

Weighted Monotone Fock Space and A Brownian Motion with the Distribution of Bożejko-Leinert-Speicher

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Abstract. We construct on a weighted monotone Fock space Φ_w with weight sequence $w = (1, c, c^2, c^3, \dots)$ an example of noncommutative Brownian motion $\{Q_t\}_{t \geq 0}$ such that the distribution $\mu_{s,t}$ of an increment $Q_t - Q_s$, $0 < s < t$, coincides with the distribution of Bożejko-Leinert-Speicher but that the process $\{Q_t\}_{t \geq 0}$ is not isomorphic to the c -free Brownian motion of Bożejko-Leinert-Speicher $\{\tilde{Q}_t\}_{t \geq 0}$.

1. Weighted Monotone Fock Space

A weighted monotone Fock space Φ_w is a deformation of the monotone Fock space Φ through a weight sequence $w = \{w_n\}_{n=0}^\infty$, $w_n > 0$. It is a special case of interacting Fock spaces [AcB, ALV]. The usual monotone Fock space corresponds to the case of trivial weight sequence $w_n := 1$, $n \geq 1$ [Lu, Mu1, Mu2]. Let us give the precise definitions.

Let $T = \mathbf{R}_+^*$ be the set of all strictly positive real numbers $s > 0$. Denote by Σ_n the set of all monotone sequences $\sigma = (s_n > s_{n-1} > \dots > s_1)$ of length n from T , which are increasing to the left. For each $n \geq 1$, Σ_n is the measure space equipped with the (induced) Lebesgue measure $d\sigma$. $\Sigma_0 = \{\Lambda\}$ is the singleton consisting of the null sequence Λ with the point mass (= Dirac measure). Denote by $\mathcal{H}_n$ the complex L^2 -space $L^2(\Sigma_n)$ with a new inner product

$$\langle u|v \rangle = w_n \int_{\Sigma_n} d\sigma \overline{u(\sigma)} v(\sigma) \quad (u, v \in \mathcal{H}_n).$$

This Hilbert space $\mathcal{H}_n := (L^2(\Sigma_n), w_n)$ is called the n -particle space. Then we put

$$\Phi_w := \mathbf{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_n \oplus \dots$$

and call it a *weighted monotone Fock space* with a weight sequence $w = \{w_n\}_{n=0}^\infty$. Here we identified $\mathcal{H}_0$ with $\mathbb{C}$ through the identification of the function $\mathbf{1} : \Lambda \rightarrow 1$ with the unit 1 of $\mathbb{C}$. Also we assume that $w_0 = 1$. We denote by $\Omega := \mathbf{1}$ the vacuum vector ($\in \mathcal{H}_0$).

For each one-particle vector $f \in \mathcal{H}_1$, the creation operator δ_f^+ on Φ_w is defined, as the left multiplication operator, by

$$(\delta_f^+ u)(t > \sigma) = f(t)u(\sigma) \quad (u = u(\sigma) \in \mathcal{H}_n).$$

The annihilation operator δ_f^- is defined as the adjoint of δ_f^+ . For the vacuum vector Ω , we have $\delta_f^- \Omega = 0$. The concrete action of δ_f^- on $u \in \mathcal{H}_n$, $n \geq 1$, is given by

$$(\delta_f^- u)(\sigma) = \frac{w_{n+1}}{w_n} \int_{t > \sigma} dt \overline{f(t)} u(t > \sigma).$$

So we put $r_n := \frac{w_n}{w_{n-1}}$, $n = 1, 2, 3, \dots$, then a weight sequence $w = \{w_n\}_{n=0}^\infty$ corresponds to a sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ in the bijective way:

$$w = (w_0, w_1, w_2, \dots) \longleftrightarrow \mathbf{r} = (r_1, r_2, r_3, \dots)$$

under the assumption $w_0 = 1$.

Let $\mathcal{A}_w$ be the C^* -algebra generated by the creation and annihilation operators $\{\delta_f^+, \delta_f^- | f \in \mathcal{H}_1\}$, and let $\phi_w(\cdot) = \langle \Omega | \cdot | \Omega \rangle$ be the vacuum state over $\mathcal{A}_w$. We will working on this C^* -probability space $(\mathcal{A}_w, \phi_w)$. We often use the short notation $\langle \cdot \rangle := \phi(\cdot)$ to mean the expectation w.r.t. a given state ϕ over a C^* -algebra.

