

THE X-METHOD FOR KLT SURFACES IN POSITIVE CHARACTERISTIC

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ABSTRACT. In this paper, we establish a weak version of the Kodaira vanishing theorem for surfaces in positive characteristic. As an application, we obtain some fundamental theorems in the minimal model theory for klt surfaces.

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0. INTRODUCTION

The X-method is a method to prove some fundamental theorems in the minimal model theory of characteristic zero. For example, in characteristic zero, we can show the basepoint free theorem by using the X-method, see for example [KMM, Chapter 3] and [Kollár-Mori, Chapter 3]. The X-method mainly depends on two tools: resolution of singularities and the Kawamata–Viehweg vanishing theorem, which is a generalization of the Kodaira vanishing theorem. In positive characteristic, we can use resolution of singularities in the case where the dimension of the variety is two or three (cf. [CP]). But, in positive characteristic, there exist counter-examples to the Kodaira vanishing theorem even in the case where the dimension of the variety is two (cf.

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[Raynaud]). Thus, we consider the following question. Can we establish a vanishing theorem in positive characteristic which is sufficient for the X-method? If the dimension of the variety is two, then we have an affirmative answer.

Theorem 0.1 (weak Kodaira vanishing theorem). *Let X be a smooth projective surface over an algebraically closed field of positive characteristic. Let A be an ample Cartier divisor. Let N be a nef Cartier divisor which is not numerically trivial. If $i > 0$ and $m \gg 0$, then*

$$H^i(X, K_X + A + mN) = 0.$$

Moreover, by a standard argument, we can generalize this theorem to a vanishing theorem of Kawamata–Viehweg type or Nadel type.

Theorem 0.2 (weak Kawamata–Viehweg vanishing theorem). *Let X be a smooth projective surface over an algebraically closed field of positive characteristic. Let A be an ample $\mathbb{R}$ -divisor whose fractional part is simple normal crossing. Let N be a nef Cartier divisor which is not numerically trivial. If $i > 0$ and $m \gg 0$, then*

$$H^i(X, K_X + \lceil A \rceil + mN) = 0.$$

Theorem 0.3 (weak Nadel vanishing theorem). *Let X be a normal projective surface over an algebraically closed field of positive characteristic. Let Δ be an $\mathbb{R}$ -divisor such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let N be a nef Cartier divisor which is not numerically trivial. Let L be a Cartier divisor such that $L - (K_X + \Delta)$ is nef and big. If $i > 0$ and $m \gg 0$, then*

$$H^i(X, \mathcal{O}_X(L + mN) \otimes \mathcal{I}_\Delta) = 0$$

where $\mathcal{I}_\Delta$ is the multiplier ideal of the pair (X, Δ) .

Using Theorem 0.3, we obtain the following basepoint free theorem (cf. [Kollár–Mori, Theorem 3.3]).

Theorem 0.4 (Basepoint free theorem). *Let X be a projective normal surface over an algebraically closed field of positive characteristic. Let Δ be a $\mathbb{Q}$ -divisor such that $\lfloor \Delta \rfloor = 0$ and $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. Let D be a nef Cartier divisor which is not numerically trivial. Assume $aD - (K_X + \Delta)$ is nef and big for some $a \in \mathbb{Z}_{>0}$. Then there exists a positive integer b_0 such that, if $b \geq b_0$, then $|bD|$ is basepoint free.*

Thus, if we can generalize the above vanishing theorems to the case of threefolds, then we can prove the above basepoint free theorem for threefolds. Unfortunately, however, there exists a counter-example to the above weak Kodaira vanishing theorem in the case where the dimension is three. We construct such counter-examples in Section 5.

By the same argument as the proof of the above weak Kodaira vanishing theorem (Theorem 0.1), we can also establish the following vanishing theorem.

Theorem 0.5. *Let $\pi : X \rightarrow S$ be a morphism over an algebraically closed field of positive characteristic from a smooth projective variety X to a projective variety S . Let A be a π -ample $\mathbb{R}$ -divisor on X whose fractional part is simple normal crossing. Set $f_{\max} := \max_{s \in S} \dim \pi^{-1}(s)$. If $i \geq f_{\max}$, then*

$$R^i \pi_* \mathcal{O}_X(K_X + \lceil A \rceil) = 0.$$

0.6 (Overview of contents). In Section 1, we summarize the notations. In Section 2, we prove Theorem 0.1, Theorem 0.2 and Theorem 0.3 by using the Frobenius maps and the Fujita vanishing theorem. In Section 3, we apply these vanishing results to the minimal model theory. In Section 4, we show Theorem 0.5 and other vanishing theorems. In Section 5, we construct counter-examples to the above vanishing results in the case where the dimension is three.

0.7 (Overview of related literature). We summarize the literature related to this paper with respect to the vanishing theorems and the basepoint free theorem.

(Vanishing theorem) Let us summarize some known results on the Kodaira vanishing theorem and its generalizations.

In characteristic zero, Kodaira establishes the Kodaira vanishing theorem. [Kawamata1] and [Viehweg] generalize this result. For detailed treatments, see [KMM, Chapter 1], [Kollár-Mori, Section 2.4, 2.5] and [Lazarsfeld, Part Three].

In positive characteristic, [Raynaud] shows that there exists a counter-example to the Kodaira vanishing theorem. [Ekedahl] and [Mukai] deeply investigate the counter-examples to the Kodaira vanishing theorem. On the other hand, there are some positive results on the Kodaira vanishing theorem in positive characteristic. For example, [Xie] shows that the Kawamata–Viehweg vanishing theorem holds for rational surfaces. In [KK], Kollár and Kovács prove the relative Kawamata–Viehweg vanishing theorem for birational morphisms between surfaces. The proof is a calculation of the cohomology for curves. We also establish this result in this paper. (See Corollary 2.7.) Our proof depends on the Frobenius maps.

(Basepoint free theorem) In characteristic zero, many people contributed to the basepoint free theorem (cf. [Benveniste] [Kawamata2] [Kawamata3] [Kawamata4] [Reid] [Shokurov]).

In positive characteristic, [Keel] shows the basepoint free theorem for $\mathbb{Q}$ -factorial threefolds with non-negative Kodaira dimension, defined

over the algebraic closure of a finite field. In this paper, we show the basepoint free theorem for klt surfaces. To prove this, we establish a weak version of the Kodaira vanishing theorem (Theorem 0.1).

Here, let us compare Theorem 0.4 with the following basepoint free theorem obtained in [T].

Theorem 0.8 (Theorem 0.3 of [T]). *Let X be a projective normal $\mathbb{Q}$ -factorial surface over an algebraically closed field of positive characteristic. Let Δ be a $\mathbb{Q}$ -divisor such that $\lfloor \Delta \rfloor = 0$. Let D be a nef Cartier divisor. Assume $aD - (K_X + \Delta)$ is nef and big for some $a \in \mathbb{Z}_{>0}$. Then D is semi-ample.*

Theorem 0.8 does not need the assumption that D is not numerically trivial. On the other hand, Theorem 0.4 does not need the $\mathbb{Q}$ -factoriality and its claim is stronger than the semi-ameness.

