

Classification of ribbon 2-knots with ribbon crossing number up to four

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1 Introduction

A ribbon 2-knot is a knotted 2-sphere in $\mathbb{R}^4$ that bounds a ribbon 3-disk, which is an immersed 3-disk with only ribbon singularities. The ribbon crossing number of a ribbon 2-knot is the minimal number of the ribbon singularities of any ribbon 3-disk bounding the knot [14]. Yasuda has classified ribbon 2-knots with ribbon crossing number up to three in [13] and has enumerated those with ribbon crossing number four in [15]. In this paper we classify these ribbon 2-knots.

Theorem 1. *The number of mutually non-isotopic ribbon 2-knots with ribbon crossing number four is either 111 or 112. Amongst them 9 or 10 knots are positive-amphicheiral. So, if each chiral pair is counted as one knot, the number of ribbon 2-knots with ribbon crossing number four is either 60 or 61; see Table 1.*

Table 1: Numbers of the ribbon 2-knots with ribbon crossing number up to four.					
Ribbon crossing number	0	1	2	3	4
(i) Number of ribbon 2-knots, each chiral pair is counted separately	1	0	3	13	111/112
(ii) Number of ribbon 2-knots, each chiral pair is counted as one knot	1	0	2	7	60/61

The ribbon 2-knots with ribbon crossing number with up to three are completely classified by the Alexander polynomial. However, those with ribbon crossing number four listed in [15] have not been classified. Theorem 1 means that there is an indistinguishable pair of ribbon 2-knots, Y43 and Y46 in Table 3, which are positive-amphicheiral; they have isomorphic knot group. Also, there is one knot, Y112 (the ribbon handlebody is shown in Fig. 1), which had been missed in [15].

Satoh [8] introduced a virtual arc presentation for a ribbon 2-knot. If a ribbon 2-knot K is presented by a virtual arc with n classical crossings, then the ribbon crossing number of K is at most n . In [2] ribbon 2-knots presented by a virtual arc with up to four crossings are enumerated, and in [6] those ribbon 2-knots are classified. There are 24 ribbon 2-knots with ribbon crossing number up to four, which are not presented by a virtual arc with up to four crossings. So, we have only to consider these knots. We have 27 sets of ribbon 2-knots $\mathcal{A}_i$ ($i = 1, 2, \dots, 17$) and $\mathcal{A}_j!$ ($j = 2, 3, 4, 7, 8, 10, 11, 12, 14, 16$), which consist of knots sharing the same Alexander polynomial; $\mathcal{A}_j!$ is the set consisting

of the mirror images of the knots in $\mathcal{A}_j$. The knots in the sets $\mathcal{A}_i$ with $i \leq 13$ (and so $\mathcal{A}_j!$ with $j \leq 12$) have been classified in [6]. Thus, we classify the knots in $\mathcal{A}_i$ with $i = 14, 15, 16, 17$ (Sec. 5). The knots in these sets are ribbon 2-knots of 1-fusion. In order to classify the knots in these sets we use the trace set, or the twisted Alexander polynomial associated to the representations to $\mathrm{SL}(2, \mathbb{C})$. The *trace set* is an invariant defined for a ribbon 2-knot of 1-fusion from the representations of the knot group to $\mathrm{SL}(2, \mathbb{C})$; see Sec. 4 in [7]. For the twisted Alexander polynomial of a ribbon 2-knot, see [4].

This paper is organized as follows: In Secs. 2 and 3, we review a ribbon handlebody presentation of a ribbon 2-knot and the stable transformations for a ribbon handlebody presentation, which were introduced in [3]. In Sec. 4 we give Yasuda's table of the ribbon 2-knots with ribbon crossing number up to four (Tables 2 and 3), which contain the 1-fusion notation of the knots. In Sec. 5 we classify the knots in $\mathcal{A}_i$, $i = 14, 15, 16, 17$, which completes the proof of Theorem 1.

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2 Ribbon handlebody presentation of a ribbon 2-knot

In this section we review a ribbon handlebody presentation of a ribbon 2-knot introduced in [3]. A *ribbon handlebody* $\mathcal{H}$ is a ribbon 2-disk, which is a 2-dimensional handlebody in $\mathbb{R}^3$ consisting of $(m+1)$ 0-handles $D_0, D_1, \dots, D_m$ and m 1-handles $B_1, B_2, \dots, B_m$ such that the preimage of each ribbon singularity consists of an arc in the interior of a 0-handle and a cocore of a 1-handle. We set $\mathcal{H} = \mathcal{D} \cup \mathcal{B}$, where $\mathcal{D} = D_0 \cup D_1 \cup \dots \cup D_m$ and $\mathcal{B} = B_1 \cup B_2 \cup \dots \cup B_m$. We associate to a ribbon handlebody $\mathcal{H}$ an immersed 3-disk $V_{\mathcal{H}}$ in $\mathbb{R}^4$ defined by

$$V_{\mathcal{H}} = \mathcal{D} \times [-2, 2] \cup \mathcal{B} \times [-1, 1]. \quad (1)$$

Then $V_{\mathcal{H}}$ is a ribbon 3-disk for the ribbon 2-knot $K_{\mathcal{H}} = \partial V_{\mathcal{H}}$ in $\mathbb{R}^4$. Conversely, for any ribbon 2-knot K in $\mathbb{R}^4$, there exists a ribbon handlebody $\mathcal{H}$ such that K is ambient isotopic to the associated 2-knot $K_{\mathcal{H}}$; see [1, 10, 12].

We suppose that each 1-handle B_q is the image of an embedding $b_q : I \times I \rightarrow \mathbb{R}^3$, $q = 1, 2, \dots, m$. Let $\beta_q : I \rightarrow \mathbb{R}^3$ be the center line of the 1-handle B_p defined by $\beta_q(t) = b_q(1/2, t)$, which is an oriented path such that

$$\begin{aligned} \beta_q(I) \cap \mathcal{D} &= \{ \beta_q(0), \beta_q(t_{q,1}), \beta_q(t_{q,2}), \dots, \beta_q(t_{q,\ell_q}), \beta_q(1) \}, \\ 0 &< t_{q,1} < t_{q,2} < \dots < t_{q,\ell_q} < 1. \end{aligned} \quad (2)$$

Let $\iota_q, \tau_q, \lambda(q, j)$ ($j = 1, 2, \dots, \ell_q$) be integers in $\{0, 1, \dots, m\}$ determined by

$$\beta_q(0) \in \partial D_{\iota_q}, \quad \beta_q(1) \in \partial D_{\tau_q}, \quad \beta_q(t_{q,j}) \in \mathrm{Int} D_{\lambda(q,j)}, \quad j = 1, 2, \dots, \ell_q. \quad (3)$$

Thus, β is an oriented path joining D_{ι_q} and D_{τ_q} . At the intersection $\beta_q(t_{q,j})$ if β_q passes from the negative side of $D_{\lambda(q,j)}$ through to the positive side we define $\epsilon(q,j) = +1$, and if it passes in the opposite direction we define $\epsilon(q,j) = -1$.

