

Asymptotic Analyses for the Emden-Fowler Equation $-\Delta u = \lambda e^u$

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§1. Introduction.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the smooth boundary $\partial\Omega$, and λ be a positive constant. For the classical solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of the Emden-Fowler equation

$$(1.1) \quad -\Delta u = \lambda e^u \quad (\text{in } \Omega), \quad u = 0 \quad (\text{on } \partial\Omega)$$

we have proven the following theorem in [3], where

$$(1.2) \quad \Sigma = \int_{\Omega} \lambda e^u dx.$$

Theorem 1. As $\lambda \downarrow 0$, the values $\{\Sigma\}$ accumulates to $8\pi m$ for some $m = 0, 1, \dots, +\infty$. The solutions $\{u\}$ behave as follows:

- (a) If $m = 0$, then $\|u\|_{\infty(\Omega)} \rightarrow 0$
- (b) If $0 < m < +\infty$, then there exists a set of m -points $\mathcal{S} = \{x_1^*, \dots, x_m^*\} \subset \Omega$ such that $\|u\|_{\infty(\bar{\Omega} \setminus \mathcal{S})} \in O(1)$ and $u|_{\mathcal{S}} \rightarrow +\infty$.
- (c) If $m = +\infty$, then $u(x) \rightarrow +\infty$ ($x \in \Omega$).

In the case (b), the limiting function of $\{u\}$, the singular limit, is given as

$$(1.3) \quad u_0(x) = 8\pi \sum_{j=1}^m G(x, x_j^*)$$

and the blow-up points $\mathcal{S}$ is located as

$$\frac{1}{2} \nabla R(y^*) + \sum_{l \neq j} \nabla_l G(x_l^*, y^*) = 0 \quad (1 \leq j \leq n),$$

where $G(x, y)$ denotes the Green function of $-\Delta$ in Ω under the homogeneous Dirichlet boundary condition and $R(x) = [G(x, y) + \frac{1}{2\lambda} \log |x - y|]_{y=\alpha}$ is the Robin function. In other words, $(x_1^*, \dots, x_m^*) \in \Omega \times \dots \times \Omega$ ($x_l \neq x_{l'}$ for $l \neq l'$) is a critical point of the function

$$(1.4) \quad K(x_1, \dots, x_m) = \frac{1}{2} \sum_{j=1}^m R(y_j) + \sum_{l \neq l'} G(x_l, x_{l'})$$

in $\Omega \times \dots \times \Omega$.

The purpose of the present article is to study the inverse problem. The following theorem can be proven:

Theorem 2. Suppose that Ω is simply connected, and let $(x_1^*, \dots, x_m^*)$ be a non-degenerate critical point of $K(x_1, \dots, x_m)$. Then there exists a family of classical solutions $\{u\}$ of (1.1) with sufficient small $\lambda > 0$ such that

$$(1.5) \quad u(x) \longrightarrow u_0(x) \quad (x \in \Omega)$$

as $\lambda \downarrow 0$, where $u_0(x)$ is the function in (1.3).

Thus, we have almost classified the solutions of (1) when $\lambda > 0$ is sufficiently small. As for the global bifurcation diagram, see

[4].

§2. Outline of Proof

Theorem 2 with $m=1$ has been proven for a generic case by V. H. Weston and J. L. Moseley by means of a Newton's iteration scheme ([5], [2]). What we shall do is that their fixed point equation can be solved by a simple iteration scheme without Newton's one, reducing their assumptions, and that the same argument on a Riemannian surface with m -leaves produces the multi-point blow-up solutions.