The proof of Theorem 0.4 and the one of Theorem 0.8 are essentially different. The proof of Theorem 0.4 depends on the above vanishing theorem (Theorem 0.3). On the other hand, the proof of Theorem 0.8 uses the minimal model theory for $\mathbb{Q}$ -factorial surfaces. In characteristic zero, [Fujino] establishes the minimal model theory for $\mathbb{Q}$ -factorial surfaces. In [T], the author establishes the minimal model theory for $\mathbb{Q}$ -factorial surfaces in positive characteristic. The arguments in [T] heavily depend on [Keel, Theorem 0.2], which holds only in positive characteristic (cf. [Keel, Section 3]). Keel's proof depends on the Frobenius maps and the theory of the algebraic spaces. For alternative proofs of [Keel, Theorem 0.2], see [CMM] and [FT]. [FT] only considers the case of surfaces.

1. NOTATIONS

We will freely use the notation and terminology of [Kollár-Mori].

Our notation will not distinguish between invertible sheaves and Cartier divisors. For example, we will write $L + M$ for invertible sheaves L and M .

For a coherent sheaf F and a Cartier divisor L , we define $F(L) := F \otimes \mathcal{O}_X(L)$.

Throughout this paper, we work over an algebraically closed field k of positive characteristic and let $\text{char } k =: p > 0$.

In this paper, a *variety* means an integral scheme which is separated and of finite type over k . A *curve* or a *surface* means a variety whose dimension is one or two, respectively.

Let X be a projective normal variety and let L be a nef $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor. We define the *numerical dimension* $\nu(X, L) \in \{0, 1, \dots, \dim X\}$

as follows. If L is numerically trivial, then we set $\nu(X, L) = 0$. If L is not numerically trivial, then we define $\nu(X, L)$ by

$$\nu(X, L) := \max\{e \in \mathbb{Z}_{\geq 1} \mid L^e \text{ is not numerically trivial}\}.$$

Note that L is not numerically trivial if and only if $\nu(X, L) \geq 1$.

2. VANISHING THEOREMS FOR SURFACES

In this section, we establish some vanishing theorems for surfaces. Proposition 2.4 is the key in this section. We prove Proposition 2.4 by using Proposition 2.3, the Fujita vanishing theorem and the Frobenius maps.

Thus, let us recall the Fujita vanishing theorem which is a generalization of the Serre vanishing theorem.

Fact 2.1 (Fujita vanishing theorem). *Let X be a smooth projective variety. Let F be a coherent sheaf and let A be an ample $\mathbb{Z}$ -divisor. Then there exists a positive integer $m(F, A)$ such that*

$$H^i(X, F(mA + N)) = 0$$

for every $i > 0$, every integer $m \geq m(F, A)$ and every nef $\mathbb{Z}$ -divisor N .

Proof. See [Fujita1, Theorem (1)] or [Fujita2, Section 5]. $\square$

Since we would like to work over $\mathbb{R}$ -divisors, let us generalize the Fujita vanishing theorem to real coefficients.

Theorem 2.2 (Fujita vanishing theorem for $\mathbb{R}$ -divisors). *Let X be a smooth projective variety. Let F be a coherent sheaf and let A be an ample $\mathbb{R}$ -divisor. Then there exists a positive real number $r(F, A)$ such that*

$$H^i(X, F(rA + N)) = 0$$

for every $i > 0$, every real number $r \geq r(F, A)$ and every nef $\mathbb{R}$ -divisor N such that $rA + N$ is a $\mathbb{Z}$ -divisor.

Proof. First, we prove that we may assume that A is a $\mathbb{Q}$ -divisor. Consider the equation:

$$A = \frac{1}{2}A + \frac{1}{2}A = A' + A''$$

where A' and A'' are ample and A' is a $\mathbb{Q}$ -divisor. Note that we can find A' and A'' by changing the coefficients of $(1/2)A$ a little. Thus we obtain the desired reduction by letting $rA + N = rA' + (N + rA'')$.

Thus we may assume that A is a $\mathbb{Q}$ -divisor. Take a positive integer m_1 such that m_1A is a $\mathbb{Z}$ -divisor. Then we obtain the assertion by Fact 2.1 and the equation $rA + N = mm_1A + ((r - mm_1)A + N)$. $\square$

Let us consider the following Serre–Fujita type vanishing theorem for surfaces.

Proposition 2.3. *Let X be a smooth projective surface and let F be a coherent sheaf on X . Let N be a nef $\mathbb{R}$ -divisor with $\nu(X, N) \geq 1$. Then there exists a positive real number $r(F, N)$ such that*

$$H^2(X, F(rN + N')) = 0$$

for every positive real number $r \geq r(F, N)$ and for every nef $\mathbb{R}$ -divisor N' such that $rN + N'$ is a $\mathbb{Z}$ -divisor.

Proof. Since X is projective, we obtain the following exact sequence:

$$\mathcal{O}_X^{\oplus s} \rightarrow F \otimes \mathcal{O}_X(A) \rightarrow 0$$

where A is a sufficiently ample $\mathbb{Z}$ -divisor. Tensoring by $\mathcal{O}_X(-A + rN + N')$, we have

$$\mathcal{O}_X(-A + rN + N')^{\oplus s} \rightarrow F(rN + N') \rightarrow 0.$$

Thus we may assume that $F =: L$ is an invertible sheaf. By Serre duality, we have

$$h^2(X, L + rN + N') = h^0(X, K_X - L - rN - N').$$

Take an ample $\mathbb{Z}$ -divisor A' . By $\nu(X, N) \geq 1$, we see $N \cdot A' > 0$. Then, for every sufficiently large number r , we obtain

$$(K_X - L - rN - N') \cdot A' < 0.$$

This implies $H^0(X, K_X - L - rN - N') = 0$. □

Now, we prove the following weak Kodaira vanishing theorem, by using the above vanishing result for H^2 .

Proposition 2.4 (weak Kodaira vanishing theorem). *Let X be a smooth projective surface and let A be an ample $\mathbb{R}$ -divisor. Let N be a nef $\mathbb{R}$ -divisor with $\nu(X, N) \geq 1$. Then there exists a positive real number $r(A, N)$ such that*

$$H^1(X, K_X + A + rN + N') = 0$$

for every positive real number $r \geq r(A, N)$ and for every nef $\mathbb{R}$ -divisor N' such that $A + rN + N'$ is a $\mathbb{Z}$ -divisor.

Proof. Consider the following exact sequence

$$0 \rightarrow \mathcal{B} \rightarrow F_*\omega_X \rightarrow \omega_X \rightarrow 0$$

where $F : X \rightarrow X$ is the Frobenius map, that is the p -th power map, and $\mathcal{B}$ is the kernel of $F_*\omega_X \rightarrow \omega_X$. Considering the composition of the pushforwards by $F, F^2, \dots, F^{e-1}$, we obtain

$$0 \rightarrow \mathcal{B}_e \rightarrow F_*^e \omega_X \rightarrow \omega_X \rightarrow 0$$

for some coherent sheaf $\mathcal{B}_e$.

Tensoring by $\mathcal{O}_X(A + rN + N')$, we have

$$0 \rightarrow \mathcal{B}_e(A + rN + N') \rightarrow F_*^e \omega_X(A + rN + N') \rightarrow \omega_X(A + rN + N') \rightarrow 0.$$

We can find a large integer $e > 0$ such that

$$H^1(X, F_*^e \omega_X(A + rN + N')) = H^1(X, \omega_X(p^e A + p^e rN + p^e N')) = 0.$$

Note that, by the Fujita vanishing theorem, we can take e independent of r and N' . By Proposition 2.3, we have

$$H^2(X, \mathcal{B}_e(A + rN + N')) = 0$$

for every large r . These imply

$$H^1(X, \omega_X(A + rN + N')) = 0.$$

□

In order to generalize the above weak Kodaira vanishing theorem to a vanishing theorem of Kawamata–Viehweg type, we recall the following covering lemma.