Then for a ribbon handlebody $\mathcal{H}$ we define a *ribbon handlebody presentation* $[X \mid R]$ consisting of:

- $X = \{x_0, x_1, \dots, x_m\}$, where each letter x_q corresponds to the 0-handle D_q ,
- $R = \{\rho_1, \rho_2, \dots, \rho_m\}$, where each relation $\rho_q : x_{\iota_q}^{w_q} = x_{\tau_q}$ (or $x_{\tau_q} = x_{\iota_q}^{w_q}$) corresponds to the 1-handle B_q that joins D_{ι_q} to D_{τ_q} passing through 0-handles according to the word w_q :

$$w_q = x_{\lambda(q,1)}^{\epsilon(q,1)} x_{\lambda(q,2)}^{\epsilon(q,2)} \cdots x_{\lambda(q,\ell_q)}^{\epsilon(q,\ell_q)}. \quad (4)$$

In particular, if $\beta_q(I) \cap \mathcal{D} = \{\beta_q(0), \beta_q(1)\}$, then $\rho_q : x_{\iota_q} = x_{\tau_q}$.

For a ribbon handlebody presentation $P = [X \mid R]$, $X = \{x_0, x_1, \dots, x_m\}$ and $R = \{\rho_1, \rho_2, \dots, \rho_m\}$ with $\rho_q : x_{\iota_q}^{w_q} = x_{\tau_q}$, $w_q \in F[X]$, we can associate an oriented labelled tree $\tilde{P} = (X, E, \lambda)$, where X is a set of vertices, E is a set of oriented edges:

$$E = \{\overrightarrow{x_{\iota_q} x_{\tau_q}} \mid q = 1, \dots, m\}, \quad (5)$$

and $\lambda : E \rightarrow F[X]$ is a labeling function defined by $\lambda(\overrightarrow{x_{\iota_q} x_{\tau_q}}) = w_q$.

Conversely, for an oriented labeled tree (X, E, λ) as above, we obtain a unique ribbon handlebody presentation $P = [X \mid R]$ and also the associated ribbon 2-knot of m -fusion, which we denote by K_P ; cf. Proposition 3.3 in [3]. Note that the knot group of K_P , $\pi_1(\mathbb{R}^4 - K_P)$, is presented by $\langle X \mid \tilde{R} \rangle$, where $\tilde{R}$ is a set of relations $\{\tilde{\rho}_1, \tilde{\rho}_2, \dots, \tilde{\rho}_m\}$ with $\tilde{\rho}_q : w_q^{-1} x_{\iota_q} w_q = x_{\tau_q}$; see [11].

Therefore, any ribbon 2-knot with ribbon crossing number r is obtained from an oriented labeled tree (X, E, λ) as above such that $\sum_{q=1}^m \ell_q = r$, where ℓ_q is the word length of the word w_q as in Eq. (4).

3 Stable transformations of a ribbon handlebody presentation

Let $P = [X \mid R]$ be a ribbon handlebody presentation, where $X = \{x_0, x_1, \dots, x_m\}$ and $R = \{\rho_1, \dots, \rho_m\}$ with

$$\rho_q : x_{\iota_q}^{w_q} = x_{\tau_q}, \quad w_q = x_{\lambda(q,1)}^{\epsilon(q,1)} x_{\lambda(q,2)}^{\epsilon(q,2)} \cdots x_{\lambda(q,\ell_q)}^{\epsilon(q,\ell_q)}, \quad \epsilon(q,s) = \pm 1. \quad (6)$$

We call the following transformations of a ribbon handlebody presentation *stable transformations*:

- S1. Replace $\rho_q : x_{\iota_q}^{w_q} = x_{\tau_q}$ by $x_{\iota_q} = x_{\tau_q}^{(w_q^{-1})}$.
- S2. Replace $\rho_q : x_{\iota_q}^{w_q} = x_{\tau_q}$ by either $x_{\iota_q}^{x_{\tau_q}^\epsilon w_q} = x_{\tau_q}$ or $x_{\iota_q}^{w_q x_{\tau_q}^\epsilon} = x_{\tau_q}$, $\epsilon = \pm 1$.
- S3. Add a generator y and a relation $y = x_p^w$ or $x_p = y^w$, where w is a word in $x_0, x_1, \dots, x_m$.

S3'. Inverse transformation of S3.

- S4. (i) Suppose $\tau_p = \iota_q$. Replace either $\rho_p : x_{\iota_p}^{w_p} = x_{\tau_p}$ or $\rho_q : x_{\iota_q}^{w_q} = x_{\tau_q}$ by $x_{\iota_p}^{w_p w_q} = x_{\tau_q}$.
(ii) Suppose $\iota_p = \iota_q$. Replace $\rho_p : x_{\iota_p}^{w_p} = x_{\tau_p}$ by $x_{\tau_q}^{w_q^{-1} w_p} = x_{\tau_p}$.
(iii) Suppose $\tau_p = \tau_q$. Replace $\rho_p : x_{\iota_p}^{w_p} = x_{\tau_p}$ by $x_{\iota_p}^{w_p w_q^{-1}} = x_{\iota_q}$.
S5. (i) Suppose $\lambda(p, s) = \tau_q$. Replace $x_{\lambda(p, s)} (= x_{\tau_q})$ in w_p in ρ_p by $w_q^{-1} x_{\iota_q} w_q$.
(ii) Suppose $\lambda(p, s) = \iota_q$. Replace $x_{\lambda(p, s)} (= x_{\iota_q})$ in w_p in ρ_p by $w_q x_{\tau_q} w_q^{-1}$.

Then we have the following (Proposition 4.1 in [3]):

Proposition 2. *Suppose that ribbon handlebody presentations P and P' are related by a finite sequence of stable transformations S1–S5. Then, the associated ribbon 2-knots K_P and $K_{P'}$ are ambient isotopic.*

We denote by

$$R(p_1, q_1, \dots, p_n, q_n), \quad p_1, q_1, \dots, p_n, q_n \in \mathbb{Z}, \quad (7)$$

a ribbon 2-knot of 1-fusion, which is presented by the ribbon handlebody presentation

$$[x, y \mid x = y^w \ (w = x^{p_1} y^{q_1} \dots x^{p_n} y^{q_n})]. \quad (8)$$

cf. [5, Sect. 2]. Then, by the transformation S1 we have:

$$R(p_1, q_1, \dots, p_n, q_n) \approx R(-q_n, -p_n, \dots, -q_1, -p_1) \quad (9)$$

$$R(p_1, q_1, \dots, p_n, q_n)! \approx R(-p_1, -q_1, \dots, -p_n, -q_n) \approx R(q_n, p_n, \dots, q_1, p_1), \quad (10)$$

where $K \approx K'$ denotes that the two 2-knots K and K' are ambient isotopic and $K!$ the mirror image of K .

Example 3. The ribbon 2-knot Y43 presented by

$$P(\text{Y43}) = [x_1, x_2, x_3 \mid \rho_1 : x_1^{x_2 x_1} = x_2, \rho_2 : x_1^{x_3 x_2} = x_3]. \quad (11)$$

is isotopic to the ribbon 2-knot of 1-fusion $R(1, 1, -1, -1, -1, -1, 1, 1)$. Thus, by Eqs. (9) and (10) Y43 is positive-amphicheiral.