2. Brwonian motion

For each $f \in \mathcal{H}_1$, the *canonical operator* Q_f is defined by $Q_f = \delta_f^+ + \delta_f^-$. By the specialization $f := \chi_{(0,t]}$ with $t \geq 0$, we obtain the creation process $D_t^+ = \delta_{\chi_{(0,t]}}^+$, the annihilation process $D_t^- = \delta_{\chi_{(0,t]}}^-$, and the pair of canonical processes $Q_t = D_t^+ + D_t^-$ and $P_t = \sqrt{-1}(D_t^+ - D_t^-)$. Here χ_I denotes the indicator function of an interval I on the real line. We are interested in the probability law of the canonical process $\{Q_t\}_{t \geq 0}$.

At first let us consider the independence structure of the process $\{Q_t\}_{t \geq 0}$. Let a C^* -probability space $(\mathcal{A}, \phi)$ and a stochastic process $\{X_t\}_{t \geq 0} \subset \mathcal{A}$ be given. Let $\mathcal{R}$ be the ring generated by all the semi-closed interval $(s, t]$ with $0 < s < t$. For each $I \in \mathcal{R}$, let $\mathcal{A}_I$ be the C^* -algebra generated by the increments $\{X_t - X_s | (s, t] \subset I\}$ supported in I . Then a process $\{X_t\}_{t \geq 0}$ is said to be a process with independent increments if, for each increasing finite sequence $I_1 < I_2 < \dots < I_n$ of elements of $\mathcal{R}$, we have

$$\phi(A_1 A_2 \cdots A_n) = \phi(A_1) \phi(A_2) \cdots \phi(A_n)$$

for all $A_i \in \mathcal{A}_{I_i}$, $i = 1, 2, \dots, n$. Of course $I < J$ means that $s < t$ for all $s \in I$ and all $t \in J$.

Proposition 2.1. $\{Q_t\}_{t \geq 0}$ is a process with independent increments.

This is a corollary of the following Proposition 2.2. For each $I \in \mathcal{R}$, let $\mathcal{A}_I^{(w)}$ be the C^* -algebra generated by $\{\delta_f^+, \delta_f^- | f \in \mathcal{H}_1; \text{supp}(f) \subset I\}$. Then the following is easily shown.

Proposition 2.2. Let $\{\mathcal{A}_I^{(w)} | I \in \mathcal{R}\}$ be the system of C^* -subalgebras of $(\mathcal{A}_w, \phi_w)$ defined above. Then for each increasing finite sequence $I_1 < I_2 < \dots < I_n$ of elements of $\mathcal{R}$, we have

$$\phi_w(A_1 A_2 \cdots A_n) = \phi_w(A_1) \phi_w(A_2) \cdots \phi_w(A_n)$$

for all $A_i \in \mathcal{A}_{I_i}^{(w)}$, $i = 1, 2, \dots, n$.

Proposition 2.2 means that also the pair process $(Q_t, P_t)_{t \geq 0}$ is an independent increments process.

Proposition 2.3. $\{Q_t\}_{t \geq 0}$ is a process with stationary increments, i.e.

$$\phi_w((Q_{t+u} - Q_{s+u})^p) = \phi_w((Q_t - Q_s)^p)$$

for all $0 \leq s < t$, all $u > 0$ and all $p = 1, 2, 3, \dots$.

The proof of Proposition 2.3 will become obvious in the later. Now we know that $\{Q_t\}_{t \geq 0}$ is a process with independent and stationary increments. Besides $\{Q_t\}_{t \geq 0}$ is shown to be a scale-invariant process in the following sense.

Let $\{X_t\}_{t \geq 0} \subset \mathcal{A}$ (resp. $\{Y_t\}_{t \geq 0} \subset \mathcal{B}$) be a stochastic process on a C^* -probability space $(\mathcal{A}, \phi)$ (resp. $(\mathcal{B}, \psi)$). Put $\mathcal{A}_0 := C^*(\{X_t | t \in T\})$ and $\mathcal{B}_0 := C^*(\{Y_t | t \in T\})$ where $C^*(E)$ denotes the C^* -subalgebra generated by a subset E . Then a process $\{X_t\}_{t \geq 0}$ is said to be isomorphic to a process $\{Y_t\}_{t \geq 0}$ if there exists some C^* -isomorphism $\pi : \mathcal{A}_0 \rightarrow \mathcal{B}_0$ such that $\pi(X_t) = Y_t$ for all $t \geq 0$ and that $\psi(\pi(X)) = \phi(X)$ for all $X \in \mathcal{A}_0$. Under this definition, we have

Theorem 2.4. For each $\lambda > 0$, the scaled process $\{Q'_t := \frac{1}{\sqrt{\lambda}} Q_{\lambda t}\}_{t \geq 0}$ is isomorphic to the original process $\{Q_t\}_{t \geq 0}$.