Proposition 2.5. *Let X be an n -dimensional smooth variety. Let D be a $\mathbb{Q}$ -divisor such that the support of the fractional part $\{D\}$ is simple normal crossing. Moreover suppose that, for the prime decomposition $\{D\} = \sum_{i \in I} \frac{b^{(i)}}{a^{(i)}} D^{(i)}$, no integers $a^{(i)}$ are divisible by p . Then there exists a finite surjective morphism $\gamma : Y \rightarrow X$ from a smooth variety Y with the following properties.*

- (1) *The field extension $K(Y)/K(X)$ is a Galois extension.*
- (2) *$\gamma^* D$ is a $\mathbb{Z}$ -divisor.*
- (3) *$\mathcal{O}_X(K_X + \lceil D \rceil) \simeq (\gamma_* \mathcal{O}_Y(K_Y + \gamma^* D))^G$, where G is the Galois group of $K(Y)/K(X)$.*
- (4) *If D' is a $\mathbb{Q}$ -divisor such that $\{D'\} = \{D\}$, then $\gamma^* D'$ is a $\mathbb{Z}$ -divisor and $\mathcal{O}_X(K_X + \lceil D' \rceil) \simeq (\gamma_* \mathcal{O}_Y(K_Y + \gamma^* D'))^G$.*

Proof. See [KMM, Theorem 1-1-1].

□

Now, we can generalize the above weak Kodaira vanishing (Proposition 2.4) to the following weak Kawamata–Viehweg vanishing.

Theorem 2.6 (weak Kawamata–Viehweg vanishing theorem). *Let X be a smooth projective surface. Let B be a nef and big $\mathbb{R}$ -divisor whose fractional part is simple normal crossing. Let N be a nef $\mathbb{R}$ -divisor with $\nu(X, N) \geq 1$. Then there exists a positive real number $r(B, N)$ such that*

$$H^i(X, K_X + \lceil B \rceil + rN + N') = 0$$

for every $i > 0$, every positive real number $r \geq r(B, N)$ and every nef $\mathbb{R}$ -divisor N' such that $rN + N'$ is a $\mathbb{Z}$ -divisor.

Proof. If $i = 2$, then the assertion follows from Proposition 2.3. Thus we assume $i = 1$.

Step 1. In this step, we assume that $B =: A$ is ample and we prove the assertion.

Since A is ample, we may assume that A is an ample $\mathbb{Q}$ -divisor and that no denominators of the coefficients of its fractional part are divisible by p . Note that the fractional part of $A + rN + N'$ is equal to the fractional part of A for an arbitrary real number r and for a nef $\mathbb{R}$ -divisor N' such that $rN + N'$ is a $\mathbb{Z}$ -divisor. Thus we can apply Proposition 2.5 for $D := A + rN + N'$ and we obtain a finite cover $\gamma : Y \rightarrow X$ with the properties in the proposition. Note that the map γ is independent of r and N' . Therefore we have

$$\begin{aligned} & H^1(X, K_X + \lceil A \rceil + rN + N') \\ &= H^1(X, K_X + \lceil (A + rN + N') \rceil) \\ &= H^1(X, \gamma_* \mathcal{O}_Y(K_Y + \gamma^*(A + rN + N')))^G \\ &= H^1(Y, K_Y + \gamma^*A + r\gamma^*N + \gamma^*N')^G \\ &= 0. \end{aligned}$$

The last equality follows from Proposition 2.4 when $r \gg 0$.

Step 2. In this step, we prove the assertion.

Let $f : Y \rightarrow X$ be a birational morphism from a smooth projective surface with the following properties: there exists an effective $\mathbb{Z}$ -divisor E such that $f^*B - \epsilon E$ is ample for $0 < \epsilon \ll 1$ and the fractional part $\{f^*B - \epsilon E\}$ is simple normal crossing. Since f has a decomposition into blow-ups of points, we consider the blow-up $g : Z \rightarrow X$ of one point P . Let C be the exceptional curve. Set $\Delta_X := \lceil B \rceil - B$ and $M := \Delta_X + B + rN + N'$. We will prove that

$$H^1(X, K_X + \Delta_X + B + rN + N') = 0.$$

Consider the exact sequences induced from the corresponding Leray spectral sequences:

$$0 \rightarrow H^1(X, K_X + M) \rightarrow H^1(Z, K_Z - C + g^*M)$$

$$0 \rightarrow H^1(X, K_X + M) \rightarrow H^1(Z, K_Z + g^*M).$$

Note that the second exact sequence is obtained by Serre duality. If $\text{mult}_P \Delta_X \geq 1$, then we set $\Delta_Z := g^*(\Delta_X) - C$ and we can reduce the problem on X to the problem on Z by the first exact sequence. If $\text{mult}_P \Delta_X < 1$, then we set $\Delta_Z := g^*(\Delta_X)$ and we can also reduce the problem on X to the problem on Z by the second exact sequence. Thus it is sufficient to prove that

$$H^1(Y, K_Y + \Delta_Y + f^*(B + rN + N')) = 0.$$

Note that $\lfloor \Delta_Y \rfloor = 0$. We see

$$\begin{aligned} & H^1(Y, K_Y + \Delta_Y + f^*(B + rN + N')) \\ &= H^1(Y, K_Y + \lceil f^*B \rceil + f^*(rN + N')) \\ &= H^1(Y, K_Y + \lceil f^*B - \epsilon E \rceil + f^*(rN + N')) \\ &= 0. \end{aligned}$$

The first equality follows from $\lfloor \Delta_Y \rfloor = 0$. The third equality follows from Step 1 when $r \gg 0$. □

By this theorem, we obtain the relative Kawamata–Viehweg vanishing theorem for non-trivial morphisms.

Corollary 2.7. *Let $\pi : X \rightarrow S$ be a proper morphism from a smooth surface X to a variety S . Let B be a π -nef and π -big $\mathbb{R}$ -divisor whose fractional part is simple normal crossing. Assume $\dim \pi(X) \geq 1$. Then*

$$R^i \pi_* \mathcal{O}_X(K_X + \lceil B \rceil) = 0$$

for every $i > 0$.

Proof. By the same argument as Step 2 of Theorem 2.6, we may assume that $B =: A$ is π -ample. We may assume that S is affine. Moreover, by taking suitable compactifications of S and $X \rightarrow S$, we may assume that X and S are projective. (See, for example, the proof of [KMM, Theorem 1-2-3].) Let A_S be an ample invertible sheaf on S and set $N := \pi^* A_S$. Then $\nu(X, N) \geq 1$. Therefore the assertion follows from Theorem 2.6 and the following Leray spectral sequence

$$\begin{aligned} E_2^{i,j} &:= H^i(S, R^j \pi_* \mathcal{O}_X(K_X + \lceil B \rceil) \otimes A_S^{\otimes m}) \\ &\Rightarrow H^{i+j}(X, K_X + \lceil B \rceil + m\pi^* A_S) =: E^{i+j}. \end{aligned}$$

□

In order to generalize the above weak Kawamata–Viehweg vanishing theorem to a vanishing theorem of Nadel type, we recall the definition of the multiplier ideals.

