

Semi-continuity of total dimension divisors for ℓ -adic sheaves (research announcement)

By

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Abstract

This is an announcement of the joint article [HY] with E. Yang, based on a talk given in December 2015 at RIMS. In this article, we give a generalization of Deligne and Laumon's lower semi-continuity property for Swan conductors of ℓ -adic sheaves on relative curves to higher relative dimensions in a geometric situation. We outline the main results and ideas in this report.

§ 1. Semi-continuity of Swan conductors

Let S be an excellent noetherian scheme, $f: X \rightarrow S$ a separated and smooth morphism of relative dimension 1, D a closed subscheme of X which is finite and flat over S , $U = X - D$ the complement and $j: U \rightarrow X$ the canonical injection. Let ℓ be a prime number invertible in S , Λ a finite field of characteristic ℓ and $\mathcal{F}$ a locally constant and constructible sheaf of Λ -modules on U .

Let s be a point of S . We say that a geometric point $\bar{s}$ of S above s is *algebraic* if $\bar{s}$ is the spectrum of an algebraic closure of $k(s)$. We denote by $X_{\bar{s}}$ and $D_{\bar{s}}$ the fibers of $f: X \rightarrow S$ and $f|_D: D \rightarrow S$ at an algebraic geometric point $\bar{s}$ of S , respectively. For each point $x \in D_{\bar{s}}$, we denote by $\text{Sw}_x(j_!\mathcal{F}|_{X_{\bar{s}}})$ the classical Swan conductor of the sheaf $j_!\mathcal{F}|_{X_{\bar{s}}}$ at x , which is an integer number [6]. We define the *total dimension* of $j_!\mathcal{F}|_{X_{\bar{s}}}$ at x the sum of $\text{Sw}_x(j_!\mathcal{F}|_{X_{\bar{s}}})$ and $\text{rank}(\mathcal{F})$ and denote it by $\text{dimtot}_x(j_!\mathcal{F}|_{X_{\bar{s}}})$. The sum

$$(1.1) \quad \sum_{x \in D_{\bar{s}}} \text{dimtot}_x(j_!\mathcal{F}|_{X_{\bar{s}}})$$

Received March 5, 2016. Revised August 4, 2016.

2010 Mathematics Subject Classification(s): 14F20, 11S15.

Key Words: Total dimension divisor, Semi-continuity.

Supported by JSPS Postdoctoral Fellowship for Foreign Researchers.

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is independent of the choice of $\bar{s}$ above s . It defines a function $\varphi: S \rightarrow \mathbb{Z}$. The semi-continuity property of Swan conductors of Deligne and Laumon is the following:

Theorem 1.1 ([3, 2.1.1]). *We take the notation and assumptions above.*

- (1) *The function $\varphi: S \rightarrow \mathbb{Z}$ is constructible and lower semi-continuous.*
- (2) *The morphism $f: X \rightarrow S$ is universally locally acyclic with respect to $j_! \mathcal{F}$ if $\varphi: S \rightarrow \mathbb{Z}$ is locally constant.*

If the morphism $f: X \rightarrow S$ has relative dimension ≥ 1 , Saito generalized the semi-continuity property (Theorem 1.1) in terms of the total dimension of vanishing cycles in the following way:

Let k be a perfect field of characteristic $p > 0$, $f: X \rightarrow S$ a morphism of k -schemes of finite type that factors through a k -scheme Y , D a closed subset of X and $U = X - D$ the complement. We put $h: X \rightarrow Y$ and put $g: Y \rightarrow S$. We assume that $g: Y \rightarrow S$ is a smooth relative curve and that D is closed subset of X which is quasi-finite over S . Let ℓ be a prime number different from p , $\mathcal{K}$ a constructible sheaf of $\mathbb{F}_\ell$ -modules on X . We assume that $f: X \rightarrow S$ is locally acyclic with respect to $\mathcal{K}$ and $h|_U: U \rightarrow Y$ is locally acyclic with respect to $\mathcal{K}|_U$. Let $\bar{s}$ be an algebraic geometric point of S and $h_{\bar{s}}: X_{\bar{s}} \rightarrow Y_{\bar{s}}$ the fiber of $h: X \rightarrow Y$ at $\bar{s}$. Notice that $Y_{\bar{s}}$ is a smooth curve over an algebraically closed field. Let x be a closed point of $D_{\bar{s}}$ and we denote by $\phi_x(\mathcal{K}|_{X_{\bar{s}}}, h_{\bar{s}})$ the stalk of the vanishing cycle complex at x . We denote by

$$(1.2) \quad \text{dimtot}(\phi_x(\mathcal{K}|_{X_{\bar{s}}}, h_{\bar{s}}))$$

the alternating sum of the total dimension of every cohomology of $\phi_x(\mathcal{K}|_{X_{\bar{s}}}, h_{\bar{s}})$. It defines a function $\varphi_{\mathcal{K},h}: D \rightarrow \mathbb{Z}$.

