (i) $\implies a(Z, -K_X|_Z) > 1$, where $Z := \overline{s(\mathcal{U})}$, and
- $\neg(\text{i}) \wedge (\text{ii}) \implies$ Stein factorization $f: Y \rightarrow X$ of s is an a -cover (i.e., $a(Y, -f^*K_X) = 1$).

Theorem 4. For Z as above, one of the following holds:

- $n \leq 5$, $\rho(X) = 2$, and Z is the exceptional locus of an elementary divisorial contraction.
- $n = 4$, $(Z, H|_Z) \cong (\mathbb{P}^2, \mathcal{O}(1))$, and Z is contracted by an elementary fiber-type contraction.

Theorem 5. For $f: Y \rightarrow X$ as above, let $\phi: Y \dashrightarrow B$ be the Iitaka fib'n for $K_Y - f^*K_X$. Then one of the following holds:

- $n = 4$, ϕ is birat to a base change of a family of H -lines,
- $\rho(X) = 2$, ϕ is birat to a base change of a MFS.

4 Base case

Theorem 6. Let $\alpha \in \text{Nef}_1(X)$ such $H \cdot \alpha = 1$.

X : general in moduli $\implies \text{Mor}(\mathbb{P}^1, X, \alpha)$: smooth irreducible.

(Idea). $\exists \mathbb{G} \supset X$: ambient variety s.t. X is a C.I. of very ample divisors on $\mathbb{G}$. Consider the family $\mathcal{X} \rightarrow W$ of C.I.s on $\mathbb{G}$, and relate the irreducibility of $\text{Mor}(\mathbb{P}^1, X, \alpha)$ and $\text{Mor}(\mathbb{P}^1, \mathbb{G}, \alpha)$. $\square$

5 Movable Bend-and-Break

Theorem 7. Suppose $X \supset \ncong$ contractible divisor.

$\alpha \in \text{Nef}_1(X)$ s.t. $H \cdot \alpha \geq 2$, $\implies \forall \text{free curve} \in \text{Mor}(\mathbb{P}^1, X, \alpha)$ deforms into a union of two free curves.

MBB is the key tool for counting the number of Manin (non-accumulating) components, and one can conclude the main theorems.

References

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