Definition 2.8. Let X be a normal surface and let Δ be an $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let $\mu : X' \rightarrow X$ be a log resolution of (X, Δ) . We define a *multiplier ideal sheaf* $\mathcal{J}_\Delta$ by

$$\mathcal{J}_\Delta := \mu_* \mathcal{O}_{X'}(K_{X'} - \lrcorner \mu^*(K_X + \Delta) \lrcorner).$$

Note that, in the case of surfaces, we can use the resolution of singularities in positive characteristic (cf. [Lipman2]). Thus, we can establish some fundamental properties (cf. [Lazarsfeld, Chapter 9]). For example, we see that $\mathcal{J}_\Delta$ is independent of log resolutions and that if $\Delta \geq 0$, then $\mathcal{J}_\Delta \subset \mathcal{O}_X$.

Now, we prove the weak Nadel vanishing theorem, which is the main theorem in this section.

Theorem 2.9 (weak Nadel vanishing theorem). *Let X be a projective normal surface and let Δ be an $\mathbb{R}$ -divisor such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let N be a nef $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor with $\nu(X, N) \geq 1$. Let L be a Cartier divisor such that $L - (K_X + \Delta)$ is nef and big. Then there exists a positive real number $r(\Delta, L, N)$ such that*

$$H^i(X, \mathcal{O}_X(L + rN + N') \otimes \mathcal{J}_\Delta) = 0$$

for every $i > 0$, every positive real number $r \geq r(\Delta, L, N)$ and every nef $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor N' such that $rN + N'$ is a Cartier divisor.

Proof. Let $\mu : X' \rightarrow X$ be a log resolution of (X, Δ) . Set

$$M := \mu^*(L + rN + N') + K_{X'} - \lrcorner \mu^*(K_X + \Delta) \lrcorner.$$

Consider the following Leray spectral sequence:

$$E_2^{i,j} := H^i(X, R^j \mu_* \mathcal{O}_{X'}(M)) \Rightarrow H^{i+j}(X', \mathcal{O}_{X'}(M)) =: E^{i+j}.$$

The assertion is equivalent to $E^{i,0} = 0$. We see

$$M = K_{X'} + \lrcorner \mu^*(L - (K_X + \Delta)) \lrcorner + r\mu^*N + \mu^*N'.$$

Thus, by Theorem 2.7, we have $E_2^{i,j} = 0$ for $j > 0$. This means $E_2^{i,0} = E^i$. Moreover, by Theorem 2.6, we see that $E^i = 0$ for $r \gg 0$. $\square$

Theorem 2.10. *Let $\pi : X \rightarrow S$ be a proper morphism from a normal surface X to a variety S . Let Δ be an $\mathbb{R}$ -divisor such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let L be a Cartier divisor such that $L - (K_X + \Delta)$ is π -nef and π -big. Assume $\dim \pi(X) \geq 1$. Then*

$$R^i \pi_*(\mathcal{O}_X(L) \otimes \mathcal{J}_\Delta) = 0$$

for every $i > 0$.

Proof. We may assume $i = 1$. Let $\mu : X' \rightarrow X$ be a log resolution of (X, Δ) . We have

$$0 \rightarrow R^1\pi_*(\mathcal{O}_X(L) \otimes \mathcal{J}_\Delta) \rightarrow R^1(\pi \circ \mu)_*(\mathcal{O}_{X'}(K_{X'} + {}^\Gamma\mu^*(L - (K_X + \Delta))^\Gamma))$$

by the exact sequence induced from the corresponding Grothendieck–Leray spectral sequence. The latter term vanishes by Corollary 2.7. $\square$

The following two results are vanishing theorems of Kawamata–Viehweg type for klt surfaces.

Theorem 2.11. *Let (X, Δ) be a projective klt surface where Δ is an effective $\mathbb{R}$ -divisor. Let N be a nef $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor with $\nu(X, N) \geq 1$. Let D be a $\mathbb{Q}$ -Cartier $\mathbb{Z}$ -divisor such that $D - (K_X + \Delta)$ is nef and big. Then there exists a positive real number $r(\Delta, D, N)$ such that*

$$H^i(X, \mathcal{O}_X(D + rN + N')) = 0$$

for every $i > 0$, every positive real number $r \geq r(\Delta, D, N)$ and every nef $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor N' such that $rN + N'$ is a Cartier divisor.

Proof. Let $\mu : X' \rightarrow X$ be a log resolution of (X, Δ) . Set

$$M := {}^\Gamma\mu^*(D + rN + N') + K_{X'} - \mu^*(K_X + \Delta)^\Gamma.$$

Consider the following Leray spectral sequence:

$$E_2^{i,j} := H^i(X, R^j\mu_*\mathcal{O}_{X'}(M)) \Rightarrow H^{i+j}(X', \mathcal{O}_{X'}(M)) =: E^{i+j}.$$

The assertion is equivalent to $E_2^{i,0} = 0$ because

$$\begin{aligned} & \mu_*\mathcal{O}_{X'}(M) \\ &= \mu_*\mathcal{O}_{X'}({}^\Gamma\mu^*(D + rN + N') + K_{X'} - \mu^*(K_X + \Delta)^\Gamma) \\ &= \mu_*\mathcal{O}_{X'}({}^\perp\mu^*(D + rN + N')^\perp + (\text{effective exceptional } \mathbb{Z}\text{-divisor})) \\ &\simeq \mathcal{O}_X(D + rN + N'). \end{aligned}$$

The above second equality holds because (X, Δ) is klt and D is a $\mathbb{Z}$ -divisor. We see

$$M = K_{X'} + {}^\Gamma\mu^*(D - (K_X + \Delta))^\Gamma + r\mu^*N + \mu^*N'.$$

Thus, by Theorem 2.7, we have $E_2^{i,j} = 0$ for $j > 0$. This means $E_2^{i,0} = E^i$. Moreover, by Theorem 2.6, we see that $E^i = 0$ for $r \gg 0$. $\square$

Theorem 2.12. *Let $\pi : X \rightarrow S$ be a proper morphism from a normal surface X to a variety S . Assume that (X, Δ) is a klt surface where Δ is an effective $\mathbb{R}$ -divisor. Let D be a $\mathbb{Q}$ -Cartier $\mathbb{Z}$ -divisor such that $D - (K_X + \Delta)$ is π -nef and π -big. Assume $\dim \pi(X) \geq 1$. Then*

$$R^i\pi_*(\mathcal{O}_X(D)) = 0$$

for every $i > 0$.

Proof. We may assume $i = 1$. Let $\mu : X' \rightarrow X$ be a log resolution of (X, Δ) . We have

$$0 \rightarrow R^1\pi_*(\mathcal{O}_X(D)) \rightarrow R^1(\pi \circ \mu)_*(\mathcal{O}_{X'}(K_{X'} + {}^\top\mu^*(D - (K_X + \Delta))^\top))$$

by the exact sequence induced from the corresponding Grothendieck–Leray spectral sequence and the proof of Theorem 2.11. The latter term vanishes by Corollary 2.7. $\square$

3. X-METHOD FOR SURFACES

In this section, we apply the vanishing theorems which are established in Section 2 to the minimal model theory. First, we see the non-vanishing theorem.

Theorem 3.1 (Non-vanishing theorem). *Let $(X, -G)$ be a projective klt surface where G is a $\mathbb{Q}$ -divisor. Note that $-G$ may not be effective. Let D be a nef Cartier divisor D such that $\nu(X, D) \geq 1$ and $aD - (K_X - G)$ is nef and big for some $a \in \mathbb{Z}_{>0}$.*

Then there exists a positive integer m_0 such that

$$H^0(X, mD + {}^\top G^\top) \neq 0.$$

for $m \geq m_0$.

