

On generalized numerical range of the Aluthge transformation

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ABSTRACT

In this report the authors show that the Aluthge transformation $\tilde{T}$ of a matrix T and a polynomial f satisfy the inclusion relation $W_C(f(\tilde{T})) \subset W_C(f(T))$ for the generalized numerical range if C is a Hermitian matrix or a rank-one matrix.

1. THE ALUTHGE TRANSFORMATION

In the development of operator theory, Aluthge [1] introduced a transformation $\tilde{T}$ for a bounded linear operator T on a complex Hilbert space H with the help of the polar decomposition $T = V|T|$ as follows:

Definition 1 (Aluthge transformation [1]). *Let $T = V|T|$ be the polar decomposition of a bounded linear operator T . Then the Aluthge transformation $\tilde{T}$ of T is defined by*

$$\tilde{T} = |T|^{\frac{1}{2}} V |T|^{\frac{1}{2}}.$$

We remark that $\tilde{T}$ is defined by using a partial isometry V and $|T|$ with $T = V|T|$ and $N(V) = N(|T|)$. But in fact, $\tilde{T}$ does not depend on the choice of V (see [19]), for example, if $T = U|T|$ is a matrix with unitary U , then $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$.

As properties of $\tilde{T}$, the following assertions are well known:

- (i) $\sigma(T) = \sigma(\tilde{T})$, where $\sigma(T)$ means the spectrum of an operator T .
- (ii) $\|T\| \geq \|\tilde{T}\|$.

(i) has been shown in [9], and we can obtain (ii) easily as follows:

$$\|\tilde{T}\| \leq \| |T|^{\frac{1}{2}} \| \cdot \|V\| \cdot \| |T|^{\frac{1}{2}} \| \leq \|T\|.$$

Recently, many authors discuss the n th iterated Aluthge transformation which is denoted by $\widetilde{T_n}$, i.e.,

$$\widetilde{T_n} = \widetilde{(\widetilde{T_{n-1}})} \quad \text{and} \quad \widetilde{T_0} = T,$$

and the following interesting property is shown in [20].

$$\lim_{n \rightarrow \infty} \|\widetilde{T_n}\| = r(T),$$

where $r(T)$ is the spectral radius of T .

2. NUMERICAL RANGE

In this section, we shall introduce the numerical range and a result on that of the Aluthge transformation.

Definition 2 (Numerical range). *For an operator T , the numerical range $W(T)$ of T is the subset of the complex numbers $\mathbb{C}$, given by,*

$$W(T) = \{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \}.$$

The following properties of the numerical range are well known.

- (i) $W(T)$ is a convex set (Hausdorff-Toeplitz).
- (ii) $\sigma(T) \subset \overline{W(T)}$.

As a result on the numerical range of $\tilde{T}$, the following result has been shown.

Theorem 2.A ([18]). *Let T be a bounded linear operator, then the following inclusion relation holds.*

$$(2.1) \quad \overline{W(\tilde{T})} \subset \overline{W(T)}.$$

Theorem 2.A was firstly shown in [10] in case T is a 2×2 matrix (in this case, $W(\tilde{T})$ and $W(T)$ are closed subsets of the complex number $\mathbb{C}$). Then one of the authors [19] proved that (2.1) holds if T admits a decomposition $T = U|T|$ for an isometry operator U . This condition is always satisfied if T is an $n \times n$ matrix, or H is finite dimensional. In [19], the relation (2.1) is shown by using the property of the numerical range

$$(2.2) \quad \overline{W(\tilde{T})} = \bigcap_{\lambda \in \mathbb{C}} \{ z \in \mathbb{C} : |z - \lambda| \leq w(T - \lambda I) \},$$

where $w(T)$ is the *numerical radius* of T , that is,

$$w(T) = \sup\{|z| : z \in W(T)\}$$

and the following characterization of $w(T) \leq 1$ by Berger and Stampfli [3]:

$$(2.3) \quad w(T) \leq 1 \text{ if and only if } \|T - zI\| \leq 1 + \sqrt{1 + |z|^2} \text{ for all } z \in \mathbb{C}.$$

In a recent paper [18], Wu showed that the inclusion (2.1) holds for every bounded linear operator T on a Hilbert space H . He showed this result by using the previous result shown in [19] and some properties of numerical range and Aluthge transformation, so this proof is not easy. In this report, we shall obtain a simplified proof of Theorem 2.A in Section 4.

