

Canonical Linear Transformation on Fock Space with an Indefinite Metric

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Abstract: We first construct a Fock space with an indefinite metric $\langle \cdot, \cdot \rangle = (\cdot, \Theta \cdot)$, where Θ is a unitary and hermitian operator. We define a Θ -selfadjoint (Segal's) field $\Phi_\varphi(f)$ which obeys the canonical commutation relations (CCR) with an indefinite metric. We consider a transformation $\Phi_\varphi(f) \rightarrow \Phi_\varphi(Tf)$ (T =real linear) which leaves the CCR invariant. We investigate the implementability of T by an operator on the Fock space.

Let $\mathcal{H}_i$ ($i=+,-$) be Hilbert spaces equipped with usual positive definite hermitian inner product $(\cdot, \cdot)_i$. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a Hilbert space equipped with the inner product $(\cdot, \cdot) = \Sigma_1 (\cdot, \cdot)_i$. Let $P_\pm$ be selfadjoint projections onto $\mathcal{H}_\pm$. Then the Hilbert space equipped with an hermitian inner product $\langle \cdot, \cdot \rangle \equiv (\cdot, \varphi)$ with $\varphi = P_+ - P_-$ is called a "Hilbert space with an indefinite metric".

Let S_n be the usual (n -fold) symmetrization operator on the n -fold tensor product space $\otimes_n \mathcal{H}$, and let

$$\mathcal{F}^{(n)} \equiv S_n[\otimes_n \mathcal{H}]$$

be the n -particle (Fock) space. The total Fock space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}$$

is also given by

$$\mathcal{F}(\mathcal{H}_+) \otimes \mathcal{F}(\mathcal{H}_-),$$

where $\mathcal{F}(\mathcal{H}_+)$ and $\mathcal{F}(\mathcal{H}_-)$ are Fock spaces constructed from $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively. For an operator A on $\mathcal{H}$, define $\Gamma(A)$ by

$$\begin{aligned} \Gamma(A) \mathcal{F}^{(n)} &\subset \mathcal{F}^{(n)}, \\ \Gamma(A)|_{\mathcal{F}^{(n)}} &= A \otimes \dots \otimes A \quad (n\text{-times}). \end{aligned}$$

Then $\Theta \equiv \Gamma(\varphi)$ is again an unitary and hermitian operator on $\mathcal{F}$.

We define an indefinite sesquilinear form in $\mathcal{F}$ by

$$\langle \cdot, \cdot \rangle = (\cdot, \Theta \cdot).$$

The adjoint of A with respect to $\langle \cdot, \cdot \rangle$ is denoted by $A^{(\Theta)}$ and equals $\Theta A^* \Theta$.

Definition 1: (1) For $f \in \mathcal{H}$, the creation operator $a^*(f)$ is defined by

$$\begin{aligned} a^*(f) : \mathcal{F}^{(n)} &\rightarrow \mathcal{F}^{(n+1)} \\ \psi &\mapsto \sqrt{n+1} S_{n+1}[f \otimes \psi]. \end{aligned}$$

(2) For $f \in \mathcal{H}$, define the Θ -selfadjoint (Segal's) field by

$$\Phi_\varphi(f) = \frac{1}{\sqrt{2}} [a^*(f) + [a^*(f)]^{(\Theta)}]^-$$

where $-$ denotes the closure.

Since $[a^*(f)]^{(\Theta)} = [a^*(\varphi f)]^*$ with $\varphi = P_+ - P_-$, Φ_φ is a normal operator. $\{\Phi_\varphi(f)\}$ obey the CCR with an indefinite metric:

$$[\Phi_\varphi(f), \Phi_\varphi(g)] = i \operatorname{Im} \langle \bar{f}, g \rangle = -i \operatorname{Re} (\bar{f}, \varphi J g)$$

where $\bar{f}$ is the complex conjugation of f and $J = \sqrt{-1}$ is a multiplication operator of i .

Definition 2: (1) An invertible real linear transformation T is called φ -symplectic if it satisfies

$$T^{(\varphi)}JT=J$$

where $T^{(\varphi)} = \varphi T^* \varphi$ and T^* is the adjoint of T with respect to $\text{Re}(\cdot, \cdot)$ in $\mathcal{H}$. (If T is complex linear, then this adjoint is equivalent to the usual adjoint with respect to $(\cdot, \cdot)$ in $\mathcal{H}$.)
 (2) $T_{\pm} = \frac{1}{2}[T \pm JTJ^{-1}]$. Especially anti-linear part T_- is called the off-diagonal part of T .