Proof. Since the proof is almost identical to that of [Kollár-Mori, Theorem 3.4], we will only discuss the necessary changes to their argument. The numbers of “Step” are the same as [Kollár-Mori, Theorem 3.4].

The argument of Step 0 works without any changes. Because we assume $\nu(X, D) \geq 1$, there is nothing to prove in Step 1. The arguments of Step 2, Step 3, Step 4 and Step 5 work without any changes.

In Step 6, we modify the argument a little. It is sufficient to prove

$$H^1(Y, K_Y + {}^\top N(b, c)^\top) = 0$$

where Y and $N(b, c)$ are the notations in [Kollár-Mori, Theorem 3.4]. Note that, by Step 4 and Step 5, we may assume that b is sufficiently large. Then, by Theorem 2.6 and the definition of $N(b, c)$, we obtain the above vanishing result for every $b \gg 0$. $\square$

Second, we prove the following basepoint free theorem.

Theorem 3.2 (Basepoint free theorem). *Let X be a projective normal surface and let Δ be a $\mathbb{Q}$ -divisor such that $\lfloor \Delta \rfloor = 0$ and $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. Let D be a nef Cartier divisor such that $\nu(X, D) \geq 1$. Assume $aD - (K_X + \Delta)$ is nef and big for some $a \in \mathbb{Z}_{>0}$. Then there exists a positive integer b_0 such that if $b \geq b_0$, then $|bD|$ is basepoint free.*

Proof. If the pair (X, Δ) is klt, then the proof of [Kollár-Mori, Theorem 3.3] works by the same modification as Theorem 3.1. Thus we assume that the pair (X, Δ) is not klt. Consider the following exact sequence:

$$0 \rightarrow \mathcal{J}_\Delta(bD) \rightarrow \mathcal{O}_X(bD) \rightarrow \mathcal{O}_{M_\Delta}(bD) \rightarrow 0$$

where M_Δ is the closed subscheme corresponding to $\mathcal{J}_\Delta$. Note that $\text{Supp} M_\Delta$ consists of the non-klt points. In particular, the dimension of M_Δ is zero. We can apply Theorem 2.9 for $L := aD$ and $N := D$. Then we see that there exists a positive integer b_1 such that if $b \geq b_1$, then $H^0(X, bD) \neq 0$ and the base locus of $|bD|$ contains no non-klt points.

The following argument is a slight modification of [Kollár-Mori, Theorem 3.3]. Fix an arbitrary prime number q . Let s be a positive integer such that

$$\text{Bs}|q^s D| = \bigcap_{l \geq 1} \text{Bs}|q^l D|.$$

Note that, since X is a noetherian scheme, we can find such an integer $s > 0$. It is sufficient to prove that $\text{Bs}|q^s D| = \emptyset$. Suppose the contrary and we derive a contradiction. Set $m := q^s$. By the above argument, we see that $\text{Bs}|mD|$ contains no non-klt points.

Let $f : Y \rightarrow X$ be a log resolution of (X, Δ) such that

- (1) $K_Y = f^*(K_X + \Delta) + \sum a_j F_j$.
- (2) $f^*(aD - (K_X + \Delta)) - \sum p_j F_j$ is ample, where $0 < p_j \ll 1$.
- (3) $f^*|mD| = |L| + \sum r_j F_j$, where $|L|$ is basepoint free and $\bigcup F_j$ is the fixed locus of $f^*|mD|$.

Since $\text{Bs}|mD|$ contains no non-klt points, for every j , the inequality $r_j > 0$ implies $a_j > -1$. We define the $\mathbb{Q}$ -divisor $N(b, c)$ by

$$\begin{aligned} N(b, c) &:= b f^* D - K_Y + \sum (-c r_j + a_j - p_j) F_j \\ &= (b - c m - a) f^* D \\ &\quad + c(m f^* D - \sum r_j F_j) \\ &\quad + f^*(aD - (K_X + \Delta)) - \sum p_j F_j. \end{aligned}$$

If $b \geq c m + a$, then $N(b, c)$ is ample. Thus, for $b \gg 0$, we have

$$H^1(Y, K_Y + \lceil N(b, c) \rceil) = 0$$

by Theorem 2.6.

By a small perturbation of p_j , we can find $c > 0$ and a prime divisor F in the fixed locus of $f^*|mD|$, which satisfy the following property: $\sum_{a_j > -1} (-c r_j + a_j - p_j) F_j =: A - F$ where $\lceil A \rceil$ is effective and F is

not a prime component of A . Note that we can find such a number c because the inequality $r_j > 0$ implies $a_j > -1$. Set

$$G := -\lceil \sum_{a_j \leq -1} (-cr_j + a_j - p_j) F_j \rceil.$$

Note that G is an effective f -exceptional $\mathbb{Z}$ -divisor and $f(G)$ consists of non-klt points. This means $\text{Supp} G \cap \text{Supp} F = \emptyset$ because $\text{Bs}|mD|$ contains no non-klt points. Then we have

$$K_Y + \lceil N(b, c) \rceil = bf^*D + \lceil A \rceil - (F + G).$$

Consider the exact sequence:

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_X(K_Y + \lceil N(b, c) \rceil) \\ &\rightarrow \mathcal{O}_X(bf^*D + \lceil A \rceil) \\ &\rightarrow \mathcal{O}_{F+G}(bf^*D + \lceil A \rceil) \rightarrow 0 \end{aligned}$$

If $b \gg 0$, then H^1 of the first term $\mathcal{O}_X(K_Y + \lceil N(b, c) \rceil)$ vanishes. Let us consider the third term $\mathcal{O}_{F+G}(bf^*D + \lceil A \rceil)$. Since F is disjoint from G , we have

$$\mathcal{O}_{F+G}(bf^*D + \lceil A \rceil) = \mathcal{O}_F(bf^*D + \lceil A \rceil) \oplus \mathcal{O}_G(bf^*D + \lceil A \rceil).$$

and

$$\mathcal{O}_F(bf^*D + \lceil A \rceil) = \mathcal{O}_F(K_F + \lceil N(b, c) \rceil).$$

H^0 of this sheaf does not vanish by the non-vanishing theorem for curves. Then we see $f(F) \not\subset \text{Bs}|bD|$. Let $b := q^l$ for $l \gg 0$. Then this is a contradiction. $\square$

Corollary 3.3. *Let X be a projective normal surface and let Δ be a $\mathbb{Q}$ -divisor such that $\lfloor \Delta \rfloor = 0$ and $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. If $K_X + \Delta$ is nef and big, then $K_X + \Delta$ is semi-ample.*

Proof. Let c be a positive integer such that $c(K_X + \Delta)$ is Cartier. Then we can apply Theorem 3.2 for $D := c(K_X + \Delta)$ and $a := 2$. Thus $|bc(K_X + \Delta)|$ is basepoint free for $b \gg 0$. $\square$

We would like to know whether the above basepoint free theorem holds for the case where $D \equiv 0$. We give the affirmative answer only for the case where X has at worst rational singularities. But our strategy is not the X-method. Let us recall the following known fact.

Fact 3.4. *Let X be a normal surface and let Δ be an effective $\mathbb{R}$ -divisor.*

- (1) *If (X, Δ) is klt, then X has at worst rational singularities.*
- (2) *If X has at worst rational singularities, then X is $\mathbb{Q}$ -factorial.*

Proof. (1) See, for example, [T, Theorem 14.4 and Remark 14.5].