Proof By the transformation S5(ii), we replace x_1 in the power of ρ_1 with $x_3 x_2 x_3 x_2^{-1} x_3^{-1}$ coming from ρ_2 . Then $P(\text{Y43})$ is deformed into

$$P(\text{Y43})_1 = [x_1, x_2, x_3 \mid \rho'_1 : x_1^{x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2, \rho_2 : x_1^{x_3 x_2} = x_3]. \quad (12)$$

By the transformation S4(ii), we replace ρ'_1 by $\rho''_1 : x_3^{(x_3 x_2)^{-1} x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2$. Then $P(\text{Y43})_1$ is deformed into

$$P(\text{Y43})_2 = [x_1, x_2, x_3 \mid \rho''_1 : x_3^{x_2^{-1} x_3^{-1} x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2, \rho_2 : x_1^{x_3 x_2} = x_3]. \quad (13)$$

By the transformation S3', $P(\text{Y43})_2$ is deformed into

$$P(\text{Y43})_3 = [x_2, x_3 \mid x_3^{x_2^{-1} x_3^{-1} x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2], \quad (14)$$

which presents $R(-1, -1, 1, 1, 1, 1, -1, -1) (\approx R(1, 1, -1, -1, -1, -1, 1, 1))$. $\square$

4 Yasuda's Table

Yasuda enumerated ribbon 2-knots with ribbon crossing number up to three in [13] and ribbon 2-knots with ribbon crossing four in [15]. He claims that any ribbon 2-knots with ribbon crossing number up to four is presented by one of the following ribbon handlebody presentations:

$$P_1(w) = [x_1, x_2 \mid x_1^w = x_2]; \quad (15)$$

$$P_2(w_1, w_2) = [x_1, x_2, x_3 \mid x_1^{w_1} = x_2, x_1^{w_2} = x_3]; \quad (16)$$

$$P_3(w_1, w_2, w_3) = [x_1, x_2, x_3, x_4 \mid x_1^{w_1} = x_2, x_2^{w_2} = x_3, x_3^{w_3} = x_4]; \quad (17)$$

$$P_4(w_1, w_2, w_3) = [x_1, x_2, x_3, x_4 \mid x_1^{w_1} = x_2, x_1^{w_2} = x_3, x_1^{w_3} = x_4]; \quad (18)$$

$$P_5(w_1, w_2, w_3, w_4) = [x_1, x_2, x_3, x_4, x_5 \mid x_1^{w_1} = x_2, x_1^{w_2} = x_3, x_1^{w_3} = x_4, x_2^{w_4} = x_5]; \quad (19)$$

$$P_6(w_1, w_2, w_3, w_4) = [x_1, x_2, x_3, x_4, x_5 \mid x_1^{w_1} = x_2, x_1^{w_2} = x_3, x_1^{w_3} = x_4, x_1^{w_4} = x_5]. \quad (20)$$

Remark 4. A ribbon 2-knot with ribbon crossing number up to four presented by the ribbon handlebody presentation

$$[x_1, x_2, x_3, x_4, x_5 \mid x_1^{w_1} = x_2, x_2^{w_2} = x_3, x_3^{w_3} = x_4, x_4^{w_4} = x_5], \quad (21)$$

$w_i \in F[x_1, x_2, x_3, x_4, x_5]$, is transformed into a ribbon 2-knot presented by one of the ribbon handlebody presentations (15)–(18).

In a similar way to Example 3, we can deform a ribbon 2-knot with up to four ribbon crossing by a finite sequence of stable transformations S1–S5 (Proposition 2) into one of the following two types:

- Type 1: a ribbon 2-knot of 1-fusion.
- Type 2: a composition of two ribbon 2-knots of 1-fusion.

In order to determine the type of a ribbon 2-knot we use the following proposition (Proposition 3.1 in [6]). Indeed, the fundamental group of a Type 2 ribbon 2-knot with ribbon crossing number up to four is isomorphic to the free product $\mathbb{Z}_3 * \mathbb{Z}_3$ (Proposition 3.2 in [6]).

Proposition 5. *The fundamental group of the 2-fold cover of S^4 branched over a ribbon 2-knot of 1-fusion K is the finite cyclic group whose order is the determinant of K , $|\Delta_K(-1)|$.*

Table 2 lists the ribbon 2-knots with ribbon crossing number up to three given by [13], and Table 3 lists the ribbon 2-knots with ribbon crossing four given by Yasuda [15]. Each column in Tables 2 and 3 shows as follows:

- The first column, Name, shows the names of the ribbon 2-knots:
 - (i) The names Ym_n , Ym_n^* ($m = 2, 3$) in Table 2 denote the knots m_n , m_n^* with ribbon crossing number m in [13]; Ym_n^* is the mirror image of Ym_n .
 - (ii) The name Yn ($1 \leq n \leq 111$) in Table 3 denotes the ribbon 2-knot K_n^2 with ribbon crossing number four in [15].

- The column, C, shows the chirality of the ribbon 2-knots:
 - (i) The symbol “a” means that the ribbon 2-knot is positive-amphicheiral.
 - (ii) In Table 3 the mirror image knot is listed.
- The column, Presentation, shows a ribbon handlebody presentation of the ribbon 2-knot: P_i is one of the ribbon handlebody presentations (15)–(20), and the symbols j and $\bar{j}$ ($j = 1, 2, 3, 4, 5$) denote the letters x_j and x_j^{-1} , respectively. For example, $P_2(21, 32)$ for the knot Y43 in Table 3 means the presentation Eq. (11) in Example 3.
- The column, Type, shows the type of the ribbon 2-knot:
 - (i) A Type 1 ribbon 2-knot is presented by a 1-fusion notation $R(p_1, q_1, \dots, p_m, q_m)$.
 - (ii) A Type 2 ribbon 2-knot is presented by a composition $R(\epsilon_1, \epsilon_2) \# R(\epsilon_3, \epsilon_4)$, $\epsilon_i = \pm 1$.
- The column, $\Delta(t)$, shows the normalized Alexander polynomial of the ribbon 2-knot in the abbreviated form: $(c_{-m} c_{-m+1} \dots c_{-1} [c_0] c_1 \dots c_{n-1} c_n) = \sum_{i=-m}^n c_i t^i$, $c_i \in \mathbb{Z}$. We normalize the Alexander polynomial of a ribbon 2-knot $\Delta(t) \in \mathbb{Z}[t^{\pm 1}]$, so that $\Delta(1) = 1$ and $(d/dt)\Delta(1) = 0$; cf. [1].
- The column, Det, shows the determinant of the ribbon 2-knot, which is given by $|\Delta(-1)|$.
- The column, Set, shows the name of the set of the ribbon 2-knots sharing the same Alexander polynomial; $\mathcal{A}_i!$ denotes the set of the mirror images of the knots in $\mathcal{A}_i$. For example, $\mathcal{A}_2 = \{\text{Y3}_1^*, \text{Y27}\}$, $\mathcal{A}_2! = \{\text{Y3}_1, \text{Y34}\}$, and the knots in the sets $\mathcal{A}_i$, $i = 1, 5, 6, 9, 13, 15, 17$, have reciprocal Alexander polynomials, and so we do not consider the set of mirror images. The sets $\mathcal{A}_i$ with $i \leq 13$ are the same sets as in [6].

The knot Y112 is missed in [15], which has the same Alexander polynomial as Y109; the ribbon handlebodies are shown as in Fig. 1.