3. C -NUMERICAL RANGE

As a generalization of the numerical range, for $n \times n$ matrices C and T , the C -numerical range of T is defined in [7] as follows:

Definition 3 (C -numerical range [7]). *For $n \times n$ matrices C and T , the C -numerical range $W_C(T)$ of T is the compact subset of complex number $\mathbb{C}$, given by,*

$$W_C(T) = \{\text{tr}(CU^*TU) : U \text{ is a unitary matrix}\}.$$

Put $C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, then $W_C(T) = W(T)$, so we can regard $W_C(T)$ as a

generalization of $W(T)$. But $W_C(T)$ is not always convex as follows:

Example ([17]). Let

$$T = C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Put unitary matrices U_1 and U_2 as follows:

$$U_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then we have

$$CU_1^*TU_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad CU_2^*TU_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix},$$

that is, $1, 2i \in W_C(T)$. But put a unitary matrix U_3 as follows:

$$U_3 = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}.$$

$$\operatorname{tr}(CU_3^*TU_3) = |u_{22}|^2 - |u_{33}|^2 + i(|u_{23}|^2 + |u_{32}|^2).$$

Assume that $\frac{1+2i}{2} \in W_C(T)$. Then the following relations hold:

$$\begin{cases} |u_{22}|^2 - |u_{33}|^2 = \frac{1}{2}, \\ |u_{23}|^2 + |u_{32}|^2 = 1. \end{cases}$$

So U_3 can not be unitary, and it is a contradiction. Hence $\frac{1+2i}{2} \notin W_C(T)$.

In fact, it is known that $W_C(T)$ is star-shaped as follows:

Theorem 3.A ([4]). *For any $n \times n$ matrices C and T , the range $W_C(T)$ is star-shaped with star center at $y = \frac{1}{n}\operatorname{tr}(C)\operatorname{tr}(T)$, i.e., if $x \in W_C(T)$, then*

$$\lambda x + (1 - \lambda)y \in W_C(T) \quad \text{for all } \lambda \in [0, 1].$$

Especially, when C is a Hermitian matrix or a rank-one matrix, the range $W_C(T)$ is a convex set (cf. [17] and [16]). In these cases, we can rephrase them in the following ways:

The case that C is a Hermitian matrix. We assume that the spectrum of the Hermitian matrix C is the set

$$c = (c_1, c_2, \dots, c_n).$$

Since C is a Hermitian matrix, there is a unitary matrix U such that

$$U^*CU = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \end{pmatrix}.$$

Hence the set $W_C(T)$ can be rewritten as follows:

$$W_C(T) = \left\{ \sum_{j=1}^n c_j \langle Tx_j, x_j \rangle : \{x_1, x_2, \dots, x_n\} \text{ is an orthonormal basis of } \mathbb{C}^n \right\},$$

which is denoted by $W_c(T)$ and we call $W_c(T)$ the c -numerical range of T . Poon [14] gave an alternative proof of the convexity of $W_c(T)$ using some type of majorization property (cf. [8, page 87-88]).

The case that C is a nonzero $n \times n$ matrix of rank one. We assume that the operator norm of C is 1. Then there exists a unitary matrix U such that

$$U^*CU = \begin{pmatrix} q & \sqrt{1-|q|^2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where q is an eigenvalue of C with $|q| \leq 1$. Hence the set $W_C(T)$ can be rewritten as follows:

$$W_C(T) = \{\langle Tx, y \rangle : x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1, \langle x, y \rangle = q\},$$

which is denoted by $W_q(T)$ and we call $W_q(T)$ the q -numerical range of T .

In this report, firstly, we shall obtain the direct proof of Theorem 2.A without using (2.2) and (2.3). Secondly, we shall generalize this result to c -numerical range in Section 5 as follows:

$$(3.1) \quad W_c(f(\tilde{T})) \subset W_c(f(T))$$

holds for all polynomial f . Lastly, we shall show the same relation (3.1) holds for q -numerical range.

4. SIMPLIFIED PROOF OF THEOREM 2.A

In this section, we shall obtain a direct proof of Theorem 2.A without using (2.2) and (2.3). To prove the this result, we prepare an obvious lemma.

Lemma 4.A ([9]). *Let A be a self-adjoint operator and B be an operator. Then AB is invertible if and only if BA is invertible. Hence $\sigma(AB) = \sigma(BA)$.*

We denote the real part of an operator A by $\Re(A) = \frac{A+A^*}{2}$.

Simplified proof of Theorem 2.A. Let $T = V|T|$ be a polar decomposition of T . Since

$$\begin{aligned} \Re(|T|V) &= \frac{|T|V + V^*|T|}{2} \\ &= \frac{V^*V|T|V + V^*|T|V^*V}{2} \\ &= V^* \frac{V|T| + |T|V^*}{2} V \\ &= V^* \Re(T) V, \end{aligned}$$

we have

$$\begin{aligned} \langle \Re(|T|V)x, x \rangle &= \langle V^* \Re(T) Vx, x \rangle \\ &= \langle \Re(T) Vx, Vx \rangle \\ &= \left\langle \Re(T) \frac{Vx}{\|Vx\|}, \frac{Vx}{\|Vx\|} \right\rangle \langle Vx, Vx \rangle. \end{aligned}$$

Hence

$$(4.1) \quad W(\Re(|T|V)) \subset W(\Re(T))W(V^*V).$$

If $0 \in W(V^*V)$, then $0 \in W(\Re(T))$. By $W(V^*V) = [0, 1]$ and Hausdorff-Toeplitz Theorem, we obtain

$$\begin{aligned}
 (4.2) \quad W(\Re(|T|V)) &\subset W(\Re(T))W(V^*V) \quad \text{by (4.1)} \\
 &= \left\{ \alpha \langle \Re(T)x, x \rangle : \|x\| = 1, \alpha \in [0, 1] \right\} \\
 &= W(\Re(T)).
 \end{aligned}$$

If $0 \notin W(V^*V)$, then V is an isometry, so $W(\Re(|T|V)) \subset W(\Re(T))$ holds by (4.1).