(2) See [Lipman1, Proposition 17.1]. $\square$

The following result is the key.

Theorem 3.5. *Let X be a projective surface whose singularities are at worst rational. Let Δ be an $\mathbb{R}$ -Weil divisor such that $\lfloor \Delta \rfloor = 0$. If $-(K_X + \Delta)$ is nef and big, then X is a rational surface.*

Proof.

Step 1. In this step, we show that we may assume that X has no curve whose self-intersection number is negative.

Suppose the contrary, that is, there exists a curve E in X such that $E^2 < 0$. Since

$$(K_X + E) \cdot E < (K_X + \Delta) \cdot E \leq 0,$$

we obtain a birational morphism $f : X \rightarrow Y$ to a projective surface whose singularities are at worst rational such that $\text{Ex} f = E$. This follows from [T, Theorem 6.2 and Theorem 20.4]. Set $\Delta_Y := f_* \Delta$. We see that the discrepancy d , defined by

$$K_X + \Delta = f^*(K_Y + \Delta_Y) + dE,$$

is non-negative. Then we can see that $-(K_Y + \Delta_Y)$ is nef and big. Moreover, if there exists a curve E_Y in Y such that $E_Y^2 < 0$, then we can repeat the same procedure as above.

Step 2. In this step, we prove that we may assume that there exists a surjective morphism $\pi : X \rightarrow Z$ to a smooth projective irrational curve Z .

Let $g : X' \rightarrow X$ be the minimal resolution and set $K_{X'} + \Delta' := g^*(K_X + \Delta)$. Since $-(K_X + \Delta)$ is big, the anti-canonical divisor

$$-K_{X'} = -g^*(K_X + \Delta) + \Delta'$$

is also big. In particular, X' is a ruled surface. If X' is rational, then there is nothing to prove. Thus we may assume that X' is an irrational ruled surface. Let $\theta : X' \rightarrow Z$ be its ruling. Because the singularities of X are at worst rational, each curve D in $\text{Ex}(g)$ is a smooth rational curve. In particular, $\theta(D)$ is one point. This means that θ factors through X . This is what we want to show.

Step 3. By Step 1 and [T, Theorem 6.8], we see that $\rho(X) \leq 2$. Moreover, by Step 2, we see that $\rho(X) = 2$. By Step 1, we see that $-(K_X + \Delta)$ is ample because, for a curve C in X , the equality $(K_X + \Delta) \cdot C = 0$ means $C^2 < 0$ by Kodaira's lemma. Thus there are two extremal rays which induce the Mori fiber space to a curve by [T, Theorem 6.8]. But this contradicts $\pi : X \rightarrow Z$ and the irrationality of Z .

□

In the case where $D \equiv 0$, the basepoint free theorem is related to the rationality of the log weak del Pezzo surfaces. Indeed, by using the above result, we prove the following basepoint free theorem.

Corollary 3.6 (Basepoint free theorem in the case where $\nu = 0$). *Let X be a projective surface whose singularities are at worst rational. Let Δ be an $\mathbb{R}$ -Weil divisor such that $\lfloor \Delta \rfloor = 0$. Let D be a numerically trivial Cartier divisor. If $-(K_X + \Delta)$ is nef and big, then $D \sim 0$.*

Proof. Let $f : X' \rightarrow X$ be a resolution and set $D' := f^*D$. Since $H^0(X, D) = H^0(X', D')$, it is sufficient to prove $D' \sim 0$. By Theorem 3.5, X' is rational. Therefore $D' \equiv 0$ means $D' \sim 0$. $\square$

Remark 3.7. In [T], a basepoint free theorem is established in the case where X is a $\mathbb{Q}$ -factorial surface (Theorem 0.8). But this result does not contain Corollary 3.6. On the other hand, a cone theorem is established under the assumption that X is a normal surface and Δ is an effective $\mathbb{R}$ -divisor such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. For more details, see [T].

4. OTHER VANISHING RESULTS

In this section, we establish some vanishing results other than the ones in Section 2. Theorem 4.2 and Theorem 4.4 are the main results in this section. Theorem 4.2 follows from a fundamental inductive argument. Theorem 4.4 follows from the same argument as Section 2.

First, we consider a generalization of Proposition 2.3.

Proposition 4.1. *Let X be an n -dimensional smooth projective variety with $n \geq 1$. Let F be a coherent sheaf on X . Let N be a nef $\mathbb{R}$ -divisor with $\nu(X, N) \geq 1$. Then there exists a positive real number $r(F, N)$ such that*

$$H^n(X, F(rN + N')) = 0$$

for every positive real number $r \geq r(F, N)$ and for every nef $\mathbb{R}$ -divisor N' such that $rN + N'$ is a $\mathbb{Z}$ -divisor.

Proof. By the same argument as Theorem 2.3, we may assume that $F =: L$ is an invertible sheaf. We prove the assertion by the induction on $n = \dim X$. If $n = 1$, then the assertion is obvious. Thus, we assume $n > 1$. Let H be a smooth hyperplane section. Consider the exact sequence:

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_X(L + rN + N') \\ &\rightarrow \mathcal{O}_X(L + rN + N' + H) \\ &\rightarrow \mathcal{O}_H(L + rN + N' + H) \rightarrow 0 \end{aligned}$$

By the hypothesis of the induction, $H^{n-1}(H, \mathcal{O}_H(L + rN + N' + H))$ vanishes. For the vanishing of $H^n(X, \mathcal{O}_X(L + rN + N' + H))$, replacing H by a large multiple, we can apply the Fujita vanishing theorem. This is what we want to show. $\square$

Second, we consider a generalization of the Kawamata–Viehweg type vanishing theorem (Theorem 2.6).

Theorem 4.2. *Let X be an n -dimensional smooth projective variety with $n \geq 2$. Let B be a nef and big $\mathbb{R}$ -divisor whose fractional part is simple normal crossing. Let N be a nef $\mathbb{R}$ -divisor with $\nu(X, N) \geq 1$. Then there exists a positive real number $r(B, N)$ such that*

$$H^{n-1}(X, K_X + \lceil B \rceil + rN + N') = 0$$

for every positive integer $r \geq r(B, N)$ and for every nef $\mathbb{R}$ -divisor N' such that $rN + N'$ is a $\mathbb{Z}$ -divisor.

Proof. If $n = 2$, then we obtain the assertion by Theorem 2.6. Then, the assertion follows from the same inductive argument as the proof of Theorem 4.1. $\square$

Next, let us recall the following known result.

Proposition 4.3. *Let $\pi : X \rightarrow S$ be a morphism from a proper variety X to a projective variety S . Let A_S be an ample Cartier divisor on S and let $N := \pi^*A_S$. Let F be a coherent sheaf on X . Set $f_{\max} := \max_{s \in S} \dim \pi^{-1}(s)$.*

If $i \geq f_{\max} + 1$, then

$$H^i(X, F(mN)) = 0$$

for an arbitrary integer $m \gg 0$.

Proof. Consider the Leray spectral sequence

$$E_2^{i,j} := H^i(S, R^j \pi_* F(mN)) \Rightarrow H^{i+j}(X, F(mN)) =: E^{i+j}.$$

Since

$$R^j \pi_* F(mN) = R^j \pi_*(F \otimes \pi^*(mA_S)) = R^j \pi_*(F) \otimes A_S^{\otimes m},$$

by Serre vanishing, we have $E_2^{i,j} = 0$ for $i > 0$ and $m \gg 0$. Thus we obtain $E_2^{0,j} = E^j$ for $m \gg 0$. If $j \geq f_{\max} + 1$, then $E_2^{0,j} = 0$. $\square$

By the same argument as Section 2, we obtain the following vanishing result.