Proof. For each fixed $\lambda > 0$, let us define a map $\Phi_w \xrightarrow{\cdot} \Phi_w : u \mapsto u' = u_\lambda$ as follows. For each one-particle vector $f \in \mathcal{H}_1$, we put

$$f(\cdot) \xrightarrow{\cdot} f_\lambda(\cdot) = \frac{1}{\sqrt{\lambda}} f\left(\frac{1}{\lambda} \cdot\right).$$

Also for each n -particle vector $u \in \mathcal{H}_n$, we put

$$u(\cdot) \xrightarrow{\cdot} u_\lambda(\cdot) = \left(\frac{1}{\lambda}\right)^{\frac{n}{2}} u\left(\frac{1}{\lambda} \cdot\right).$$

Besides for the vacuum vector, we put

$$\Omega \xrightarrow{\prime} \Omega.$$

Then this map $u \xrightarrow{\prime} u' = u_\lambda$ defines a unitary operator on Φ_w , because we have

$$\begin{aligned} \langle u'|v' \rangle &= w_n \int_{\sigma} d\sigma \overline{u'}(\sigma) v'(\sigma) \\ &= w_n \int_{\sigma} d\sigma \overline{\left(\frac{1}{\lambda}\right)^{\frac{n}{2}} u\left(\frac{1}{\lambda}\sigma\right) \left(\frac{1}{\lambda}\right)^{\frac{n}{2}} v\left(\frac{1}{\lambda}\sigma\right)} \\ &= w_n \int_{\sigma} \frac{1}{\lambda^n} d\sigma \overline{u\left(\frac{1}{\lambda}\sigma\right) v\left(\frac{1}{\lambda}\sigma\right)} \\ &= w_n \int_{\tau} d\tau \overline{u(\tau) v(\tau)} \\ &= \langle u|v \rangle, \end{aligned}$$

where in the 4th equality we put $\tau = \frac{1}{\lambda}\sigma$, and used $d\sigma = \lambda^n d\tau$ because of $\sigma = \lambda\tau$. This unitary operator $\Phi_w \ni u \mapsto u' \in \Phi_w$ naturally induces the transformation of operators $T \mapsto T'$ as

$$\begin{array}{ccc} \Phi_w & \xrightarrow{\prime} & \Phi_w \\ \downarrow T & & \downarrow T' \\ \Phi_w & \xrightarrow{\prime} & \Phi_w \end{array} \quad \begin{array}{ccc} u & \xrightarrow{\prime} & u' \\ \downarrow T & & \downarrow T' \\ v & \xrightarrow{\prime} & v' \end{array}$$

Then first we know $(\delta_f^+)' = \delta_{f_\lambda}^+$, because we have

$$\begin{aligned} ((\delta_f^+)'u')(t > \sigma) &= (\delta_f^+ u)'(t > \sigma) \\ &= \left(\frac{1}{\lambda}\right)^{\frac{n+1}{2}} (\delta_f^+ u)\left(\frac{1}{\lambda}(t > \sigma)\right) \\ &= \left(\frac{1}{\lambda}\right)^{\frac{1}{2}} f\left(\frac{1}{\lambda}t\right) \cdot \left(\frac{1}{\lambda}\right)^{\frac{n}{2}} u\left(\frac{1}{\lambda}\sigma\right) \\ &= f_\lambda(t) \cdot u_\lambda(\sigma) \\ &= (\delta_{f_\lambda}^+ u_\lambda)(t > \sigma) \\ &= (\delta_{f_\lambda}^+ u')(t > \sigma). \end{aligned}$$

Besides we know $(\delta_f^-)' = \delta_{f_\lambda}^-$, because we have

$$\begin{aligned} ((\delta_f^-)'u')(\sigma) &= (\delta_f^- u)'(\sigma) \\ &= \left(\frac{1}{\lambda}\right)^{\frac{n-1}{2}} (\delta_f^- u)\left(\frac{1}{\lambda}\sigma\right) \\ &= \left(\frac{1}{\lambda}\right)^{\frac{n-1}{2}} \frac{w_n}{w_{n-1}} \int_{t > \frac{1}{\lambda}\sigma} dt \bar{f}(t) u\left(t > \frac{1}{\lambda}\sigma\right) \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{\lambda}\right)^{-1} \frac{w_n}{w_{n-1}} \int_{t > \frac{1}{\lambda}\sigma} \frac{1}{\lambda} d(\lambda t) \left(\frac{1}{\lambda}\right)^{\frac{1}{2}} \bar{f}\left(\frac{1}{\lambda} \cdot \lambda t\right) \cdot \left(\frac{1}{\lambda}\right)^{\frac{n}{2}} u\left(\frac{1}{\lambda}(\lambda t > \sigma)\right) \\
&= \frac{w_n}{w_{n-1}} \int_{s > \sigma} ds \bar{f}_\lambda(s) \cdot u_\lambda(s > \sigma) \\
&= (\delta_{f_\lambda}^- u_\lambda)(\sigma) \\
&= (\delta_{f_\lambda}^- u')(\sigma),
\end{aligned}$$