On the other hand, for any two operators H and K , the following relation is easily obtained:

$$(4.3) \quad \Re\{\Re(H)K\} = \frac{1}{2}\{\Re(HK) + \Re(K^*H)\}.$$

Therefore we have

$$\begin{aligned}
 \overline{W(\Re(\tilde{T}))} &= \overline{W(|T|^{\frac{1}{2}}\Re(V)|T|^{\frac{1}{2}})} \\
 &= \overline{\text{conv}\sigma(|T|^{\frac{1}{2}}\Re(V)|T|^{\frac{1}{2}})} \\
 &= \overline{\Re\text{conv}\sigma(|T|^{\frac{1}{2}}\Re(V)|T|^{\frac{1}{2}})} \\
 &= \overline{\Re\text{conv}\sigma(\Re(V)|T|)} \quad \text{by Lemma 4.A} \\
 &\subset \overline{\Re W(\Re(V)|T|)} \\
 &= \overline{W(\Re\{\Re(V)|T|\})} \\
 &= \frac{1}{2}\overline{W(\Re(V|T|) + \Re(|T|V))} \quad \text{by (4.3)} \\
 &\subset \frac{1}{2}\left\{ \overline{W(\Re(T))} + \overline{W(\Re(|T|V))} \right\} \\
 &\subset \frac{1}{2}\left\{ \overline{W(\Re(T))} + \overline{W(\Re(T))} \right\} \quad \text{by (4.2)} \\
 &= \overline{W(\Re(T))},
 \end{aligned}$$

where $\text{conv}\sigma(T)$ means the convex hull of $\sigma(T)$.

Since $\widetilde{e^{i\theta}T} = e^{i\theta}\tilde{T}$ holds for each $\theta \in [0, 2\pi)$, we have

$$\overline{W(\Re\{e^{i\theta}\tilde{T}\})} \subset \overline{W(\Re\{e^{i\theta}T\})} \quad \text{for all } \theta \in [0, 2\pi),$$

so that we obtain (2.1). □

Remark. In our proof of Theorem 2.A, the equation (4.3) plays an important role. (4.3) is also useful to extend the relation (2.1) to c -numerical range or q -numerical range.

5. c -NUMERICAL RANGE OF THE ALUTHGE TRANSFORMATION OF A MATRIX

In this section, we shall generalize Theorem 2.A to c -numerical range and T to $f(T)$ where f is a polynomial.

Theorem 5.1. *Let T be an $n \times n$ matrix, f be a polynomial and $c = (c_1, c_2, \dots, c_n)$ be a finite real sequence. Then the following inclusion holds:*

$$W_c(f(\tilde{T})) \subset W_c(f(T)).$$

In this result, we may assume that $c = (c_1, c_2, \dots, c_n)$ is a finite real sequence arranged in the decreasing order $c_1 \geq c_2 \geq \dots \geq c_n$ by the definition of $W_c(T)$.

To prove Theorem 5.1, we shall prepare the following results:

Theorem 5.A ([12]). *Let T be an $n \times n$ matrix and $c = (c_1, c_2, \dots, c_n)$ is a finite real sequence arranged in the decreasing order $c_1 \geq c_2 \geq \dots \geq c_n$. Then*

$$(5.1) \quad \max \{ \Re(z e^{i\theta}) : z \in W_c(T) \} = \sum_{j=1}^n c_j \lambda_j (\Re(e^{i\theta} T)),$$

holds for every $0 \leq \theta \leq 2\pi$, where $\lambda_j(S)$ means the j th eigenvalue of an $n \times n$ Hermitian matrix S :

$$\lambda_1(S) \geq \lambda_2(S) \geq \dots \geq \lambda_n(S).$$

Lemma 5.B ([6], [13, page 237])). *Suppose that T is an $n \times n$ complex matrix and $\{\Re \lambda_1(T), \Re \lambda_2(T), \dots, \Re \lambda_n(T)\}$ denotes the set of real parts of eigenvalues of T arranged in the decreasing order. Then the inequality*

$$\sum_{j=1}^k \Re \lambda_j(T) \leq \sum_{j=1}^k \lambda_j (\Re(T))$$

holds for every $1 \leq k \leq n - 1$.