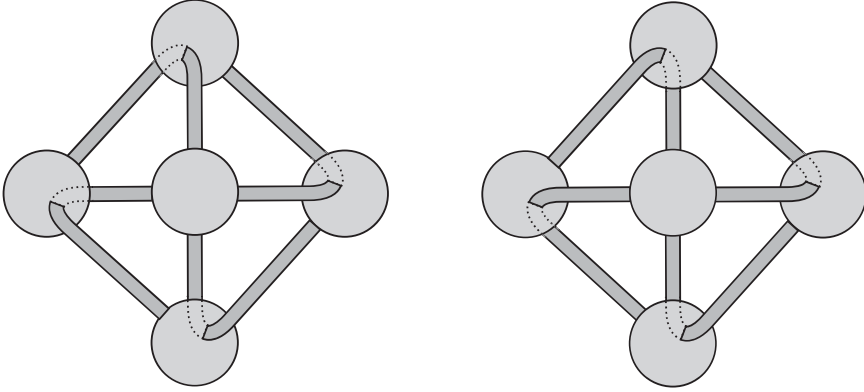


Figure 1: Ribbon handlebodies presenting Y109 and Y112.

Table 2: Ribbon 2-knots with up to three crossings.

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y0	a		Trivial knot	$([1])$	1	$\mathcal{A}_1$
Y2.1	a	$P_1(21)$	$R(1, 1)$	$(1 \begin{bmatrix} -1 & 1 \end{bmatrix})$	3	
Y2.2		$P_1(2\bar{1})$	$R(1, -1)$	$([0] \begin{bmatrix} 2 & -1 \end{bmatrix})$	3	$\mathcal{A}_3!$
Y2.2*		$P_1(\bar{2}1)$	$R(-1, 1)$	$(-1 \begin{bmatrix} 2 & [0] \end{bmatrix})$	3	$\mathcal{A}_3$
Y3.1		$P_1(211)$	$R(1, 2)$	$(1 \begin{bmatrix} -1 & [0] & 1 \end{bmatrix})$	1	$\mathcal{A}_2!$
Y3.1*		$P_1(221)$	$R(-1, -2)$	$(1 \begin{bmatrix} [0] & -1 & 1 \end{bmatrix})$	1	$\mathcal{A}_2$
Y3.2		$P_1(2\bar{1}\bar{1})$	$R(1, -2)$	$([0] \begin{bmatrix} 1 & 1 & -1 \end{bmatrix})$	1	
Y3.2*		$P_1(\bar{2}11)$	$R(-1, 2)$	$(-1 \begin{bmatrix} 1 & 1 & [0] \end{bmatrix})$	1	
Y3.3		$P_2(31, 2)$	$R(-1, 1, 1, 1)$	$(-1 \begin{bmatrix} 2 & -1 & [1] \end{bmatrix})$	5	
Y3.3*		$P_2(\bar{3}\bar{1}, \bar{2})$	$R(1, -1, -1, -1)$	$([1] \begin{bmatrix} -1 & 2 & -1 \end{bmatrix})$	5	
Y3.4		$P_2(31, \bar{2})$	$R(1, 1, -1, 1)$	$(1 \begin{bmatrix} -2 & [2] \end{bmatrix})$	5	
Y3.4*		$P_2(3\bar{1}, \bar{2})$	$R(-1, -1, 1, -1)$	$([2] \begin{bmatrix} -2 & 1 \end{bmatrix})$	5	
Y3.5	a	$P_2(3\bar{1}, \bar{2})$	$R(1, -1, -1, 1)$	$(-1 \begin{bmatrix} [3] & -1 \end{bmatrix})$	5	
Y3.6		$P_3(3, 4, 2)$	$R(-1, 1, -1, -1, 1, 1)$	$(1 \begin{bmatrix} -3 & 3 & [0] \end{bmatrix})$	5	
Y3.6*		$P_3(\bar{3}, \bar{4}, \bar{2})$	$R(1, -1, 1, 1, -1, -1)$	$([0] \begin{bmatrix} 3 & -3 & 1 \end{bmatrix})$	5	
Y3.7		$P_3(3, 4, \bar{2})$	$R(-1, 1, 1, -1, 1, 1)$	$(-1 \begin{bmatrix} 3 & [-2] & 1 \end{bmatrix})$	5	
Y3.7*		$P_3(3, \bar{4}, \bar{2})$	$R(1, -1, -1, 1, -1, -1)$	$(1 \begin{bmatrix} -2 & 3 & -1 \end{bmatrix})$	5	

Table 3: Ribbon 2-knots with four crossings.