Theorem 1.2 ([5, Proposition 2.16]). *The function $\varphi_{\mathcal{K},h}: D \rightarrow \mathbb{Z}$ is constructible. If $f|_D: D \rightarrow S$ is étale, the function*

$$(1.3) \quad h_*(\varphi_{\mathcal{K},h}): S \rightarrow \mathbb{Z}, \quad s \mapsto \sum_{x \in D_{\bar{s}}} \varphi_{\mathcal{K},h}(x)$$

is locally constant on S .

Theorem 1.2 is used to define the coefficients of characteristic cycles of ℓ -adic sheaves (cf. [5]). In [HY], we give another generalization of Theorem 1.1. We consider that D is Cartier divisor of X relative to S . When $f: X \rightarrow S$ has higher relative dimension, we replace the target of the function $\varphi: S \rightarrow \mathbb{Z}$ in Theorem 1.1 by a family of divisors on the fibers of $f: X \rightarrow S$ that we call the *total dimension divisors*. We formulate our main result by a semi-continuity property of this family of divisors. First of all, we recall the ramification theory of Abbes and Saito which is used to define the notion of total dimension divisor.

§ 2. Ramification theory of Abbes and Saito

In the following of this report, we fix a prime number $p > 0$, an algebraically closed field k of characteristic p and a prime number ℓ different from p .

In this section, let K be a complete discrete valuation field of characteristic $p > 0$, $\overline{K}$ a separable closure of K , G_K the Galois group of $\overline{K}$ over K , $\mathcal{O}_K$ the integer ring of K and F the residue field of $\mathcal{O}_K$. Abbes and Saito defined a decreasing filtration $\{G_K^r\}_{r \in \mathbb{Q}_{\geq 1}}$ of G_K which is called the *ramification filtration* [1]. For each $r \geq 1$, we put $G_K^{r+} = \overline{\bigcup_{s>r} G_K^s}$. Then $\{G_K^r\}_{r \in \mathbb{Q}_{\geq 1}}$ has the following properties (cf. [1, 2, 4]):

- (i) G_K^1 is the inertia subgroup of G_K ;
- (ii) G_K^{1+} is wild inertia subgroup of G_K ;
- (iii) for each $r \in \mathbb{Q}_{>1}$, the quotient G_K^r/G_K^{r+} is abelian and killed by p ;
- (iv) if the residue field F is perfect, the ramification filtration coincides with the canonical upper numbering filtration shifted by one.

Let M be a finitely generated $\mathbb{F}_\ell$ -module with a continuous G_K -action. Then, the module M has a decomposition $M = \bigoplus_{r \geq 1} M^{(r)}$, such that $M^{(1)} = M^{G_K^{1+}}$ and that, for each $r > 1$,

$$(2.1) \quad (M^{(r)})^{G_K^r} = 0 \quad \text{and} \quad (M^{(r)})^{G_K^{r+}} = M^{(r)}.$$

The decomposition is called the *slope decomposition*. The *total dimension* of M is defined by

$$(2.2) \quad \dim_{\text{tot}_K} M = \sum_r r \cdot \dim_{\mathbb{F}_\ell} M^{(r)}$$

It coincides with the classical total dimension (§1) if the residue field F is perfect (cf. (iv)).

Let Y be a smooth k -scheme, E a reduced Cartier divisor of Y , $\{E_i\}_{i \in I}$ the set of irreducible components of E , $V = Y - E$ the complement and $u : V \rightarrow Y$ the canonical injection. We denote by ξ_i a geometric generic point of E_i ($i \in I$), by $Y_{(\xi_i)}$ the strict localization of Y at ξ_i , by K_i the fraction field of $Y_{(\xi_i)}$, by $\overline{K}_i$ a separable closure of K_i and by η_i the generic point of $Y_{(\xi_i)}$. Notice that $Y_{(\xi_i)}$ is a spectrum of henselian discrete valuation ring. Let $\mathcal{G}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$ -modules on V . The restriction $\mathcal{G}|_{\eta_i}$ associates to a finitely generated $\mathbb{F}_\ell$ -module with continuous $\text{Gal}(\overline{K}_i/K_i)$ -action. We define the *total dimension divisor* $\text{DT}_Y(u_! \mathcal{G})$ of $u_! \mathcal{G}$ by

$$(2.3) \quad \text{DT}_Y(u_! \mathcal{G}) = \sum_{i \in I} \dim_{\text{tot}_{K_i}} (\mathcal{G}|_{\eta_i}) \cdot E_i$$

It has integer coefficients (cf. [4, Proposition 3.10]). When Y is a smooth curve, the degree of $\mathrm{DT}_Y(u_! \mathcal{G})$ is the sum of total dimensions of $\mathcal{G}$ at critical points (cf. (1.1)).