Lemma 5.C ([5], [13, page 241])). *Suppose that G and H are $n \times n$ Hermitian matrices. Then the inequality*

$$\sum_{j=1}^k \lambda_j (G + H) \leq \sum_{j=1}^k \{ \lambda_j(G) + \lambda_j(H) \}$$

holds for every $1 \leq k \leq n - 1$.

Lemma 5.2. *Let A be a positive invertible matrix and X be an arbitrary matrix. Then for each polynomial f and real θ , there exists a matrix S such that*

$$\begin{aligned} e^{i\theta} f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) &= A^{\frac{1}{2}} S A^{\frac{1}{2}}, \\ e^{i\theta} f(XA) &= SA, \\ e^{i\theta} f(AX) &= AS. \end{aligned}$$

Proof. Let $f(z) = f(0) + g(z)z$, where $g(z)$ is also a polynomial. By using the equation

$$(A^{\frac{1}{2}} X A^{\frac{1}{2}})^n = A^{\frac{1}{2}} (XA)^{n-1} X A^{\frac{1}{2}},$$

we obtain the following equation:

$$\begin{aligned} f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) &= f(0)I + g(A^{\frac{1}{2}} X A^{\frac{1}{2}}) A^{\frac{1}{2}} X A^{\frac{1}{2}} \\ (5.2) \quad &= f(0)I + A^{\frac{1}{2}} g(XA) X A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} \{f(0)A^{-1} + g(XA)X\} A^{\frac{1}{2}}. \end{aligned}$$

By setting

$$S = e^{i\theta} \{f(0)A^{-1} + g(XA)X\},$$

we have

$$\begin{aligned} e^{i\theta} f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) &= e^{i\theta} A^{\frac{1}{2}} \{f(0)A^{-1} + g(XA)X\} A^{\frac{1}{2}} \quad \text{by (5.2)} \\ &= A^{\frac{1}{2}} S A^{\frac{1}{2}}, \\ SA &= e^{i\theta} \{f(0)A^{-1} + g(XA)X\} A \\ &= e^{i\theta} \{f(0)I + g(XA)XA\} \\ &= e^{i\theta} f(XA) \end{aligned}$$

and

$$\begin{aligned} AS &= e^{i\theta} A \{f(0)A^{-1} + g(XA)X\} \\ &= e^{i\theta} \{f(0)I + Ag(XA)X\} \\ &= e^{i\theta} f(AX) \quad \text{by } A(XA)^n X = (AX)^{n+1}. \end{aligned}$$

Hence the proof is complete. □

Proof of Theorem 5.1. We use a polar decomposition $T = U|T|$ where U is a unitary matrix. Put $A = |T| \geq 0$ and $X = U$. By perturbing A to $A + \varepsilon I$ for small $\varepsilon > 0$, we need only to prove Theorem 5.1 for a positive invertible A . By Theorem 5.A, we shall show the following inequality

$$(5.3) \quad \sum_{j=1}^n c_j \lambda_j \left(\Re \{e^{i\theta} f(\tilde{T})\} \right) \leq \sum_{j=1}^n c_j \lambda_j \left(\Re \{e^{i\theta} f(T)\} \right)$$

for every $0 \leq \theta \leq 2\pi$. Moreover by the following equations

$$\begin{aligned}
\sum_{j=1}^n c_j \lambda_j \left(\Re \{ e^{i\theta} f(\tilde{T}) \} \right) &= \sum_{j=1}^{n-1} (c_j - c_{j+1}) \sum_{k=1}^j \lambda_k \left(\Re \{ e^{i\theta} f(\tilde{T}) \} \right) + c_n \sum_{k=1}^n \lambda_k \left(\Re \{ e^{i\theta} f(\tilde{T}) \} \right), \\
\sum_{j=1}^n c_j \lambda_j \left(\Re \{ e^{i\theta} f(T) \} \right) &= \sum_{j=1}^{n-1} (c_j - c_{j+1}) \sum_{k=1}^j \lambda_k \left(\Re \{ e^{i\theta} f(T) \} \right) + c_n \sum_{k=1}^n \lambda_k \left(\Re \{ e^{i\theta} f(T) \} \right), \\
\sum_{k=1}^n \lambda_k \left(\Re \{ e^{i\theta} f(\tilde{T}) \} \right) &= \sum_{k=1}^n \lambda_k \left(\Re \{ e^{i\theta} f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) \} \right) \\
&= \sum_{k=1}^n \lambda_k \left(\Re (A^{\frac{1}{2}} S A^{\frac{1}{2}}) \right) \quad \text{by Lemma 5.2} \\
&= \text{tr} \left(\Re (A^{\frac{1}{2}} S A^{\frac{1}{2}}) \right) = \Re \left\{ \text{tr} (A^{\frac{1}{2}} S A^{\frac{1}{2}}) \right\} = \Re \left(\text{tr} (S A) \right) \\
&= \text{tr} \left(\Re (S A) \right) = \sum_{k=1}^n \lambda_k \left(\Re \{ e^{i\theta} f(X A) \} \right) \quad \text{by Lemma 5.2} \\
&= \sum_{k=1}^n \lambda_k \left(\Re \{ e^{i\theta} f(T) \} \right),
\end{aligned}$$