where in the 5th equality we put $s = \lambda t$. So we have $(Q_f)' = Q_{f_\lambda}$. By the specialization $f := \chi_{[0,t]}$, we get

$$(\chi_{[0,t]})_\lambda(s) = \frac{1}{\sqrt{\lambda}} \chi_{[0,t]} \left(\frac{1}{\lambda} s \right) = \frac{1}{\sqrt{\lambda}} \chi_{[0,\lambda t]}(s),$$

and hence

$$Q_{(\chi_{[0,t]})_\lambda} = Q_{\frac{1}{\sqrt{\lambda}} \chi_{[0,\lambda t]}} = \frac{1}{\sqrt{\lambda}} Q_{\chi_{[0,\lambda t]}}.$$

So we obtain

$$(Q_t)' = \frac{1}{\sqrt{\lambda}} Q_{\lambda t}.$$

Besides it is easy to see that $T \mapsto T'$ is a C^* -algebra automorphism of $\mathcal{A}_w$ satisfying $\phi_w(T') = \phi_w(T)$. $\square$

Corollary 2.5. *For each $t_1, t_2, \dots, t_l \in T$, we have*

$$\langle Q_{t_1} Q_{t_2} \cdots Q_{t_l} \rangle = \langle Q'_{t_1} Q'_{t_2} \cdots Q'_{t_l} \rangle.$$

Proof. $T \mapsto T'$ is a C^* -algebra automorphism of $\mathcal{A}_w$ satisfying $\phi_w(T') = \phi_w(T)$. $\square$

Proposition 2.6. $\langle Q_s Q_t \rangle = \min\{s, t\}$ for $s, t \in T$.

By Propositions 2.1, 2.3 and 2.4, it is natural to interpret the process $\{Q_t\}_{t \geq 0}$ as a noncommutative analogue of Brownian motion.

3. Moments of Canonical Operators

Let a weighted monotone Fock space Φ_w with the weight sequence $w = \{w_n\}$ ($\leftrightarrow \mathbf{r} = \{r_n\}$) be given. In this section, we derive some recurrence relations concerning the moments of the distribution $\mu_{f,w} = \mu_{f,\mathbf{r}}$ of the canonical operator Q_f on Φ_w under the vacuum state ϕ_w . For simplicity we assume that $\|f\|_{L^2} = 1$. Put $\mu := \mu_{f,\mathbf{r}}$.

Let us treat the moments of μ :

$$m_p = \phi_w(Q_f^p) = \int_{-\infty}^{+\infty} x^p d\mu, \quad p = 0, 1, 2, 3, \dots$$

The p th moment m_p can be expanded as

$$\begin{aligned}
m_p &= \phi(Q_f^p) = \phi((D_f^+ + D_f^-)^p) \\
&= \sum_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p \in \{+, -\}} \phi(D_f^{\varepsilon_p} \cdots D_f^{\varepsilon_2} D_f^{\varepsilon_1}) \\
&= \sum_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p \in \{+, -\}} \langle \Omega | D_f^{\varepsilon_p} \cdots D_f^{\varepsilon_2} D_f^{\varepsilon_1} \Omega \rangle.
\end{aligned}$$

Besides it is easy to see that the contributing terms in the last expression are given by the sequences of signatures $(\varepsilon_p, \dots, \varepsilon_2, \varepsilon_1)$ satisfying the following two conditions

$$\begin{cases} \#\{i \mid 1 \leq i \leq l, \varepsilon_i = +\} \geq \#\{i \mid 1 \leq i \leq l, \varepsilon_i = -\}, & l = 1, \dots, p, \\ \#\{i \mid 1 \leq i \leq p, \varepsilon_i = +\} = \#\{i \mid 1 \leq i \leq p, \varepsilon_i = -\}. \end{cases}$$