Our purpose is to investigate an operator which is expected to implement $U_T \Phi_{\varphi}(f) U_T^{-1} = \Phi_{\varphi}(Tf)$, and to investigate the new vacuum $\Omega_T = U_T^{-1} \Omega$. Here $\Omega \in \mathcal{F}^{(0)} = \mathbb{C}$ is the Fock vacuum. Since $\Phi_{\varphi}(f) \rightarrow \Phi_{\varphi}(Tf)$ leaves the CCR invariant, one may expect that U_T is a Θ -unitary (bijective Θ -isometric) operator.

Definition 3: (1) T is called Θ -unitarily implementable if there is a Θ -unitary (bijective Θ -isometric) operator U_T which implements $U_T \Phi_{\varphi}(f) U_T^{-1} = \Phi_{\varphi}(Tf)$.

(2) T is called weakly Θ -unitarily implementable if there exist a Θ -isometric (not necessarily bounded) operator U_T^{-1} and a cyclic vector $\Omega_T \in \mathcal{F}$ such that

$$U_T^{-1} P(\Phi_{\varphi}(f)) \Omega = P(\Phi_{\varphi}(Tf)) \Omega_T,$$

where $P(\Phi_{\varphi}(f)) = P(\Phi_{\varphi}(f_1), \dots, \Phi_{\varphi}(f_n))$ is any polynomial of $\{\Phi_{\varphi}(f_i)\}$.

(3) T is called Θ -unitarily quasi-implementable if the Fredholm determinant $\det[1 + T_-^{(\varphi)} T_-]$ uniformly converges to a non-vanishing finite value in $(0, \infty)$.

When $\varphi=1$ (namely when $\Theta=1$), three notions in this definition coincide each other [1,3,4]. For the implementability, the author proved [1]:

Theorem 1: T is Θ -unitarily implementable if and only if T_- is Hilbert-Schmidt and $[T, \varphi] = 0$. In this case U_T is a unitary operator commuting with Θ .

Theorem 2: Let $U_T^{-1} \Omega = \Omega_T \in \mathcal{H}$. Then

- (i) $T_- \in \text{H.S.}$ (H.S. denotes the Hilbert-Schmidt class),
- (ii) $(-\infty, 0]$ is in the resolvent set of $T_+^{(\varphi)} T_+ = 1 + T_-^{(\varphi)} T_-$.

In order to obtain a sufficient condition, we propose a φ -polar decomposition of T , namely a decomposition of T in terms of a φ -selfadjoint operator and a φ -unitary operator.

Theorem 3: Let a φ -symplectic operator T satisfy the conditions in Theorem 2. Then T has a decomposition

$$T = UH,$$

where U is a φ -unitary operator (which commutes with J) and H is a φ -selfadjoint φ -symplectic operator with its spectrum in the right half plane.

Definition 4: φ -selfadjoint φ -symplectic operator S is called a generalized φ -scaling if S leaves K and JK invariant where $\mathcal{H} = K \oplus JK$ and $\oplus$ refers the orthogonality with respect to both $\text{Re}(\cdot, \cdot)$ and $\text{Re}\langle \cdot, \cdot \rangle$.

A generalized φ -scaling S takes the following form on $K \oplus JK$:

$$\begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}.$$

Here $ChC = \bar{h} = h$, where C is a complex conjugation operator:

$$K = \{x \in \mathcal{H} ; Cx = x\}.$$

Is H in Theorem 3 always similar to a generalized φ -scaling S

through suitable φ -unitary operator V ? (This holds if $\varphi=1$ [1,3,4].)

$$H=VS V^{-1}.$$

If this is the case, we have a decomposition

$$T=V_1 S V_2$$

under the conditions of Theorem 2, where V_1 are φ -unitary. But V seems unbounded in general.

For a generalized φ -scaling S , we can obtain rather concrete theorems [1]. It sometimes suffices to consider generalized φ -scalings for physical applications [1,2].

Theorem 4: For a generalized φ -scaling S , if $S_- \in H.S.$, and if $\alpha_r \equiv$ selfadjoint part of φ -selfadjoint operator $h^{-2} > 0$, then

(i) both S and S^{-1} are weakly θ -unitarily implementable.

(ii) The overlap between Ω and Ω_S is given by

$$\begin{aligned} |\langle \Omega, \Omega_S \rangle| &= \det^{-1/4} [1 + S_-^{(\varphi)} S_-] \\ &= \det^{-1/4} [1 + \frac{1}{4} (h - h^{-1})^2]. \end{aligned}$$

This is non-vanishing finite.