Theorem 4.4. *Let $\pi : X \rightarrow S$ be a morphism from a smooth projective variety X to a projective variety S . Let A be a π -ample $\mathbb{R}$ -divisor on X whose fractional part is simple normal crossing. Let A_S be an ample Cartier divisor on S and let $N := \pi^*A_S$. Set $f_{\max} := \max_{s \in S} \dim \pi^{-1}(s)$.*

(1) *If $i \geq f_{\max}$, then*

$$H^i(X, \mathcal{O}_X(K_X + \lceil A \rceil + mN)) = 0$$

for an arbitrary integer $m \gg 0$.

(2) *If $i \geq f_{\max}$, then*

$$R^i \pi_* \mathcal{O}_X(K_X + \lceil A \rceil) = 0.$$

Proof. By the usual spectral sequence argument, (2) follows from (1). Thus we only prove (1). By the assumption, we may assume that A is an ample $\mathbb{R}$ -divisor whose fractional part is simple normal crossing. Moreover, by Proposition 2.5, we may assume that A is an ample $\mathbb{Z}$ -divisor. Then, the assertion follows from the same arguments as Proposition 2.4 by using Proposition 4.3 instead of Proposition 2.3. $\square$

5. EXAMPLES IN DIMENSION THREE

It is natural to consider the following question.

Question 5.1. Can we generalize the vanishing results in Section 2 to higher dimensional varieties?

Unfortunately, the answer is NO. In this section, we construct counterexamples.

Example 5.2 (cf. Proposition 2.3). *There exists a smooth projective 3-fold X , a coherent sheaf F and a semi-ample and big $\mathbb{Z}$ -divisor B which satisfy the following property.*

There exists a positive integer m_0 such that for an arbitrary integer $m \geq m_0$

$$H^2(X, F(mB)) \neq 0.$$

Construction. Let X_0 be an arbitrary smooth projective 3-fold and let $x_0 \in X_0$ be an arbitrary point. Let $f : X \rightarrow X_0$ be the blowup at x_0 . Let E be the exceptional divisor and let $B := f^*A_0$ where A_0 is an ample $\mathbb{Z}$ -divisor on X_0 . We define F by

$$F := \mathcal{O}_X(K_X + E).$$

Consider the exact sequence:

$$0 \rightarrow \mathcal{O}_X(K_X + mB) \rightarrow \mathcal{O}_X(K_X + E + mB) \rightarrow \mathcal{O}_E(K_E + mB) \rightarrow 0.$$

$H^3(X, \mathcal{O}_X(K_X + mB))$ vanishes for an arbitrary large integer $m \gg 0$ by Proposition 4.1. Consider $H^2(E, \mathcal{O}_E(K_E + mB))$. Since $B = f^*(A)$ and $f(E)$ is one point, we have

$$h^2(E, K_E + mB) = h^2(E, K_E) = h^0(E, \mathcal{O}_E) = 1 \neq 0$$

for an arbitrary integer $m \in \mathbb{Z}$. These mean

$$H^2(X, \mathcal{O}_X(K_X + E + mB)) \neq 0$$

for an arbitrary large integer $m \gg 0$. This is what we want to show. $\square$

In the construction of the following three examples, we use a counter-example to the Kodaira vanishing theorem (cf. [Raynaud]).

Example 5.3 (cf. Theorem 2.4). *There exists a smooth projective 3-fold X , an ample $\mathbb{Z}$ -divisor A and a semi-ample $\mathbb{Z}$ -divisor N with $\nu(X, N) = 1$ which satisfy the following property.*

There exists a positive integer m_0 such that for an arbitrary integer $m \geq m_0$

$$H^1(X, K_X + A + mN) \neq 0.$$

Construction. Let Z be a smooth projective surface and let A_Z be an ample $\mathbb{Z}$ -divisor such that

$$H^1(Z, K_Z + A_Z) \neq 0.$$

Let C be an arbitrary smooth projective curve. Set $X := Z \times C$ and let π_Z and π_C be their projections respectively. Take two distinct points $c_0 \in C$ and $c_1 \in C$ and let $Z_0 := Z \times \{c_0\}$ and $Z_1 := Z \times \{c_1\}$. Note that $Z_0 \simeq Z$ and $Z_1 \simeq Z$. Since c_0 and c_1 are ample $\mathbb{Z}$ -divisors on C ,

$$A := \pi_Z^*(A_Z) + Z_0 + Z_1 \text{ and } A - Z_0$$

are ample $\mathbb{Z}$ -divisors on X . We show

$$H^1(X, K_X + A + mZ_1) \neq 0$$

for an arbitrary integer $m \gg 0$. Consider the exact sequence:

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_X(K_X + A - Z_0 + mZ_1) \\ &\rightarrow \mathcal{O}_X(K_X + A + mZ_1) \\ &\rightarrow \mathcal{O}_{Z_0}(K_X + A + mZ_1) \rightarrow 0. \end{aligned}$$

By Theorem 4.2, $H^2(X, \mathcal{O}_X(K_X + A - Z_0 + mZ_1))$ vanishes for an arbitrary large integer $m \gg 0$. Let us calculate $H^1(Z_0, \mathcal{O}_{Z_0}(K_X + A + mZ_1))$. By $Z_0 \cap Z_1 = \emptyset$, we see

$$H^1(Z_0, K_X + A + mZ_1) = H^1(Z_0, K_{Z_0} + \pi_Z^* A_Z) = H^1(Z, K_Z + A_Z) \neq 0$$

for an arbitrary integer $m \in \mathbb{Z}$. These mean

$$H^1(X, K_X + A + mZ_1) \neq 0$$

for $m \gg 0$. This is what we want to show. $\square$

Example 5.4 (cf. Theorem 2.4). *There exists a smooth projective 3-fold X , an ample $\mathbb{Z}$ -divisor A and a semi-ample and big $\mathbb{Z}$ -divisor B which satisfy the following property.*

There exists a positive integer m_0 such that for an arbitrary integer $m \geq m_0$

$$H^1(X, K_X + A + mB) \neq 0.$$

Construction. By Proposition 2.5 and Step 1 in the proof of Theorem 2.6, it is sufficient that we construct

$$H^1(X, K_X + \Delta + A + mN) \neq 0$$

for an ample $\mathbb{Q}$ -divisor A and a simple normal crossing $\mathbb{Q}$ -divisor Δ such that $0 \leq \Delta \leq 1$ and that $\Delta + A$ is a $\mathbb{Z}$ -divisor.

Let $Z \subset \mathbb{P}^N$ be a smooth projective surface and let A_Z be an ample $\mathbb{Z}$ -divisor such that

$$H^1(Z, K_Z + A_Z) \neq 0.$$

Let $Y \subset \mathbb{P}^{N+1}$ be the projective cone over Z and let $f : X \rightarrow Y$ be the blowup of the vertex of Y .