Such sequences $(\varepsilon_p, \dots, \varepsilon_2, \varepsilon_1)$ correspond to the noncrossing pair partitions (=NCPP) of the p points set $\{p, p-1, \dots, 2, 1\}$ in the bijective way. Besides the noncrossing pair partitions are identified with the noncrossing diagrams in the natural way. For example we have

$$(-, -, -, +, +, +, -, -, +, -, +, +) \longleftrightarrow \text{Diagram with 12 points and 6 arcs: } \overbrace{1 \dots 2}^{\text{arc}}, \overbrace{3 \dots 4}^{\text{arc}}, \overbrace{5 \dots 6}^{\text{arc}}, \overbrace{7 \dots 8}^{\text{arc}}, \overbrace{9 \dots 10}^{\text{arc}}, \overbrace{11 \dots 12}^{\text{arc}}$$

For a noncrossing diagram g which is corresponding to a sequence of signatures $(\varepsilon_p, \dots, \varepsilon_2, \varepsilon_1)$, we put $V_{\mathbf{r}}(g) := \langle \Omega | D_f^{\varepsilon_p} \cdots D_f^{\varepsilon_2} D_f^{\varepsilon_1} \Omega \rangle$. Then we obtain a formula for the even moments m_{2k}

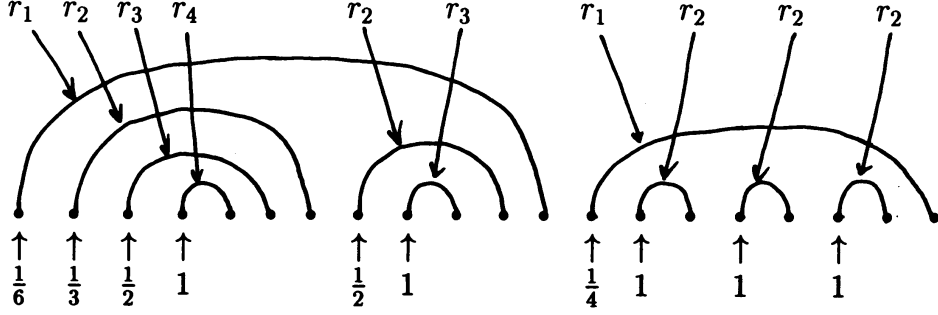
$$m_{2k} = \sum_{\substack{g: \text{NCPP of} \\ 2k \text{ points set}}} V_{\mathbf{r}}(g)$$

Besides we can see that the following recurrence formula for $\langle g \rangle_{\mathbf{r}} := V_{\mathbf{r}}(g)$ hold.

Recurrence relations

$$\begin{aligned}
\text{(i)} \quad & \langle \overbrace{1 \dots 2}^{g_1} \overbrace{3 \dots 4}^{g_2} \cdots \overbrace{2j-1 \dots 2j}^{g_j} \rangle_{\mathbf{r}} = \langle \overbrace{1 \dots 2}^{g_1} \rangle_{\mathbf{r}} \langle \overbrace{3 \dots 4}^{g_2} \rangle_{\mathbf{r}} \cdots \langle \overbrace{2j-1 \dots 2j}^{g_j} \rangle_{\mathbf{r}} \\
\text{(ii)} \quad & \langle \overbrace{1 \dots 2}^g \rangle_{\mathbf{r}} = \frac{r_1}{|g|+1} \langle g \rangle_{\mathbf{r}'} \\
\text{(iii)} \quad & \langle \overbrace{1 \dots 2}^{\text{arc}} \rangle_{\mathbf{r}} = r_1
\end{aligned}$$

Here $|g|$ denotes the number of lines in a diagram g . $\mathbf{r}'$ denotes the sequence obtained by the shift of $\mathbf{r} = (r_1, r_2, r_3, \dots)$, that is, $\mathbf{r}' := (r_2, r_3, r_4, \dots)$. For example, the following figure explains the rule for the calculation of $V_{\mathbf{r}}(g)$.



Also we note here that $2k$ th moment $m_{2k}(\mathbf{r})$ is a homogeneous polynomial of degree k in variables $r_1, r_2, \dots, r_k$. So we have for each $c > 0$

$$m_{2k}(c r_1, c r_2, c r_3, \dots, c r_n, \dots) = c^k m_{2k}(r_1, r_2, r_3, \dots, r_n, \dots).$$