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y1	Y7	$P_1(2111)$	$R(1, 3)$	$(1 \ -1 \ 0 \ [0] \ 1)$	3	
Y2	a	$P_1(2121)$	$R(1, 1, 1, 1)$	$(1 \ -1 \ [1] \ -1 \ 1)$	5	
Y3	Y8	$P_1(2\bar{1}\bar{1}\bar{1})$	$R(3, -1)$	$([0] \ 1 \ 0 \ 1 \ -1)$	3	$\mathcal{A}_4!$
Y4	Y9	$P_1(2\bar{1}2\bar{1})$	$R(1, -1, 1, -1)$	$([0] \ 0 \ 3 \ -2)$	5	
Y5	a	$P_1(2211)$	$R(2, 2)$	$(1 \ 0 \ [-1] \ 0 \ 1)$	1	
Y6	Y10	$P_1(22\bar{1}\bar{1})$	$R(2, -2)$	$([0] \ 0 \ 2 \ 0 \ -1)$	1	
Y7	Y1	$P_1(2221)$	$R(3, 1)$	$(1 \ [0] \ 0 \ -1 \ 1)$	3	
Y8	Y3	$P_1(\bar{2}111)$	$R(-1, 3)$	$(-1 \ 1 \ 0 \ 1 \ [0])$	3	$\mathcal{A}_4$
Y9	Y4	$P_1(\bar{2}1\bar{2}1)$	$R(-1, 1, -1, 1)$	$(-2 \ 3 \ 0 \ [0])$	5	
Y10	Y6	$P_1(\bar{2}\bar{2}11)$	$R(-2, 2)$	$(-1 \ 0 \ 2 \ 0 \ [0])$	1	
Y11	Y18	$P_2(231, \bar{2})$	$R(2, 1, -1, 1)$	$(1 \ 0 \ [-2] \ 2)$	3	
Y12	Y17	$P_2(23\bar{1}, \bar{2})$	$R(2, 1, -1, -1)$	$([1] \ 1 \ -2 \ 1)$	3	
Y13	Y16	$P_2(2\bar{3}1, \bar{2})$	$R(2, -1, -1, 1)$	$([0] \ 2 \ -1)$	3	$\mathcal{A}_3!$
Y14	Y15	$P_2(2\bar{3}\bar{1}, \bar{2})$	$R(2, -1, -1, -1)$	$([0] \ 1 \ 0 \ 1 \ -1)$	3	$\mathcal{A}_4!$
Y15	Y14	$P_2(\bar{2}31, 2)$	$R(-2, 1, 1, 1)$	$(-1 \ 1 \ 0 \ 1 \ [0])$	3	$\mathcal{A}_4$
Y16	Y13	$P_2(\bar{2}3\bar{1}, 2)$	$R(1, -1, -1, 2)$	$(-1 \ 2 \ [0])$	3	$\mathcal{A}_3$
Y17	Y12	$P_2(\bar{2}\bar{3}1, 2)$	$R(-2, -1, 1, 1)$	$(1 \ -2 \ 1 \ [1])$	3	
Y18	Y11	$P_2(\bar{2}\bar{3}\bar{1}, 2)$	$R(-2, -1, 1, -1)$	$(2 \ [-2] \ 0 \ 1)$	3	
Y19	Y31	$P_2(311, 2)$	$R(-1, 1, 1, 2)$	$(-1 \ 2 \ -1 \ 0 \ [1])$	3	
Y20	Y33	$P_2(313, 2)$	$R(-1, 1, 1, 1, -1, 1)$	$(-1 \ 2 \ -2 \ 2 \ [0])$	7	
Y21	Y32	$P_2(313, \bar{2})$	$R(1, 1, -1, 1, 1, 1)$	$(1 \ -2 \ 2 \ [-1] \ 1)$	7	
Y22	Y30	$P_2(3\bar{1}3, 2)$	$R(1, -1, -1, 1, -1, 1)$	$(-2 \ 4 \ [-1])$	7	
Y23	a	$P_2(3\bar{1}\bar{3}, \bar{2})$	$R(1, 1, -1, -1, 1, 1)$	$(2 \ [-3] \ 2)$	7	$\mathcal{A}_5$
Y24	Y37	$P_2(321, 2)$	$R(-1, 1, 2, 1)$	$(-1 \ 2 \ 0 \ [-1] \ 1)$	1	
Y25	Y36	$P_2(32\bar{1}, 2)$	$R(1, -2, -1, 1)$	$(-1 \ [2] \ 1 \ -1)$	1	
Y26	Y35	$P_2(3\bar{2}1, \bar{2})$	$R(1, 1, -2, 1)$	$([1])$	1	$\mathcal{A}_1$
Y27	Y34	$P_2(3\bar{2}\bar{1}, \bar{2})$	$R(1, 2, -1, -1)$	$(1 \ [0] \ -1 \ 1)$	1	$\mathcal{A}_2$
Y28	Y39	$P_2(331, 2)$	$R(-1, 2, 1, 1)$	$(-1 \ 1 \ 1 \ -1 \ [1])$	1	
Y29	Y38	$P_2(33\bar{1}, \bar{2})$	$R(1, 2, -1, 1)$	$(1 \ -1 \ -1 \ 2])$	1	
Y30	Y22	$P_2(\bar{3}1\bar{3}, \bar{2})$	$R(1, -1, -1, 1, 1, -1)$	$([-1] \ 4 \ -2)$	7	
Y31	Y19	$P_2(\bar{3}\bar{1}\bar{1}, \bar{2})$	$R(2, 1, 1, -1)$	$([1] \ 0 \ -1 \ 2 \ -1)$	3	
Y32	Y21	$P_2(\bar{3}\bar{1}\bar{3}, 2)$	$R(1, 1, 1, -1, 1, 1)$	$(1 \ [-1] \ 2 \ -2 \ 1)$	7	
Y33	Y20	$P_2(\bar{3}\bar{1}\bar{3}, \bar{2})$	$R(1, -1, 1, 1, 1, -1)$	$([0] \ 2 \ -2 \ 2 \ -1)$	7	
Y34	Y27	$P_2(\bar{3}21, 2)$	$R(-1, -1, 2, 1)$	$(1 \ -1 \ [0] \ 1)$	1	$\mathcal{A}_2!$
Y35	Y26	$P_2(\bar{3}2\bar{1}, 2)$	$R(1, -2, 1, 1)$	$([1])$	1	$\mathcal{A}_1$
Y36	Y25	$P_2(\bar{3}\bar{2}1, \bar{2})$	$R(1, -1, -2, 1)$	$(-1 \ 1 \ [2] \ -1)$	1	
Y37	Y24	$P_2(\bar{3}\bar{2}\bar{1}, \bar{2})$	$R(1, 2, 1, -1)$	$(1 \ [-1] \ 0 \ 2 \ -1)$	1	
Y38	Y29	$P_2(\bar{3}\bar{3}\bar{1}, 2)$	$R(-1, -2, 1, -1)$	$([2] \ -1 \ -1 \ 1)$	1	
Y39	Y28	$P_2(\bar{3}\bar{3}\bar{1}, \bar{2})$	$R(1, -2, -1, -1)$	$([1] \ -1 \ 1 \ 1 \ -1)$	1	

Table 3: Ribbon 2-knots with four crossings (cont'd).