First, we observe that the singular limit $u_0(x)$ in (1.3) with $m=1$, is related to the conformal mapping $h: \Omega \rightarrow D \equiv \{ |z| < 1 \}$ satisfying $h(x_j^*) = 0$ when $\Omega \subset \mathbb{R}^2$ is simply connected. In fact we have

$$(2.1) \quad e^{-u_0} = |R|^2.$$

The condition $\nabla R(x_j^*) = 0$ is equivalent to $R''(x_j^*) = 0$. Here, we deduce the equation satisfied the function $v = e^{-u}$, that is,

$$(2.2) \quad v \Delta v - |\nabla v|^2 = \frac{\lambda}{2} \quad (\text{in } \Omega), \quad v = 1 \quad (\text{on } \partial\Omega).$$

We want to construct a family of non-negative solutions $\{v\}$ for small $\lambda > 0$ so that

$$(2.3) \quad v \rightarrow |R|^2 \quad \text{as } \lambda \downarrow 0.$$

Introducing the inverse mapping $g = h^{-1}: D \rightarrow \Omega$, we pull-back these relations into D so that $V = g^*v$ satisfies

$$(2.4) \quad V\Delta V - |\nabla V|^2 = \frac{1}{2}|g'|^2 \quad (\text{in } D), \quad V = 1 \quad (\text{on } \partial D)$$

with

$$(2.5) \quad V \rightarrow |z|^2 \quad \text{as } \lambda \downarrow 0.$$

We note the relation

$$(2.6) \quad g''(0) = 0$$

implied by $h''(\lambda_1^*) = 0$.

The left-hand side of (2.4) is a quadratic form of V , which we write as $Q(V)$ to define

$$(2.7) \quad Q\{V, W\} = \frac{1}{2}(V\Delta W + W\Delta V) - \nabla V \cdot \nabla W.$$

A calculation yields

$$(2.8) \quad Q\{|p|^2, |g|^2\} = 2|\omega_z(p, g)|^2$$

for holomorphic functions $p(z)$ and $g(z)$, where $\omega_z(p, g)$ denotes the Wronskian;

$$(2.9) \quad \omega_z(p, g) = pg' - p'g.$$

In particular we have $Q(|p|^2) = 0$.

Now we can introduce the integral for the first relation of (2.4), which was essentially discovered by J. Liouville [1]:

$$(2.10) \quad V = |z/G|^2 + \frac{1}{8}|M/G|^2,$$

where $G(z)$ and $M(z)$ are holomorphic functions. From (2.8) we have that

$$\begin{aligned} Q(V) &= \frac{\lambda}{4} Q\{|z/G|^2, |M/G|^2\} = \frac{\lambda}{2} |\omega_z(z/G, M/G)|^2 \\ &= \frac{\lambda}{2} |\omega_z(z, M)/G^2|^2. \end{aligned}$$

We have only to solve that

$$(2.11) \quad \omega_z(z, M) = g'G^2 \quad (\text{in } D) \quad \text{with} \quad |G|^2 = 1 + \frac{\lambda}{8} |M|^2 \quad (\text{on } \partial D).$$

Given $K(z)$, the solvability of

$$(2.12) \quad \omega_z(z, M) = K$$

is equivalent to $K'(0) = 0$. Then the solution $M(z)$ is given as

$$(2.13) \quad M = d(K) + az,$$

where $a \in \mathbb{C}$ denotes the integral constant and $d(K) = \sum_{n \neq 1} \frac{K_n}{n-1} z^n$

for $K(z) = \sum K_n z^n$. The problem (2.11) is reduced to

$$(2.14) \quad |G|^2 = 1 + \frac{\lambda}{8} |d(g'G^2) + az|^2 \quad \text{on } \partial D \quad \text{with} \quad G'(0) = 0.$$

The relation (2.5) is realized by

$$(2.15) \quad G \rightarrow 1 \quad \text{as} \quad \lambda \downarrow 0.$$

Here, we put $G = 1 + \lambda H$ to deduce

$$(2.16) \quad H + \overline{H} = \frac{1}{8} |d(g') + az|^2 + \lambda \Phi(H, a, \lambda) \quad \text{on } \partial D \quad \text{with} \quad H'(0) = 0,$$

where

$$(2.17) \quad \Phi(H, a, \lambda) = -|H|^2 + \frac{1}{8} \{ (d(g') + az) \overline{(2d(g'H) + \lambda d(g'H^2))} \}$$

$$+ \overline{(\alpha(g') + az)} (2\alpha(g'H) + \lambda \alpha(g'H^2)) + |2\alpha(g'H) + \lambda \alpha(g'H^2)|^2 \}.$$