Theorem 5: In Theorem 4, if $\inf \text{spec}(\alpha_r) < 0$, then the vector Ω_S which satisfies

$$\langle \Omega_S, P(\Phi_\varphi(f)) \Omega_S \rangle = \langle \Omega, P(\Phi_\varphi(Sf)) \Omega \rangle$$

cannot be in the Fock space: $\|\Omega_S\| = \infty$.

As is well known, when $\varphi=1$, the necessary and sufficient condition for T to be unitarily implementable is $T_- \in H.S.$ Then for $\varphi=1$, the overlap of the vacua does not vanish if and only

if T is unitarily implemented. In fact when $\varphi=1$, we have $T=U_1 SU_2$ where U_1 are unitaries commuting with J . Further since transformations $\Phi_{\varphi=1}(f) \rightarrow \Phi_{\varphi=1}(U_1 f)$ are implemented by unitaries $\Gamma(U_1)$ on the Fock space, we have $U_T = \Gamma(U_1) U_S \Gamma(U_2)$. Then $\Omega_T = \Gamma(U_2)^{-1} \Omega_S$ and $(\Omega, \Omega_T) = (\Omega, \Omega_S)$.

For given S , let $T = V_1 S V_2$ where V_1 are φ -unitaries. Then

$$S_- \in \text{H.S.} \quad \leftrightarrow \quad T_- \in \text{H.S.}$$

and

$$\det[1+S_-^{(\varphi)} S_-] = \det[1+T_-^{(\varphi)} T_-].$$

Since $\Gamma(V_1)$ are not bounded operators, T is not necessarily weakly Θ -unitarily implementable even if S is weakly Θ -unitarily implementable. But the above equation means that the formal overlap $\det^{-1/4}[1+T_-^{(\varphi)} T_-]$ is an invariant quantity under φ -unitaries. Furthermore if $\varphi \neq 1$, $\det^{-1/4}[1+S_-^{(\varphi)} S_-]$ can converge to a non-vanishing (finite) quantity even if $S_- \notin \text{H.S.}$ Then Definition 3 (3) implies that the formally defined overlap is non-vanishing (finite), which is equivalent to the unitarily implementability of S when $\varphi=1$.

(Sketch of the proof of Theorem 4)

Let $K = K_+ \oplus K_-$ ($K_{\pm} = P_{\pm} K$) and let $\{e_i\}$ be complete orthonormal basis in K with respect to both $\text{Re}(\cdot, \cdot)$ and $\text{Re}\langle \cdot, \cdot \rangle$. We use the following unitary transformation W :

$$W\mathcal{K} = L^2(Q; d\mu_0),$$

$$Q = \mathbb{R}^{\infty}, \quad d\mu_0 = \prod_{i=1}^{\infty} \exp[-q_i^2] \frac{dq_i}{\sqrt{\pi}},$$

$$W\Omega = 1,$$

$$W[\Phi_{\varphi}(e_i)]W^{-1} = \begin{cases} q_i & e_i \in K_+ \\ -iq_i & e_i \in K_- \end{cases},$$

$$W[\Phi_\varphi(Je_1)]W^{-1} = \begin{cases} -i\partial/\partial q_1 + iq_1 & e_1 \in K_+ \\ -\partial/\partial q_1 + q_1 & e_1 \in K_- \end{cases}.$$

Note that

$$[a^*(e_1)]^{(\theta)} = \frac{1}{\sqrt{2}}[\Phi_\varphi(e_1) + i\Phi_\varphi(Je_1)].$$

Since the transformed vacuum should satisfy

$$[\Phi_\varphi(S^{-1}e_1) + i\Phi_\varphi(S^{-1}Je_1)]\Omega_S = 0,$$

$$\langle \Omega_S, \Omega_S \rangle = 1,$$

we have [1]

$$\Omega_S = [\det(\alpha)]^{1/4} \exp[-\frac{1}{2}(q, (\alpha-1)q)]$$

where

$$(q, \alpha q) = \sum_{ij} q_i \alpha_{ij} q_j$$

and

$$\alpha_{ij} = (e_i, \psi^* h^{-2} \psi e_j), \quad \psi = P_+ + iP_-.$$

Remark that α is a φ -selfadjoint symmetric matrix.