§ 3. Main results

Let S be an irreducible k -scheme of finite type, $f: X \rightarrow S$ a smooth morphism of finite type, $\{D\}_{i \in I}$ a set of effective Cartier divisors of X relative to S , D the sum of all D_i ($i \in I$), $U = X - D$ the complement and $j: U \rightarrow X$ the canonical injection. For each $i \in I$, we assume that D_i is irreducible and $f|_{D_i}: D_i \rightarrow S$ is smooth. For an algebraic geometric point $\bar{t} \rightarrow S$, we denote by $D_{\bar{t}}$ the pull-back of the relative Cartier divisor D on the smooth scheme $X_{\bar{t}}$. Let $\mathcal{F}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$ -modules on U .

The main result of [HY] is the following theorem:

Theorem 3.1 ([HY, Theorem 4.5]). *Let $\bar{\eta}$ be an algebraic geometric generic point of S . Let R be the unique Cartier divisor of X supported on D such that $R_{\bar{\eta}} = \mathrm{DT}_{X_{\bar{\eta}}}(j_! \mathcal{F}|_{X_{\bar{\eta}}})$. Then,*

1. *For each algebraic geometric point $\bar{t} \rightarrow S$, the difference $R_{\bar{t}} - \mathrm{DT}_{X_{\bar{t}}}(j_! \mathcal{F}|_{X_{\bar{t}}})$ is an effective Cartier divisor on $X_{\bar{t}}$.*
2. *There exists an open dense subset W of S such that, for any algebraic geometric point $\bar{t} \rightarrow W$, we have $R_{\bar{t}} = \mathrm{DT}_{X_{\bar{t}}}(j_! \mathcal{F}|_{X_{\bar{t}}})$.*

The first step of proving Theorem 3.1 is the following proposition that compares the pull-back of the total dimension divisor of an ℓ -adic sheaf and the total dimension divisor of the pull-back of the sheaf.

Proposition 3.2 ([HY, Proposition 4.2]). *Let Y be a smooth k -scheme, E an reduced Cartier divisor on Y , $V = Y - E$ the complement, $u: V \rightarrow Y$ the canonical injection and $\mathcal{G}$ a locally constant and constructible sheaf of $\mathbb{F}_\ell$ -modules on V . Let C be a smooth k -curve and $\iota: C \rightarrow Y$ an immersion. We assume that C intersects E properly at a closed point $y \in Y$. Then, we have*

$$(3.1) \quad (\mathrm{DT}_Y(u_! \mathcal{G}), C)_y \geq \dim_{\mathrm{tot}_y}(u_! \mathcal{G}|_C).$$

This inequality extends a similar one due to Saito [4, Proposition 3.9], where we need extra conditions that E is smooth at x and that the ramification of $\mathcal{G}$ along E behaves well at x . The second step is to reduce Theorem 3.1 to the relative curve case using Proposition 3.2 and a part of [4, Proposition 3.9] saying that (3.1) is an equality under some mild ramification and transversal conditions. Using Theorem 1.1, we obtain the following two properties:

- (i) For any closed point $t \in S$, the difference $R_t - \mathrm{DT}_{X_t}(j!\mathcal{F}|_{X_t})$ is an effective divisor on X_t ;
- (ii) There exists an open dense subset W of S such that, for any closed point $t \in W$, the difference $\mathrm{DT}_{X_t}(j!\mathcal{F}|_{X_t}) - R_t$ is an effective divisor on X_t .

Finally, we obtain Theorem 3.1 as a consequence of properties (i) and (ii).

Acknowledgement

The author of the announcement would like to express his gratitude to the organizers of the conference "Algebraic Number Theory and Related Topics 2015" held at RIMS for providing him the opportunity to present this topic. The author would like to express his deepest thanks to Professor Takeshi Saito for many discussions during the preparation of this work. The author is supported by JSPS Postdoctoral Fellowship for Foreign Researchers during his stay at the University of Tokyo. The author is grateful to these institutions.

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