it is sufficient to prove the inequality

$$\sum_{j=1}^k \lambda_j \left(\Re \{ e^{i\theta} f(\tilde{T}) \} \right) \leq \sum_{j=1}^k \lambda_j \left(\Re \{ e^{i\theta} f(T) \} \right)$$

holds for $0 \leq \theta \leq 2\pi$ and every $k = 1, 2, \dots, n-1$.

By using Lemma 5.2 and Fan's two inequalities, we have

$$\begin{aligned}
& \sum_{j=1}^k \lambda_j \left(\Re \{ e^{i\theta} f(\tilde{T}) \} \right) \\
&= \sum_{j=1}^k \lambda_j \left(\Re \{ e^{i\theta} f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) \} \right) \\
&= \sum_{j=1}^k \lambda_j \left(\Re(A^{\frac{1}{2}} S A^{\frac{1}{2}}) \right) \quad \text{by Lemma 5.2} \\
&= \sum_{j=1}^k \lambda_j \left(A^{\frac{1}{2}} \Re(S) A^{\frac{1}{2}} \right) \\
&= \sum_{j=1}^k \Re \lambda_j \left(A^{\frac{1}{2}} \Re(S) A^{\frac{1}{2}} \right) \\
&= \sum_{j=1}^k \Re \lambda_j \left(\Re(S) A \right) \quad \text{by Lemma 4.A} \\
&\leq \sum_{j=1}^k \lambda_j \left(\Re \{ \Re(S) A \} \right) \quad \text{by Lemma 5.B} \\
&= \frac{1}{2} \sum_{j=1}^k \lambda_j \left(\Re(SA) + \Re(AS) \right) \quad \text{by (4.3)} \\
&\leq \frac{1}{2} \left\{ \sum_{j=1}^k \lambda_j \left(\Re(SA) \right) + \sum_{j=1}^k \lambda_j \left(\Re(AS) \right) \right\} \quad \text{by Lemma 5.C} \\
&= \frac{1}{2} \left\{ \sum_{j=1}^k \lambda_j \left(\Re \{ e^{i\theta} f(XA) \} \right) + \sum_{j=1}^k \lambda_j \left(\Re \{ e^{i\theta} f(AX) \} \right) \right\} \quad \text{by Lemma 5.2} \\
&= \frac{1}{2} \left\{ \sum_{j=1}^k \lambda_j \left(\Re \{ e^{i\theta} f(T) \} \right) + \sum_{j=1}^k \lambda_j \left(U^* \Re \{ e^{i\theta} f(T) \} U \right) \right\} \\
&= \sum_{j=1}^k \lambda_j \left(\Re \{ e^{i\theta} f(T) \} \right).
\end{aligned}$$

Hence the proof of Theorem 5.1 is complete. □

The case $f(z) = z$, we obtain the following corollary.

Corollary 5.3. *Let T be an $n \times n$ matrix and $c = (c_1, c_2, \dots, c_n)$ be a finite real sequence. Then the following inclusion holds:*

$$W_c(\tilde{T}) \subset W_c(T).$$

6. q -NUMERICAL RANGE OF THE ALUTHGE TRANSFORMATION OF A MATRIX

It is known that there is a close relationship between the family of q -numerical ranges $W_q(T)$ ($0 \leq q \leq 1$) of a matrix T and the Davis-Wielandt shell $W(T, T^*T)$ of T . The latter is defined by

$$W(T, T^*T) = \{(\langle Tx, x \rangle, \langle T^*Tx, x \rangle) \in \mathbb{C} \times \mathbb{R} : x \in \mathbb{C}^n, \|x\| = 1\}.$$

It is shown that the range $W(T, T^*T)$ is convex if T is an $n \times n$ matrix for $n \geq 3$ in [2]. In the case T is a 2×2 matrix, the range $W(T, T^*T)$ is convex if its affine hull is 2-dimensional, and the range $W(T, T^*T)$ is the boundary of a convex set if its affine hull is 3-dimensional. The following lemma provides a tool to compare the q -numerical ranges of two matrices.

Lemma 6.A ([11, page 389, Theorem 2.1]). *Suppose that A is an $n \times n$ matrix and B is an $m \times m$ matrix. Then the following two conditions are mutually equivalent:*

- (i) *The inclusion $W_q(B) \subset W_q(A)$ holds for every $0 \leq q \leq 1$.*
- (ii) *The inclusion $W(B) \subset W(A)$ and the inequality*

$$\max\{h : (z, h) \in W(B, B^*B)\} \leq \max\{h : (z, h) \in W(A, A^*A)\}$$

hold for every $z \in W(B)$.