Under the conditions of Theorem 4, we can prove that $\Omega_S = \Omega_S(q) \in L^2(Q, d\mu_0) = \mathcal{F}$ and the cyclicity of Ω_S [1]. Further

$$\Omega_{S^{-1}} = [\det(\alpha^{-1})]^{1/4} \exp[-\frac{1}{2}(q, (\alpha^{-1}-1)q)].$$

Let $\alpha = \alpha_r + i\alpha_i$ where α_r and α_i are selfadjoint real matrices (this follows from the properties of α). If α_r is positive (then strictly positive since $\alpha_r - 1$ is H.S.), since

$$(\alpha^{-1})_r = (\alpha_r + \alpha_i \alpha_r^{-1} \alpha_i)^{-1},$$

$$(\alpha^{-1})_i = -\alpha_r^{-1} \alpha_i (\alpha_r + \alpha_i \alpha_r^{-1} \alpha_i)^{-1},$$

then $(\alpha^{-1})_r$ is again a (strictly) positive operator. Thus

$\Omega_{S^{-1}} \in \mathcal{F}$. The θ -isometricity of U_S^{-1} follows from

$$\langle \Omega_S, P(\Phi_\varphi(f))\Omega_S \rangle = \langle \Omega, P(\Phi_\varphi(Sf))\Omega \rangle$$

which is proved in [1]. Finally

$$\langle \Omega, \Omega_S \rangle = \langle \Omega_S, \Omega \rangle = \int \Omega_S(q) d\mu_0 = \det^{-1/4} [1 + S_-^{(\varphi)} S_-].$$

□

From the above proof, the reader can guess that $\alpha_r > 0$ is needed to ensure $\|\Omega_S\| < \infty$.

(Sketch of the proof of Theorem 5)

Since α is a φ -selfadjoint operator, α takes the following form on $JK_+ \oplus JK_-$:

$$\begin{pmatrix} (\alpha_r)_{++} & i(\alpha_1)_{+-} \\ i(\alpha_1)_{-+} & (\alpha_r)_{--} \end{pmatrix}, \quad \alpha_{ij} = P_i \alpha P_j.$$

First assume that $f \in JK_+$ be an eigenvector of α_r belonging to the eigenvalue $-\lambda < 0$. Since $\Phi_\varphi(f)$ is selfadjoint,

$$\|\exp[i\Phi_\varphi(f)]\| = 1.$$

Note

$$\langle \Omega_S, \exp[i\Phi_\varphi(f)] \Omega_S \rangle = \langle \Omega, \exp[i\Phi_\varphi(Sf)] \Omega \rangle$$

$$= \exp\left[-\frac{1}{4} \langle Sf, Sf \rangle\right] = \exp\left[-\frac{\lambda}{4} \|f\|^2\right].$$

If $\lambda > 0$, the right hand side can be made arbitrarily large, which contradicts

$$|\langle \Omega_S, \exp[i\Phi_\varphi(f)] \Omega_S \rangle| \leq \|\Omega_S\|^2 < \infty.$$

The case of $f \in JK_-$ is similarly discussed.

□

Our theory can be applied for quantum electrodynamics-type models where $\mathcal{F}(\mathcal{H}_-)$ is the Fock space of the gaugeon (ghost particle which has a negative norm) and $\mathcal{F}(\mathcal{H}_+)$ is the Fock space of physical particles (photon, etc.). In these

models, the Hamiltonian H is expected to be Θ -selfadjoint (namely $H\Theta$ is selfadjoint). As a simple example, let H be Θ -selfadjoint and bilinear with respect to creation and annihilation operators. Let H be diagonalized [1,2] by a transformation defined by $\Phi_\varphi(f) \rightarrow \Phi_\varphi(Tf)$ for any $f \in \mathcal{K}$. Then Ω_T is the physical vacuum of the Hamiltonian. If T is weakly Θ -unitarily implementable, then $\rho_T(\dots) = \langle \Omega_T, \dots \Omega_T \rangle$ is a normalized Θ -selfadjoint linear functional on the field algebra, which typically appears in QED-type models. ρ_T is called a Lorentz state in [2].

Theorem 5 implies that the linear functional ρ_T defined by

$$\rho_T(P(\Phi_\varphi(f))) = \langle \Omega, P(\Phi_\varphi(Tf)) \Omega \rangle$$

cannot be a continuous state in general on the C^* -algebra generated by $\{\exp[i\Phi(f)]; f \in \mathcal{K}\}$, where $\Phi(f)$ is the selfadjoint Segal's field.

The converse problem, namely to obtain a representation (or T) from the expectation values, is the problem which must be solved to construct a QED-type model in a mathematically rigorous way [2]. This corresponds to a generalization of the GNS-construction. This will be discussed someday.

-References-

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