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y40	a	$P_2(21, 31)$	$R(1, 1) \# R(1, 1)$	$(1 \ -2 \ [3] \ -2 \ 1)$	9	$\mathcal{A}_6$
Y41	Y42	$P_2(21, 3\bar{1})$	$R(1, 1) \# R(1, -1)$	$([2] \ -3 \ 3 \ -1)$	9	$\mathcal{A}_7$
Y42	Y41	$P_2(21, \bar{3}1)$	$R(1, 1) \# R(-1, 1)$	$(-1 \ 3 \ -3 \ [2])$	9	$\mathcal{A}_7!$
Y43	a	$P_2(21, 32)$	$R(-1, -1, 1, 1, 1, 1, -1, -1)$	$(1 \ -2 \ [3] \ -2 \ 1)$	9	$\mathcal{A}_6$
Y44	Y45	$P_2(21, 3\bar{2})$	$R(1, -1, 1, 1, -1, 1, 1, -1)$	$([2] \ -3 \ 3 \ -1)$	9	$\mathcal{A}_7$
Y45	Y44	$P_2(21, \bar{3}2)$	$R(-1, 1, 1, -1, 1, 1, -1, 1)$	$(-1 \ 3 \ -3 \ [2])$	9	$\mathcal{A}_7!$
Y46	a	$P_2(21, 3\bar{2})$	$R(1, 1, 1, -1, -1, 1, 1, 1)$	$(1 \ -2 \ [3] \ -2 \ 1)$	9	$\mathcal{A}_6$
Y47	Y52	$P_2(2\bar{1}, 3\bar{1})$	$R(1, -1) \# R(1, -1)$	$([0] \ 0 \ 4 \ -4 \ 1)$	9	$\mathcal{A}_8$
Y48	a	$P_2(2\bar{1}, \bar{3}1)$	$R(1, -1) \# R(-1, 1)$	$(-2 \ [5] \ -2)$	9	$\mathcal{A}_9$
Y49	Y53	$P_2(2\bar{1}, 32)$	$R(-1, -1, 1, 1, 1, -1, -1, -1)$	$([2] \ -3 \ 3 \ -1)$	9	$\mathcal{A}_7$
Y50	Y55	$P_2(2\bar{1}, 3\bar{2})$	$R(1, -1, 1, 1, -1, -1, 1, -1)$	$([0] \ 0 \ 4 \ -4 \ 1)$	9	$\mathcal{A}_8$
Y51	Y54	$P_2(2\bar{1}, \bar{3}2)$	$R(-1, 1, 1, -1, 1, -1, -1, 1)$	$(-2 \ [5] \ -2)$	9	$\mathcal{A}_9$
Y52	Y47	$P_2(2\bar{1}, \bar{3}1)$	$R(-1, 1) \# R(-1, 1)$	$(1 \ -4 \ 4 \ 0 \ [0])$	9	$\mathcal{A}_8!$
Y53	Y49	$P_2(\bar{2}1, 32)$	$R(-1, -1, -1, 1, 1, 1, 1, -1, -1)$	$(-1 \ 3 \ -3 \ [2])$	9	$\mathcal{A}_7!$
Y54	Y51	$P_2(\bar{2}1, 3\bar{2})$	$R(1, -1, -1, 1, -1, 1, 1, -1)$	$(-2 \ [5] \ -2)$	9	$\mathcal{A}_9$
Y55	Y50	$P_2(\bar{2}1, \bar{3}2)$	$R(-1, 1, -1, -1, 1, 1, -1, 1)$	$(1 \ -4 \ 4 \ 0 \ [0])$	9	$\mathcal{A}_8!$
Y56	Y58	$P_3(3, \bar{1}\bar{4}, \bar{2})$	$R(1, 1, -1, 1, -1, -1)$	$(2 \ [-3] \ 2)$	7	$\mathcal{A}_5$
Y57	a	$P_3(3, \bar{1}\bar{4}, 2)$	$R(1, -1, -1, -1, -1, 1)$	$(-1 \ 2 \ [-1] \ 2 \ -1)$	7	
Y58	Y56	$P_3(\bar{3}, \bar{1}\bar{4}, 2)$	$R(-1, -1, 1, -1, 1, 1)$	$(2 \ [-3] \ 2)$	7	$\mathcal{A}_5$
Y59	Y66	$P_4(31, 4, 2)$	$R(-1, -1, 1, 1, -1, 1, 1, 1)$	$(1 \ -3 \ 3 \ -1 \ [1])$	9	
Y60	Y64	$P_4(31, 4, \bar{2})$	$R(1, -1, -1, 1, 1, 1, -1, 1)$	$(-1 \ 3 \ -3 \ [2])$	9	$\mathcal{A}_7!$
Y61	Y63	$P_4(31, \bar{4}, \bar{2})$	$R(1, 1, -1, 1, 1, -1, -1, 1)$	$(1 \ -3 \ [4] \ -1)$	9	$\mathcal{A}_{10}!$
Y62	Y65	$P_4(3\bar{1}, 4, 2)$	$R(-1, -1, 1, 1, -1, 1, 1, -1)$	$(1 \ -3 \ [4] \ -1)$	9	$\mathcal{A}_{10}!$
Y63	Y61	$P_4(3\bar{1}, 4, \bar{2})$	$R(1, -1, -1, 1, 1, 1, -1, -1)$	$(-1 \ [4] \ -3 \ 1)$	9	$\mathcal{A}_{10}$
Y64	Y60	$P_4(3\bar{1}, \bar{4}, \bar{2})$	$R(1, 1, -1, 1, 1, -1, -1, -1)$	$([2] \ -3 \ 3 \ -1)$	9	$\mathcal{A}_7$
Y65	Y62	$P_4(\bar{3}1, 4, \bar{2})$	$R(1, 1, -1, -1, 1, -1, -1, 1)$	$(-1 \ [4] \ -3 \ 1)$	9	$\mathcal{A}_{10}$
Y66	Y59	$P_4(\bar{3}1, \bar{4}, 2)$	$R(1, 1, -1, -1, 1, -1, -1, -1)$	$([1] \ -1 \ 3 \ -3 \ 1)$	9	
Y67	Y82	$P_3(23, 4, 1)$	$R(1, -1, -1, -1, 1, 1, 1, 1, -1, -1)$	$(1 \ [-2] \ 4 \ -3 \ 1)$	11	$\mathcal{A}_{11}!$
Y68	Y81	$P_3(23, 4, \bar{1})$	$R(1, 1, -1, -1, 1, 1, 1, -1, -1, -1)$	$([0] \ 3 \ -4 \ 3 \ -1)$	11	
Y69	Y80	$P_3(23, \bar{4}, 1)$	$R(1, -1, -1, -1, -1, 1, 1, 1, -1, -1)$	$(-1 \ 3 \ [-3] \ 3 \ -1)$	11	$\mathcal{A}_{13}$
Y70	Y79	$P_3(23, \bar{4}, \bar{1})$	$R(1, 1, -1, -1, -1, 1, 1, -1, -1, -1)$	$(1 \ [-2] \ 4 \ -3 \ 1)$	11	$\mathcal{A}_{11}!$
Y71	Y78	$P_3(\bar{2}\bar{3}, 4, 1)$	$R(-1, -1, 1, -1, 1, 1, -1, 1, 1, -1)$	$(2 \ [-4] \ 4 \ -1)$	11	$\mathcal{A}_{12}$
Y72	Y77	$P_3(\bar{2}\bar{3}, 4, \bar{1})$	$R(-1, 1, 1, -1, 1, 1, -1, -1, 1, -1)$	$([-1] \ 5 \ -4 \ 1)$	11	
Y73	Y76	$P_3(\bar{2}\bar{3}, \bar{4}, 1)$	$R(-1, -1, 1, -1, -1, 1, -1, 1, 1, -1)$	$(-2 \ 5 \ [-3] \ 1)$	11	
Y74	Y75	$P_3(\bar{2}\bar{3}, \bar{4}, \bar{1})$	$R(-1, 1, 1, -1, -1, 1, -1, -1, 1, -1)$	$(2 \ [-4] \ 4 \ -1)$	11	$\mathcal{A}_{12}$
Y75	Y74	$P_3(\bar{2}\bar{3}, 4, 1)$	$R(1, -1, -1, 1, 1, -1, 1, 1, -1, 1)$	$(-1 \ 4 \ [-4] \ 2)$	11	$\mathcal{A}_{12}!$
Y76	Y73	$P_3(\bar{2}\bar{3}, 4, \bar{1})$	$R(1, 1, -1, 1, 1, -1, 1, -1, -1, 1)$	$(1 \ [-3] \ 5 \ -2)$	11	
Y77	Y72	$P_3(\bar{2}\bar{3}, \bar{4}, 1)$	$R(1, -1, -1, 1, -1, -1, 1, 1, -1, 1)$	$(1 \ -4 \ 5 \ [-1])$	11	
Y78	Y71	$P_3(\bar{2}\bar{3}, \bar{4}, \bar{1})$	$R(1, 1, -1, 1, -1, -1, 1, -1, -1, 1)$	$(-1 \ 4 \ [-4] \ 2)$	11	$\mathcal{A}_{12}!$
Y79	Y70	$P_3(\bar{2}\bar{3}, 4, 1)$	$R(-1, -1, 1, 1, 1, -1, -1, 1, 1, 1)$	$(1 \ -3 \ 4 \ [-2] \ 1)$	11	$\mathcal{A}_{11}$
Y80	Y69	$P_3(\bar{2}\bar{3}, 4, \bar{1})$	$R(-1, 1, 1, 1, 1, -1, -1, -1, 1, 1)$	$(-1 \ 3 \ [-3] \ 3 \ -1)$	11	$\mathcal{A}_{13}$
Y81	Y68	$P_3(\bar{2}\bar{3}, \bar{4}, 1)$	$R(-1, -1, 1, 1, -1, -1, -1, 1, 1, 1)$	$(-1 \ 3 \ -4 \ 3 \ [0])$	11	
Y82	Y67	$P_3(\bar{2}\bar{3}, \bar{4}, \bar{1})$	$R(-1, 1, 1, 1, -1, -1, -1, -1, 1, 1)$	$(1 \ -3 \ 4 \ [-2] \ 1)$	11	$\mathcal{A}_{11}$

Table 3: Ribbon 2-knots with four crossings (cont'd).