In this section, we shall prove the following theorem.

Theorem 6.1. *Suppose that T is an $n \times n$ matrix and $f(z)$ is a polynomial in z . Then the inclusion*

$$(6.1) \quad W_q(f(\tilde{T})) \subset W_q(f(T))$$

holds for every complex number q with $|q| \leq 1$.

To prove Theorem 6.1, we have an alternative condition of (ii) in the above Lemma 6.A.

Lemma 6.2. *Suppose that A is an $n \times n$ matrix and B is an $m \times m$ matrix. Then the following two conditions are mutually equivalent:*

- (i): *The inclusion $W_q(B) \subset W_q(A)$ holds for every $0 \leq q \leq 1$.*
- (iii): *The inequality*

$$\lambda_1(B^*B + k \Re(e^{i\theta} B)) \leq \lambda_1(A^*A + k \Re(e^{i\theta} A))$$

holds for every $0 \leq \theta \leq 2\pi$ and $k \geq 0$.

Proof. We prove the equivalence of condition (ii) of Lemma 6.A and condition (iii) of Lemma 6.2. We compare the following two compact convex sets:

$$A_0 = \{(z, t) \in \mathbb{C} \times \mathbb{R} : z \in W(A), 0 \leq t \leq \max\{h : (z, h) \in W(A, A^*A)\}\}$$

and

$$B_0 = \{(z, t) \in \mathbb{C} \times \mathbb{R} : z \in W(B), 0 \leq t \leq \max\{h : (z, h) \in W(B, B^*B)\}\}.$$

For every $0 \leq \theta \leq 2\pi$, we consider the projection $\Pi = \Pi_\theta$ given by

$$(z, t) = (\Re(z), \Im(z), t) \rightarrow \Re(e^{i\theta}z) + it = (\cos \theta \Re(z) - \sin \theta \Im(z)) + it.$$

Then (ii) of Lemma 6.A holds if and only if the condition $B_0 \subset A_0$ holds, and also this condition is equivalent to

$$(6.2) \quad \Pi_\theta(B_0) \subset \Pi_\theta(A_0)$$

for every $0 \leq \theta \leq 2\pi$, where the compact convex sets $\Pi_\theta(A_0)$ and $\Pi_\theta(B_0)$ are characterized by

$$\Pi_\theta(A_0) = \text{conv}(W(\Re(e^{i\theta}A)), W(\Re(e^{i\theta}A) + iA^*A))$$

and

$$\Pi_\theta(B_0) = \text{conv}(W(\Re(e^{i\theta}B)), W(\Re(e^{i\theta}B) + iB^*B)).$$

Each of these sets contains its projection onto the real line. These sets are contained in the closed upper half plane $\Im(z) \geq 0$. Thus, for each $0 \leq \theta \leq 2\pi$, the inclusion relation (6.2) is equivalent to the inequality

$$(6.3) \quad \begin{aligned} & \max\{\Im(z) + k \Re(z) : z \in W(\Re(e^{i\theta}B) + iB^*B)\} \\ & \leq \max\{\Im(z) + k \Re(z) : z \in W(\Re(e^{i\theta}A) + iA^*A)\} \end{aligned}$$

for every $k \in \mathbb{R}$ (cf. [15, page 81, Theorem A]). By basic properties of the numerical range, we have

$$\begin{aligned} & \max\{\Im(z) + k \Re(z) : z \in W(\Re(e^{i\theta}A) + iA^*A)\} \\ & = \max W(A^*A + k \Re(e^{i\theta}A)) \\ & = \lambda_1(A^*A + k \Re(e^{i\theta}A)) \end{aligned}$$

and

$$\begin{aligned} & \max\{\Im(z) + k \Re(z) : z \in W(\Re(e^{i\theta}B) + iB^*B)\} \\ & = \lambda_1(B^*B + k \Re(e^{i\theta}B)) \end{aligned}$$

(cf. [8, page 9-11]), so that (6.3) is equivalent to

$$\lambda_1(B^*B + k \Re(e^{i\theta}B)) \leq \lambda_1(A^*A + k \Re(e^{i\theta}A))$$

for every $0 \leq \theta \leq 2\pi$ and $k \in \mathbf{R}$. By replacing θ by $\theta + \pi$, we may restrict the range of k as $k \geq 0$. Thus the condition (ii) of Lemma 6.1 and the condition (iii) of Lemma 6.2 are equivalent. $\square$

Proof of Theorem 6.1. Since the equation

$$W_{cq}(S) = c W_q(S)$$

holds for any complex numbers c, q with $|c| = 1$ and $|q| \leq 1$, it is sufficient to prove (6.1) for $0 \leq q \leq 1$. Therefore we have only to prove the inequality

$$(6.4) \quad \lambda_1 \left(f(\tilde{T})^* f(\tilde{T}) + k \Re \{ e^{i\theta} f(\tilde{T}) \} \right) \leq \lambda_1 \left(f(T)^* f(T) + k \Re \{ e^{i\theta} f(T) \} \right)$$

for every $0 \leq \theta \leq 2\pi$ and $k \geq 0$ by Lemma 6.2.