Let us derive a functional equation satisfied by the generating function $f(s) = f(s; \mathbf{r})$ for the even moments $\{m_{2k}(\mathbf{r})\}$ of the distribution $\mu = \mu_{\mathbf{r}} = \mu_{f, \mathbf{r}}$:

$$f(s; \mathbf{r}) = \sum_{k=0}^{\infty} m_{2k}(\mathbf{r}) s^k.$$

Using the recurrence relations for the moments (3.1), the generating function $f(s; \mathbf{r})$ can be rewritten as

$$\begin{aligned} f(s; \mathbf{r}) &= \sum_{k=0}^{\infty} m_{2k}(\mathbf{r}) s^k \\ &= 1 + \sum_{k=1}^{\infty} \sum_{j=1}^k \sum_{\substack{k_1 + \dots + k_j = k \\ k_1 \geq 1, \dots, k_j \geq 1}} \frac{r_1}{k_1} m_{2(k_1-1)}(\mathbf{r}') s^{k_1} \dots \frac{r_1}{k_j} m_{2(k_j-1)}(\mathbf{r}') s^{k_j} \\ &= 1 + \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{r_1}{k} m_{2(k-1)}(\mathbf{r}') s^k \right)^j. \end{aligned}$$

Now we put $g(s; \mathbf{r}) := \sum_{k=1}^{\infty} \frac{r_1}{k} m_{2(k-1)}(\mathbf{r}') s^k$, then this quantity satisfies

$$f(s; \mathbf{r}) = \frac{1}{1 - g(s; \mathbf{r})}.$$

Also this quantity $g(s; \mathbf{r})$ can be rewritten as

$$\begin{aligned} g(s; \mathbf{r}) &= \sum_{k=1}^{\infty} \frac{r_1}{k} m_{2(k-1)}(\mathbf{r}') s^k \\ &= r_1 \sum_{k=1}^{\infty} \int_0^s ds m_{2(k-1)}(\mathbf{r}') s^{k-1} \\ &= r_1 \int_0^s ds \sum_{l=0}^{\infty} m_{2l}(\mathbf{r}') s^l \\ &= r_1 \int_0^s ds \sum_{k=0}^{\infty} m_{2k}(\mathbf{r}') s^k \\ &= r_1 \int_0^s ds f(s; \mathbf{r}'). \end{aligned}$$

So we get

$$g(s; \mathbf{r}) = r_1 \int_0^s ds f(s; \mathbf{r}'). \quad (3.3)$$

Therefore the moment generating function $f(s; \mathbf{r})$ satisfies the following functional equation:

$$\begin{cases} f(s; \mathbf{r}) = \frac{1}{1 - r_1 \int_0^s ds f(s; \mathbf{r}')}, \\ f(0) = 1. \end{cases}$$

4. An example – the distribution of Bożejko-Leinert-Speicher

In this section, we give an example of weighted monotone Fock space Φ_w such that the probability distribution μ_t of its associated Brownian motion $\{Q_t\}_{t \geq 0}$ can be explicitly obtained. This example corresponds to the weight sequence w given by

$$\mathbf{r} = (r_1, r_2, r_3, \dots) := (1, c, c, c, \dots).$$

In this case, the quantity $g(s; \mathbf{r})$ is given by

$$\begin{aligned} g(s; 1, c, c, c, \dots) &= 1 \cdot \int_0^s ds f(s; c, c, c, \dots) \\ &= \int_0^s ds \sum_{k=0}^{\infty} m_{2k}(c, c, c, \dots) s^k \end{aligned}$$

from (3.3). By the way, since in general the $2k$ th moment $m_{2k}(r_1, r_2, r_3, \dots)$ is a homogeneous polynomial of degree k in k variables $r_1, r_2, \dots, r_k$, we have

$$m_{2k}(c, c, c, \dots) = c^k m_{2k}(1, 1, 1, \dots).$$

Note that $a_{2k} := m_{2k}(1, 1, 1, \dots)$ is just the $2k$ th moment of the arcsine law with mean 0 and variance 1 because the weight sequence $(1, 1, 1, \dots)$ corresponds to the usual monotone Fock space [Mu1, Mu2]. Now $g(s; \mathbf{r})$ can be rewritten as

$$\begin{aligned} g(s; 1, c, c, c, \dots) &= \int_0^s ds \sum_{l=0}^{\infty} m_{2l}(1, 1, 1, \dots) (cs)^l \\ &= \int_0^s ds f(cs; 1, 1, 1, \dots), \end{aligned}$$

where $f(s; 1, 1, 1, \dots)$ is just the generating function $a(s) := \frac{1}{\sqrt{1-2s}}$ for the even moments of the arcsine law. Hence we have

$$\begin{aligned} g(s; 1, c, c, c, \dots) &= \int_0^s ds a(cs) = \int_0^s ds \frac{1}{\sqrt{1-2cs}} \\ &= \left[-\frac{1}{c} (1-2cs)^{\frac{1}{2}} \right]_0^s = \frac{1}{c} - \frac{1}{c} (1-2cs)^{\frac{1}{2}}. \end{aligned}$$