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y83	Y90	$P_3(43, 1, 2)$	$R(1, 1, 1, 1, -1, -1, -1, 1)$	$(1 -2 [3] -2 1)$	9	$\mathcal{A}_6$
Y84	Y89	$P_3(43, 1, \bar{2})$	$R(1, -1, 1, 1, -1, 1, -1, -1)$	$([3] -4 2)$	9	
Y85	Y88	$P_3(43, \bar{1}, 2)$	$R(1, 1, 1, -1, -1, -1, -1, 1)$	$(-1 [3] -2 2 -1)$	9	
Y86	Y87	$P_3(43, \bar{1}, \bar{2})$	$R(1, -1, 1, -1, -1, 1, -1, -1)$	$([1] -2 4 -2)$	9	
Y87	Y86	$P_3(\bar{43}, 1, 2)$	$R(-1, 1, -1, 1, 1, -1, 1, 1)$	$(-2 4 -2 [1])$	9	
Y88	Y85	$P_3(\bar{43}, 1, \bar{2})$	$R(-1, -1, -1, 1, 1, 1, 1, -1)$	$(-1 2 -2 [3] -1)$	9	
Y89	Y84	$P_3(\bar{43}, \bar{1}, 2)$	$R(-1, 1, -1, -1, 1, -1, 1, 1)$	$(2 -4 [3])$	9	
Y90	Y83	$P_3(\bar{43}, \bar{1}, \bar{2})$	$R(-1, -1, -1, -1, 1, 1, 1, -1)$	$(1 -2 [3] -2 1)$	9	$\mathcal{A}_6$
Y91	Y106	$P_5(5, 2, 3, 4)$	$R(-1, -1, 1, -1, -1, 1, 1, -1, 1, 1, -1, 1)$	$(-1 4 -5 [3])$	13	$\mathcal{A}_{14}$
Y92	Y105	$P_5(5, 2, 3, \bar{4})$	$R(-1, -1, 1, 1, -1, 1, 1, -1, 1, -1, -1, 1)$	$(1 -4 [6] -2)$	13	
Y93	Y104	$P_5(5, 2, \bar{3}, 4)$	$R(-1, 1, 1, -1, -1, -1, 1, 1, 1, 1, -1, -1)$	$(1 -3 [5] -3 1)$	13	$\mathcal{A}_{15}$
Y94	Y103	$P_5(5, 2, \bar{3}, \bar{4})$	$R(-1, 1, 1, 1, -1, -1, 1, 1, 1, -1, -1, -1)$	$(-1 [4] -4 3 -1)$	13	$\mathcal{A}_{16}$
Y95	Y102	$P_5(5, \bar{2}, 3, 4)$	$R(1, -1, -1, -1, 1, 1, 1, -1, -1, 1, 1, 1)$	$(1 -3 [5] -3 1)$	13	$\mathcal{A}_{15}$
Y96	Y101	$P_5(5, \bar{2}, 3, \bar{4})$	$R(1, -1, -1, 1, 1, 1, 1, -1, -1, -1, 1, 1)$	$(-1 [4] -4 3 -1)$	13	$\mathcal{A}_{16}$
Y97	Y100	$P_5(5, \bar{2}, \bar{3}, 4)$	$R(1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1, -1)$	$([3] -5 4 -1)$	13	$\mathcal{A}_{14}!$
Y98	Y99	$P_5(5, \bar{2}, \bar{3}, \bar{4})$	$R(1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, -1)$	$([1] -2 5 -4 1)$	13	
Y99	Y98	$P_5(\bar{5}, 2, 3, 4)$	$R(-1, -1, 1, -1, -1, 1, -1, -1, 1, 1, -1, 1)$	$(1 -4 5 -2 [1])$	13	
Y100	Y97	$P_5(\bar{5}, 2, 3, \bar{4})$	$R(-1, -1, 1, 1, -1, 1, -1, -1, 1, -1, -1, 1)$	$(-1 4 -5 [3])$	13	$\mathcal{A}_{14}$
Y101	Y96	$P_5(\bar{5}, 2, \bar{3}, 4)$	$R(-1, 1, 1, -1, -1, -1, -1, 1, 1, 1, -1, -1)$	$(-1 3 -4 [4] -1)$	13	$\mathcal{A}_{16}!$
Y102	Y95	$P_5(\bar{5}, 2, \bar{3}, \bar{4})$	$R(-1, 1, 1, 1, -1, -1, -1, 1, 1, -1, -1, -1)$	$(1 -3 [5] -3 1)$	13	$\mathcal{A}_{15}$
Y103	Y94	$P_5(\bar{5}, \bar{2}, 3, 4)$	$R(1, -1, -1, -1, 1, 1, -1, -1, -1, 1, 1, 1)$	$(-1 3 -4 [4] -1)$	13	$\mathcal{A}_{16}!$
Y104	Y93	$P_5(\bar{5}, \bar{2}, 3, \bar{4})$	$R(1, -1, -1, 1, 1, 1, -1, -1, -1, -1, 1, 1)$	$(1 -3 [5] -3 1)$	13	$\mathcal{A}_{15}$
Y105	Y92	$P_5(\bar{5}, \bar{2}, \bar{3}, 4)$	$R(1, 1, -1, -1, 1, -1, -1, 1, -1, -1, 1, -1)$	$(-2 [6] -4 1)$	13	
Y106	Y91	$P_5(\bar{5}, \bar{2}, \bar{3}, \bar{4})$	$R(1, 1, -1, 1, 1, -1, -1, 1, -1, -1, 1, -1)$	$([3] -5 4 -1)$	13	$\mathcal{A}_{14}!$
Y107	Y111	$P_6(3, 4, 5, 2)$	$R(-1, -1, 1, -1, -1, 1, 1, 1, -1, -1, 1, 1, -1, 1)$	$(-1 4 -6 4 [0])$	15	
Y108	Y110	$P_6(3, 4, 5, \bar{2})$	$R(1, -1, -1, -1, 1, 1, -1, 1, 1, -1, -1, 1, 1, 1)$	$(1 -4 6 [-3] 1)$	15	
Y109	a	$P_6(3, 4, \bar{5}, \bar{2})$	$R(1, 1, -1, -1, 1, -1, -1, 1, 1, 1, -1, 1, 1, -1)$	$(-1 4 [-5] 4 -1)$	15	$\mathcal{A}_{17}$
Y110	Y108	$P_6(3, \bar{4}, \bar{5}, \bar{2})$	$R(-1, 1, 1, 1, -1, -1, 1, -1, -1, 1, 1, -1, -1, -1)$	$(1 [-3] 6 -4 1)$	15	
Y111	Y107	$P_6(\bar{3}, \bar{4}, \bar{5}, \bar{2})$	$R(1, 1, -1, 1, 1, -1, -1, -1, 1, 1, -1, -1, 1, -1)$	$([0] 4 -6 4 -1)$	15	
Y112	a	$P_6(3, \bar{4}, 5, \bar{2})$	$R(1, -1, -1, 1, 1, 1, -1, 1, 1, -1, -1, -1, 1, 1)$	$(-1 4 [-5] 4 -1)$	15	$\mathcal{A}_{17}$

5 Classification of the knots

The ribbon 2-knots in the sets $\mathcal{A}_i$, $i = 1, 2, \dots, 13$, have been classified in [6] except for the pair Y43 and Y46 in $\mathcal{A}_6$, which have isomorphic knot group; see Sect. 7 in [6], where $Y43 = R_{8,6}^8$ and $Y46 = R_{8,1}^8$. In this section we classify the ribbon 2-knots in each of the sets $\mathcal{A}_i$, $i = 14, 15, 16, 17$.

5.1 Classification of the knots in $\mathcal{A}_{14}$

The set $\mathcal{A}_{14}$ consists of the two knots Y91 and Y100, which share the same Alexander polynomial $-t^{-3} + 4t^{-2} - 5t^{-1} + 3$. Since they have different trace sets as shown in Table 4, we obtain $Y91 \not\approx Y100$.