To prove the inequality (6.4), we shall prove that the following inequality holds for a positive matrix A and an arbitrary X :

$$(6.5) \quad \begin{aligned} & \lambda_1 \left(f(A^{\frac{1}{2}} X A^{\frac{1}{2}})^* f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) + k \Re \{ e^{i\theta} f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) \} \right) \\ & \leq \frac{1}{2} \lambda_1 \left(f(XA)^* f(XA) + k \Re \{ e^{i\theta} f(XA) \} \right) \\ & \quad + \frac{1}{2} \lambda_1 \left(f(AX)^* f(AX) + k \Re \{ e^{i\theta} f(AX) \} \right). \end{aligned}$$

By perturbing A to $A + \varepsilon I$ for small $\varepsilon > 0$, it suffices to prove (6.5) for a positive invertible matrix A .

using Lemma 5.2, the Cauchy-Schwarz inequality and the Arithmetic-Geo inequality, we have

$$\begin{aligned}
& \lambda_1 \left(f(A^{\frac{1}{2}} X A^{\frac{1}{2}})^* f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) + k \Re \{ e^{i\theta} f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) \} \right) \\
&= \lambda_1 \left(A^{\frac{1}{2}} S^* A S A^{\frac{1}{2}} + k \Re \{ A^{\frac{1}{2}} S A^{\frac{1}{2}} \} \right) \quad \text{by Lemma 5.2} \\
&= \lambda_1 \left(A^{\frac{1}{2}} S^* A S A^{\frac{1}{2}} + k \{ A^{\frac{1}{2}} \Re(S) A^{\frac{1}{2}} \} \right) \\
&= \Re \lambda_1 \left(S^* A S A + k \Re(S) A \right) \quad \text{by Lemma 4.A} \\
&\leq \lambda_1 \left(\Re \{ S^* A S A + k \Re(S) A \} \right) \quad \text{by Lemma 5.B} \\
&= \max_{x \in \mathbb{C}^n, \|x\|=1} \left[\Re \langle S^* A S x, x \rangle + \frac{k}{2} \langle \{ \Re(SA) + \Re(AS) \} x, x \rangle \right] \quad \text{by (4.3)} \\
&= \max_{x \in \mathbb{C}^n, \|x\|=1} \left[\Re \langle S A x, A S x \rangle + \frac{k}{2} \langle \{ \Re(SA) + \Re(AS) \} x, x \rangle \right] \\
&\leq \max_{x \in \mathbb{C}^n, \|x\|=1} \left[\langle A S^* S A x, x \rangle^{\frac{1}{2}} \langle S^* A^2 S x, x \rangle^{\frac{1}{2}} + \frac{k}{2} \langle \{ \Re(SA) + \Re(AS) \} x, x \rangle \right] \\
&\leq \max_{x \in \mathbb{C}^n, \|x\|=1} \left[\frac{1}{2} \langle f(XA)^* f(XA) x, x \rangle + \frac{1}{2} \langle f(AX)^* f(AX) x, x \rangle \right. \\
&\quad \left. + \frac{k}{2} \langle \Re \{ e^{i\theta} f(XA) \} x, x \rangle + \frac{k}{2} \langle \Re \{ e^{i\theta} f(AX) \} x, x \rangle \right] \\
&\leq \frac{1}{2} \max_{x \in \mathbb{C}^n, \|x\|=1} \left[\langle f(XA)^* f(XA) x, x \rangle + k \langle \Re \{ e^{i\theta} f(XA) \} x, x \rangle \right] \\
&\quad + \frac{1}{2} \max_{x \in \mathbb{C}^n, \|x\|=1} \left[\langle f(AX)^* f(AX) x, x \rangle + k \langle \Re \{ e^{i\theta} f(AX) \} x, x \rangle \right] \\
&= \frac{1}{2} \lambda_1 \left(f(XA)^* f(XA) + k \Re \{ e^{i\theta} f(XA) \} \right) \\
&\quad + \frac{1}{2} \lambda_1 \left(f(AX)^* f(AX) + k \Re \{ e^{i\theta} f(AX) \} \right).
\end{aligned}$$

shall use a polar decomposition $T = U|T|$ where U is a unitary matrix and $A = |T|$, $X = U$ in (6.5). Since $(|T|U)^n = U^* T^n U$ for every integer $n \geq$