Using the basic relation $f(s) = \frac{1}{1-g(s)}$, we obtain the explicit form of the generating function $f(s) = f(s; \mathbf{r})$ for the even moments of the distribution $\mu = \mu_{f, \mathbf{r}}$ associated to the Fock space Φ_w with the weight sequence $\mathbf{r} = (1, c, c, c, \dots)$, as

$$f(s) = \frac{(c-1) - \sqrt{1-2cs}}{(c-2) + 2s}.$$

Then the Cauchy transform $G_\mu(z) = \int_{-\infty}^{+\infty} \frac{1}{z-\xi} d\mu(\xi)$ of the measure μ is given by

$$G_\mu(z) = \frac{1}{z} f\left(\frac{1}{z^2}\right) = \frac{(1-c)z + \sqrt{z^2 - 2c}}{(2-c)z^2 - 2}. \quad (4.1)$$

Here we remark that the expression (4.1) is obtained as the specialization $\alpha := 1$ & $\beta := \sqrt{\frac{c}{2}}$ of the Cauchy transform $G_{\nu_{\alpha, \beta}}(z)$ of the distribution of Bożejko-Leinert-Speicher $\nu_{\alpha, \beta}$, which is defined as follows [BLS]:

$$\begin{aligned} \nu_{\alpha, \beta} &= \tilde{\nu}_{\alpha, \beta} + a(\delta_{x_1} + \delta_{x_2}), \\ d\tilde{\nu}_{\alpha, \beta}(x) &= \chi_{[-2\beta, 2\beta]}(x) \frac{1}{2\pi} \frac{\alpha^2 \sqrt{4\beta^2 - x^2}}{\alpha^4 - (\alpha^2 - \beta^2)x^2} dx, \quad (\text{abs. conti. part}) \\ x_1 &= -\frac{\alpha^2}{\sqrt{\alpha^2 - \beta^2}}, \quad x_2 = \frac{\alpha^2}{\sqrt{\alpha^2 - \beta^2}}, \quad (\text{atomic part}) \\ a &= \begin{cases} \frac{1}{2} \frac{\alpha^2 - 2\beta^2}{\alpha^2 - \beta^2} & \left(0 \leq \frac{\beta^2}{\alpha^2} \leq \frac{1}{2}\right), \\ 0 & \left(\frac{1}{2} \leq \frac{\beta^2}{\alpha^2}\right). \end{cases} \end{aligned}$$

The Cauchy transform of $\nu_{\alpha, \beta}$ is known to be

$$G(z) = \frac{z(\frac{1}{2}\alpha^2 - \beta^2) + \frac{1}{2}\alpha^2 \sqrt{z^2 - 4\beta^2}}{z^2(\alpha^2 - \beta^2) - \alpha^4}.$$

Hence we obtain the explicit form of the measure μ as follows.

$$\begin{aligned} \mu &= \tilde{\mu} + b(\delta_{\xi_1} + \delta_{\xi_2}), \\ d\tilde{\mu}(x) &= \chi_{[-\sqrt{2c}, \sqrt{2c}]}(x) \frac{1}{\pi} \frac{\sqrt{2c - x^2}}{2 + (c-2)x^2} dx, \quad (\text{abs. conti. part}) \\ \xi_1 &= -\sqrt{\frac{2}{2-c}}, \quad \xi_2 = \sqrt{\frac{2}{2-c}}, \quad (\text{atomic part}) \\ b &= \begin{cases} \frac{1-c}{2-c} & (0 \leq c \leq 1), \\ 0 & (1 \leq c). \end{cases} \end{aligned}$$

Now let us remove the assumption of $\|f\|_{L^2} = 1$. For general $f \in \mathcal{H}_1$, put $f = \|f\|_{L^2} \cdot u$ with $\|u\|_{L^2} = 1$, then we have $\langle Q_f^p \rangle = (\|f\|_{L^2})^p \langle Q_u^p \rangle$. Put $\mu_t := \mu_{\chi_{(0,t]}, \mathbf{r}}$, then we see that $\mu_t(dx) = \mu(\frac{dx}{\sqrt{t}})$, and hence μ_t equals to $\nu_{\sqrt{t}, \sqrt{\frac{ct}{2}}}$.

Since the distribution of Q_f depends only on $\|f\|_{L^2}$, the distribution $\mu_{s,t}$ of an increment $Q_t - Q_s$ coincides with μ_{t-s} .