Then, by [Hartshorne, Chapter V, Example 2.11.4], we see that $X = \text{Proj}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(1))$. Let $\pi : X \rightarrow Z$ be the natural projection and let $\mathcal{O}_X(1)$ be the canonically defined π -ample invertible sheaf. Let Z_0 and Z_1 be the sections defined by the following surjections respectively

$$\mathcal{O}_Z \oplus \mathcal{O}_Z(1) \rightarrow \mathcal{O}_Z \rightarrow 0$$

$$\mathcal{O}_Z \oplus \mathcal{O}_Z(1) \rightarrow \mathcal{O}_Z(1) \rightarrow 0.$$

By the definition of Z_0 and Z_1 , we have $\mathcal{O}_X(1)|_{Z_0} = \mathcal{O}_{Z_0}$ and $\mathcal{O}_X(1)|_{Z_1} = \mathcal{O}_{Z_1}(1)$ where $\mathcal{O}_{Z_1}(1)$ is a very ample invertible sheaf defined by $Z \simeq Z_1$ and $\mathcal{O}_Z(1)$. By the same argument as [Hartshorne, Chapter V, Proposition 2.6], we see $\mathcal{O}_X(Z_1) \simeq \mathcal{O}_X(1)$. Moreover, by a direct calculation, we see that Z_0 is the exceptional locus of f . (See [Hartshorne, Chapter V, Example 2.11.4].) Note that Z_0 and Z_1 are disjoint because $\mathcal{O}_X(1)|_{Z_0} = \mathcal{O}_{Z_0}$.

We fix a small positive rational number $\epsilon_0 \in \mathbb{Q}_{>0}$ such that the $\mathbb{Q}$ -divisor

$$A := \pi^* A_Z + \epsilon_0 Z_1$$

is ample. Set

$$\Delta := (1 - \epsilon_0)Z_1 + Z_0.$$

Note that $\Delta + A$ is a $\mathbb{Z}$ -divisor. Let A_Y be an ample invertible sheaf on Y and let

$$B := f^*A_Y.$$

Consider the exact sequence:

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_X(K_X + \Delta + A + mB - Z_0) \\ &\rightarrow \mathcal{O}_X(K_X + \Delta + A + mB) \\ &\rightarrow \mathcal{O}_{Z_0}(K_X + \Delta + A + mB) \rightarrow 0. \end{aligned}$$

First let us calculate $H^2(X, \mathcal{O}_X(K_X + \Delta + A + mB - Z_0))$:

$$\begin{aligned} &H^2(X, K_X + \Delta + A + mB - Z_0) \\ &= H^2(X, K_X + (1 - \epsilon_0)Z_1 + A + mB) = 0 \end{aligned}$$

for an arbitrary large integer $m \gg 0$ by Theorem 4.2. Second let us calculate $H^1(Z_0, \mathcal{O}_{Z_0}(K_X + \Delta + A + mB))$. Since $f(Z_0)$ is one point, we see

$$\begin{aligned} &\mathcal{O}_{Z_0}(K_X + \Delta + A + mB) \\ &= \mathcal{O}_{Z_0}(K_X + Z_1 + Z_0 + \pi^*A_Z + mf^*A_Y) \\ &= \mathcal{O}_{Z_0}(K_{Z_0} + \pi^*A_Z) \end{aligned}$$

for an arbitrary integer $m \in \mathbb{Z}$. By $Z_0 \simeq Z$, we see

$$\begin{aligned} H^1(Z_0, K_X + \Delta + A + mB) &= H^1(Z_0, K_{Z_0} + \pi^*A_Z) \\ &= H^1(Z, K_Z + A_Z) \\ &\neq 0. \end{aligned}$$

Therefore we obtain

$$H^1(X, K_X + \Delta + A + mB) \neq 0$$

for an arbitrary large integer $m \gg 0$. This is what we want to show. $\square$

Example 5.5 (cf. Theorem 2.4). *There exists a smooth projective 3-fold W , an ample $\mathbb{Z}$ -divisor A_W and a semi-ample $\mathbb{Z}$ -divisor N_W with $\nu(W, N_W) = 2$ which satisfy the following property.*

There exists a positive integer m_0 such that for an arbitrary integer $m \geq m_0$

$$H^1(W, K_W + A_W + mN_W) \neq 0.$$

Construction. By Proposition 2.5 and Step 1 in the proof of Theorem 2.6, it is sufficient that we construct

$$H^1(W, K_W + \Delta_W + A_W + mN_W) \neq 0$$

for an ample $\mathbb{Q}$ -divisor A_W and a simple normal crossing $\mathbb{Q}$ -divisor Δ_W such that $0 \leq \Delta_W \leq 1$ and that $\Delta_W + A_W$ is a $\mathbb{Z}$ -divisor.

We use the same notations $X, Y, Z, A, \dots$ as Example 5.4. Let $y_0 \in Y$ be the vertex as a projective cone. There exists a finite morphism $\theta : Y \rightarrow \mathbb{P}^3$. Fix an open dense subset $\mathbb{A}^3 \subset \mathbb{P}^3$ such that $\theta(y_0) \in \mathbb{A}^3$. Take an arbitrary projection $\mathbb{A}^3 \rightarrow \mathbb{A}^2 =: U$ and fix its projectivication $U \subset \mathbb{P}^2 =: P$. Now, we have the following morphisms

$$X \xrightarrow{f} Y \xrightarrow{\theta} \mathbb{P}^3 \supset \mathbb{A}^3 \rightarrow \mathbb{A}^2 = U \subset \mathbb{P}^2 = P.$$

Here, by considering the composition of the above dominant rational maps, we obtain a dominant rational map $g : X \dashrightarrow P$. Note that this is a morphism on $(\theta \circ f)^{-1}(\mathbb{A}^3)$. By its construction, we see $Z_0 \subset (\theta \circ f)^{-1}(\mathbb{A}^3)$ since $f(Z_0) = y_0$. By taking a log resolution of indeterminacy $h : W \rightarrow X$, we obtain a surjective morphism $l : W \rightarrow P$ from a smooth projective 3-fold W such that $h(\text{Ex}(h)) \subset X \setminus (\theta \circ f)^{-1}(\mathbb{A}^3)$ and $h^{-1}(Z_1) \cup \text{Ex}(h)$ is simple normal crossing (cf. [CP]). Then $h : h^{-1}(Z_0) \rightarrow Z_0$ is an isomorphism and let $Z_W := h^{-1}(Z_0)$. Let

$$A_W := h^*A - E = h^*\pi^*A_Z + \epsilon_0 h^*Z_1 - E$$

be an ample $\mathbb{Q}$ -divisor on W where E is an h -exceptional $\mathbb{Q}$ -divisor with $0 \leq E < 1$. Note that we can find such a divisor E by [Kollár-Mori, Lemma 2.62]. Let

$$\Delta_W := Z_W + \lceil A_W \rceil - A_W.$$

Since $\text{Supp}(h^*Z_1 \cup \text{Ex}(h))$ is disjoint from Z_W we see that

$$\mathcal{O}_{Z_W}(\Delta_W + A_W) = \mathcal{O}_{Z_W}(Z_W + h^*\pi^*A_Z).$$

Let A_P be an ample $\mathbb{Z}$ -divisor on P and let

$$N_W := l^*A_P.$$

Consider the exact sequence:

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_W(K_W + \Delta_W + A_W + mN_W - Z_W) \\ &\rightarrow \mathcal{O}_W(K_W + \Delta_W + A_W + mN_W) \\ &\rightarrow \mathcal{O}_{Z_W}(K_W + \Delta_W + A_W + mN_W) \rightarrow 0. \end{aligned}$$

Then by the same calculation as Example 5.4, we obtain the desired result. $\square$

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