have the equation $f(|T|U) = U^*f(T)U$ for any polynomial f , so that

$$\begin{aligned}
& \lambda_1 \left(f(\tilde{T})^* f(\tilde{T}) + k \Re \{ e^{i\theta} f(\tilde{T}) \} \right) \\
& \leq \frac{1}{2} \lambda_1 \left(f(T)^* f(T) + k \Re \{ e^{i\theta} f(T) \} \right) \\
& \quad + \frac{1}{2} \lambda_1 \left(f(|T|U)^* f(|T|U) + k \Re \{ e^{i\theta} f(|T|U) \} \right) \\
& = \frac{1}{2} \lambda_1 \left(f(T)^* f(T) + k \Re \{ e^{i\theta} f(T) \} \right) \\
& \quad + \frac{1}{2} \lambda_1 \left(U^* f(T)^* f(T) U + k U^* \Re \{ e^{i\theta} f(T) \} U \right) \\
& = \lambda_1 \left(f(T)^* f(T) + k \Re \{ e^{i\theta} f(T) \} \right).
\end{aligned}$$

Hence the proof of Theorem 6.1 is complete. $\square$

In particular, by putting $q = 1$ in Theorem 6.1, we have the following relation.

Corollary 6.3. *If T is an $n \times n$ matrix. Then*

$$W(f(\tilde{T})) \subset W(f(T)) \quad \text{holds for all polynomial } f.$$

Moreover, we obtain the inequalities on the numerical radius and the spectral norm.

Corollary 6.4. *Let T be an $n \times n$ matrices. Then the following assertions hold:*

- (i) $w(f(\tilde{T})) \leq w(f(T))$ for all polynomials f .
- (ii) $\|f(\tilde{T})\| \leq \|f(T)\|$ for all polynomials f ,

where $\|\cdot\|$ means the spectral norm.

Corollary 6.4 is easily obtained by the following Proposition 6.5.

Proposition 6.5. *Let A and B be $n \times n$ matrices. Then the following assertions are mutually equivalent:*

- (i) $W(f(A)) \subset W(f(B))$ for all polynomials f .
- (ii) $w(f(A)) \leq w(f(B))$ for all polynomials f .
- (iii) $\|f(A)\| \leq \|f(B)\|$ for all polynomials f ,

where $\|\cdot\|$ means the spectral norm.

Proof. Proofs of (ii) $\implies$ (i) and (iii) $\implies$ (i) are obvious by

$$W(A) = \bigcap_{\mu \in \mathbb{C}} \{z : |z - \mu| \leq w(A - \mu)\}$$

and

$$W(A) = \bigcap_{\mu \in \mathbb{C}} \{z : |z - \mu| \leq \|A - \mu\|\}.$$

Proof of (i) $\implies$ (ii) is also obvious. Hence we shall show (i) $\implies$ (iii). In fact, we have only to show that

$$W(f(A)) \subset W(f(B)) \quad \text{for all polynomials } f \implies \|A\| \leq \|B\|.$$

So we shall show

$$\|B\| < 1 \implies \|A\| \leq 1.$$

Let $r(A)$ be the spectral radius of A . Since

$$r(A) \leq w(A) \leq w(B) \leq \|B\| < 1$$

hold, the inverses $(1+A)^{-1}$ and $(1+B)^{-1}$ exist, and we can consider the Cayley transform of A and B as follows:

$$\Phi(A) \equiv (1-A)(1+A)^{-1}, \quad \Phi(B) \equiv (1-B)(1+B)^{-1}.$$

On the other hand, setting

$$g_n(z) \equiv 1 + 2 \sum_{k=1}^n (-1)^k z^k,$$

we have

$$\Phi(A) = \lim_{n \rightarrow \infty} g_n(A), \quad \Phi(B) = \lim_{n \rightarrow \infty} g_n(B)$$

since

$$\frac{1-z}{1+z} = 1 + 2 \sum_{k=1}^{\infty} (-1)^k z^k$$

holds. By the assumption, we have

$$W(g_n(A)) \subset W(g_n(B)) \quad (n = 1, 2, \dots),$$

then we obtain

$$W(\Phi(A)) \subset W(\Phi(B)).$$

On the other hand, since B is a contraction, we have $\Re(\Phi(B)) \geq 0$, that is, $W(\Phi(B))$ is included in the right-half plane. Then $W(\Phi(A))$ is also included in the right-half plane, that is, $\Re(\Phi(A)) \geq 0$ holds. Therefore, $1 + \Phi(A)$ is invertible, and $A = \Phi(\Phi(A))$ is a contraction, so that the proof is complete. $\square$

Proof of Corollary 6.4. Put $A = \tilde{T}$ and $B = T$ in Proposition 6.5, and we have Corollary 6.4 by Corollary 6.3. $\square$

Lastly, we summarize Theorems 5.1 and 6.1 as follows:

Theorem 6.6. Suppose that T and C are $n \times n$ complex matrices and f is a complex polynomial. If C is a Hermitian matrix or a rank-one matrix, then the following inclusion relation holds:

$$W_C(f(\tilde{T})) \subset W_C(f(T)).$$

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