After all we have

Proposition 4.1. *Let $\{Q_t\}_{t \geq 0}$ be the canonical process on a weighted monotone Fock space Φ_w with weight sequence $w = (1, c, c^2, c^3, \dots)$. Then, under the vacuum state ϕ_w , the probability distribution $\mu_{s,t}$ of an increment $Q_t - Q_s$, $0 < s < t$, of the process $\{Q_t\}_{t \geq 0}$ is the distribution of Bożejko-Leinert-Speicher $\nu_{\alpha,\beta}$ with parameter $\alpha = \sqrt{t-s}$ and $\beta = \sqrt{\frac{c(t-s)}{2}}$.*

Remark 4.2. Note that, by the specializations $c = 1$ and $c = 2$ for μ , we get the arcsine law and the Wigner semicircle law, respectively.

$$\begin{cases} c = 1 \Rightarrow p(x) = \frac{1}{\pi \sqrt{2-x^2}} & (\text{arcsine law}) \\ c = 2 \Rightarrow p(x) = \frac{1}{\pi} \sqrt{1 - \left(\frac{x}{2}\right)^2} & (\text{Wigner semi-circle law}) \end{cases}$$

Remark 4.3. The distribution of Bożejko-Leinert-Speicher $\nu_{\alpha,\beta}$ was obtained in [BSp, BLS] as the central limit distribution in the c -free central limit theorem. Also the distribution of its associated Brownian motion $\{\tilde{Q}_t\}_{t \geq 0}$ is given by the distribution of Bożejko-Leinert-Speicher. We remark here that our Brownian motion $\{Q_t\}_{t \geq 0}$ is not isomorphic to the c -free Brownian motion of Bożejko-Speicher $\{\tilde{Q}_t\}_{t \geq 0}$ in [BSp] although they have the same distribution $\mu_t = \nu_{\alpha,\beta}$, for each time $t \geq 0$, with $\alpha = \sqrt{t}$ and $\beta = \sqrt{\frac{ct}{2}}$. The reason is that the correlation function of $\{Q_t\}_{t \geq 0}$ is different from the correlation function of $\{\tilde{Q}_t\}_{t \geq 0}$. For our Brownian motion $\{Q_t\}_{t \geq 0}$, the correlation $\langle Q_s Q_t Q_t Q_s \rangle$ is not symmetric in two variables s and t . Indeed, for $0 < s < t$, we have

$$\langle Q_s Q_t Q_t Q_s \rangle = w_2 \left\{ \frac{1}{2} s^2 + s(t-s) \right\} + w_1^2 s^2, \quad (4.2)$$

whereas we have

$$\langle Q_t Q_s Q_s Q_t \rangle = w_2 \left\{ \frac{1}{2} s^2 \right\} + w_1^2 s^2. \quad (4.3)$$

Hence we know $\langle Q_s Q_t Q_t Q_s \rangle \neq \langle Q_t Q_s Q_s Q_t \rangle$, and recognize the non-symmetry in the roles played by the past s and the future t ($0 \leq s < t$). On the other hand, for the c -free Brownian motion of Bożejko-Speicher $\{\tilde{Q}_t\}_{t \geq 0}$ in [BSp], it can be checked that

$$\langle \tilde{Q}_s \tilde{Q}_t \tilde{Q}_t \tilde{Q}_s \rangle = \langle \tilde{Q}_t \tilde{Q}_s \tilde{Q}_s \tilde{Q}_t \rangle$$

for $0 < s < t$. This concludes that $\{Q_t\}_{t \geq 0}$ is not isomorphic to $\{\tilde{Q}_t\}_{t \geq 0}$. Note that the expressions (4.2) and (4.3) hold for the Brownian motion $\{Q_t^{(w)}\}_{t \geq 0}$ of general

weight sequence w . Now let $\Phi^{(\lambda)}$ be the interacting free Fock space over the one-particle space $\mathcal{H}_1 := L^2(\mathbf{R}_+)$, with the weight sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, such that the probability measure $\mu^{(\lambda)}$ of its canonical operator $\bar{Q}_1$ coincides with μ_w . Such a sequence λ always exists (see [AcB]). Then we can check that

$$\langle \bar{Q}_s \bar{Q}_t \bar{Q}_t \bar{Q}_s \rangle = \langle \bar{Q}_t \bar{Q}_s \bar{Q}_s \bar{Q}_t \rangle .$$

for $0 < s < t$. So we observe that also, for each w , the Brownian motion $\{Q_t^{(w)}\}$ on Φ_w is not isomorphic to the Brownian motion $\{\bar{Q}_t^{(\lambda)}\}$ on the corresponding interacting free Fock space $\Phi^{(\lambda)}$ although they have the same distribution $\nu_{\sqrt{t}, \sqrt{\frac{at}{2}}}$ for each time $t \geq 0$.

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