Table 4: Trace sets of the knots in $\mathcal{A}_{14}$, $\mathcal{A}_{16}$, and $\mathcal{A}_{17}$.

Set	Knot	Trace set
$\mathcal{A}_{14}$	Y91	$\{0, 0, 0, 0, 0, 0, (\delta + \epsilon\sqrt{5})/2 \mid \delta, \epsilon = \pm 1\}$
	Y100	$\{0, 0, 0, 0, 0, 0, \delta\sqrt{5(3 + \epsilon\sqrt{3})/6} \mid \delta, \epsilon = \pm 1\}$
$\mathcal{A}_{16}$	Y94	$\{0, 0, 0, 0, 0, 0\}$
	Y96	$\{0, 0, 0, 0, 0, 0, \pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$
$\mathcal{A}_{17}$	Y109	$\left\{ \begin{array}{l} \mathbb{C} - \{\pm\sqrt{3}\}, \pm\sqrt{2}, 0, 0, 0, 0, 0, 0, \\ \mathbb{C} - \{\pm\sqrt{5}\}, \mathbb{C} - \{\pm\sqrt{5}\}, \pm 1, \\ (\delta + \epsilon\sqrt{13})/2, (\delta + \epsilon\sqrt{13})/2 \ (\delta, \epsilon = \pm 1), \\ \pm\beta_1, \pm\beta_2, \pm\beta_3, \pm\beta_4 \end{array} \right\}$
	Y112	$\left\{ \begin{array}{l} \mathbb{C} - \{\pm\sqrt{3}\}, \pm\sqrt{2}, 0, 0, 0, 0, 0, 0, \\ \gamma_1, \gamma_2, \gamma_3, \gamma_4 \end{array} \right\}$

- The numbers α_k , $k = 1, 2, 3$, are the roots of the cubic equation $1 - x - 2x^2 + x^3 = 0$ with $-1 < \alpha_1 < 0 < \alpha_2 < 1, 2 < \alpha_3 < 3$.
- The complex numbers β_k , $k = 1, 2, 3, 4$, are the roots of the quartic equation $5 - 2x - 4x^2 + x^3 + x^4 = 0$; $\beta_1, \beta_2 = 1.25 \pm 0.27i$, $\beta_3, \beta_4 = -1.75 \pm 0.17i$.
- The complex numbers γ_k , $k = 1, 2, 3, 4$, are the roots of the quartic equation $5 - 4x^2 + x^4 = 0$; $\gamma_k = \pm 1.46 \pm 0.34i$.

5.2 Classification of the knots in $\mathcal{A}_{15}$

The set $\mathcal{A}_{15}$ consists of the four knots Y93, Y95, Y102(= Y95!), and Y104(= Y93!), which share the same Alexander polynomial $t^{-2} - 3t^{-1} + 5 - 3t + t^2$. Table 5 lists the trace sets of the irreducible representations to $SL(2, \mathbb{C})$ of the knot groups of Y93 and Y95, and the associated twisted Alexander polynomials, which show these four knots are mutually non-isotopic. In fact, since the twisted Alexander polynomials are not reciprocal, the knots Y93 and Y95 are not positive-amhicheiral.

Table 5: Twisted Alexander polynomials of Y93 and Y95 in $\mathcal{A}_{15}$.

Set	Knot	$(s + s^{-1}, u)$	Twisted Alexander polynomial
$\mathcal{A}_{15}$	Y93	$\left(\frac{\epsilon}{\sqrt{2}}, \frac{3}{2}\right) (\epsilon = \pm 1)$	$1 - \epsilon\sqrt{2}t + \frac{5}{2}t^2 - \epsilon\frac{3}{\sqrt{2}}t^3 + \frac{5}{2}t^4 - \epsilon\sqrt{2}t^5 + t^6$
		$(0, \alpha_1)$	$1 + \beta_1t^2 + \gamma_2t^4 + t^6$
		$(0, \alpha_2)$	$1 + \beta_5t^2 + \gamma_1t^4 + t^6$
		$(0, \alpha_3)$	$1 + \beta_4t^2 + \gamma_1t^4 + t^6$
		$(0, \alpha_4)$	$1 + \beta_2t^2 + \gamma_3t^4 + t^6$
		$(0, \alpha_5)$	$1 + \beta_3t^2 + \gamma_2t^4 + t^6$
	Y95	$(0, \alpha_6)$	$1 + \beta_6t^2 + \gamma_3t^4 + t^6$
		$(0, \alpha_1)$	$1 + \beta_1t^2 + \beta_3t^4 + t^6$
		$(0, \alpha_2)$	$1 + \beta_2t^2 + \beta_1t^4 + t^6$
		$(0, \alpha_3)$	$1 + \beta_5t^2 + \beta_4t^4 + t^6$
		$(0, \alpha_4)$	$1 + \beta_6t^2 + \beta_2t^4 + t^6$
		$(0, \alpha_5)$	$1 + \beta_6t^2 + \beta_4t^4 + t^6$
		$(0, \alpha_6)$	$1 + \beta_5t^2 + \beta_3t^4 + t^6$

- The numbers α_k , $k = 1, \dots, 6$, are the roots of the 6th order equation $13 - 91x + 182x^2 - 156x^3 + 65x^4 - 13x^5 + x^6 = 0$ with $0 < \alpha_1 < 0.5 < \alpha_2 < 1 < \alpha_3 < 2 < \alpha_4 < 3 < \alpha_5 < 3.5 < \alpha_6 < 4$.
- The numbers β_k , $k = 1, \dots, 6$, are the roots of the 6th order equation $-1 - 81x + 201x^2 - 178x^3 + 73x^4 - 14x^5 + x^6 = 0$ with $-1 < \beta_1 < 0 < \beta_2 < 1$, $2 < \beta_3 < 2.4 < \beta_4 < 2.8 < \beta_5 < 3$, $5 < \beta_6 < 6$.
- The numbers γ_k , $k = 1, 2, 3$, are the roots of the cubic equation $-5 + 12x - 7x^2 + x^3 = 0$ with $0 < \gamma_1 < 1 < \gamma_2 < 2$, $4 < \gamma_3 < 5$.

5.3 Classification of the knots in $\mathcal{A}_{16}$

The set $\mathcal{A}_{16}$ consists of the two knots Y94 and Y96, which share the same Alexander polynomial $-t^{-1} + 4 - 4t + 3t^2 - t^3$. Since they have different trace sets as shown in Table 4, we obtain $Y94 \not\approx Y96$.

5.4 Classification of the knots in $\mathcal{A}_{17}$

The set $\mathcal{A}_{17}$ consists of the two knots Y109 and Y112, which share the same Alexander polynomial $-t^{-2} + 4t^{-1} - 5 + 4t - t^2$. Since they have different trace sets as shown in Table 4, we obtain $Y109 \not\approx Y112$.

Remark 6. According to Toshio Sumi [9], we can also distinguish the knots Y109 and Y112 in the following ways:

- They have distinct twisted Alexander polynomials associated to the nonabelian representations to $SL(2, 2)$ as listed in Table 6.
- They have distinct numbers of the irreducible representations to $SL(2, 7)$.

Table 6: Twisted Alexander polynomials of the knots in $\mathcal{A}_{17}$. .

Set	Knot	$\rho : \pi K \rightarrow \mathrm{SL}(2, 2)$	$\Delta_{K, \rho}$
$\mathcal{A}_{17}$	Y109	$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$1 + t^6$
	Y112	$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$1 + t^2 + t^4 + t^6$

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