Putting $\alpha(g') = c_0 + z^2 I_0(z)$ with $c_0 \in \mathbb{C}$, we have

$$|\alpha(g') + az|^2 = |a|^2 + 2 \operatorname{Re} \{ (a \bar{c}_0 + \bar{a} I_0(z)) z \} + |\alpha(g')|^2 \text{ if } |z|=1.$$

Therefore, the first relation of (2.4) is equivalent to

$$(2.18) \quad 2 \operatorname{Re} H = \frac{|a|^2}{8} + \frac{1}{4} \operatorname{Re} \{ (a \bar{c}_0 + \bar{a} I_0(z)) z \} + \frac{1}{8} |\alpha(g')|^2 + \lambda \Phi \text{ (on } \partial D).$$

Utilizing Schwarz's formula we have

$$(2.19) \quad H(z) = \frac{|a|^2}{16} + \frac{1}{8} \{ (a \bar{c}_0 + \bar{a} I_0(z)) z \} \\ + \frac{1}{2\lambda \Gamma_1} \int_{\partial D} \left\{ \frac{|\alpha(g')|^2}{8} + \lambda \Phi(H, a, \lambda) \right\} \left\{ \frac{1}{5-z} + \frac{1}{25} \right\} d\zeta \text{ for } z \in D.$$

Hence $H'(0)=0$, the second relation of (2.16), is realized by

$$(2.20) \quad a \bar{c}_0 + \bar{a} I_0(0) = \mathcal{F}(H, a, \lambda),$$

where

$$(2.21) \quad \mathcal{F}(H, a, \lambda) = \frac{1}{2\lambda \Gamma_1} \int_{\partial D} \left\{ \frac{|\alpha(g')|^2}{8} + \lambda \Phi(H, a, \lambda) \right\} \frac{d\zeta}{\zeta^2}.$$

For given holomorphic function $H(z)$ with $H'(0)=0$, the equation (2.20) is a linear equation of $(a, \bar{a})$ if $\lambda=0$. In the case of $|I_0(0)/c_0| \neq 1$, which is equivalent to

$$(2.22) \quad |g''(0)/g'(0)| \neq 2,$$

the equation (2.20) is solved with respect to a if $\lambda=0$, and so is true for small $\lambda>0$ by the implicit function theorem.

Thus we obtain

$$(2.23) \quad \alpha = \alpha(H, \lambda) \quad \text{if} \quad 0 < \lambda < \frac{1}{2}(\|H\|_{L^\infty(D)})^{-1} < 1.$$

Substituting this into the right-hand side of (2.19), we get the fixed point equation

$$(2.24) \quad H = N(H, \lambda).$$

Here, $N(H, \lambda)'(0) = 0$ follows from $H'(0) = 0$ as is imposed in (2.20).

$H \equiv 0$ solves (2.24) when $\lambda = 0$.

The formal consideration given above will be realized if we can prepare appropriate function spaces. Let H_L be the Hardy-Lebesgue class:

$$H_L = \{f(z) \mid \text{holomorphic in } D \text{ with } \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < +\infty\}.$$

We put

$$H_{L_0} = \{f \in H_L \mid f'(0) = 0\} \quad \text{and} \quad H_{L_0}^\perp = \{f \in H_{L_0} \mid f' \in H_L\}.$$

The functional α defined before is an isomorphism from H_{L_0} onto $H_{L_0}^\perp$. The inclusion $H_{L_0}^\perp \subset X = \{f(z) \mid \text{holomorphic in } D, \text{ continuous on } \bar{D}, \text{ and } f'(0) = 0\}$ assures us of the nonlinear operator

$$N = N(\cdot, \lambda): B \rightarrow B$$

defined above, where B denotes the unit ball of the Banach space X , provided that $\lambda > 0$ is sufficiently small. Furthermore, then the mapping $N(\cdot, \lambda)$ is a contraction. Hence it has a fixed point.

We note that the condition (2.22) is equivalent to the non-degeneracy of the critical point α_j^* of the Robin function $R(z)$.

The multi-point blow-up solutions can be constructed in the following way. As before, we first observe that the singular limit $u_0(z)$ of (1.3) satisfies

$$(2.25) \quad e^{-g^* u_0} = |w|^2,$$

where $w = \prod_{j=1}^n \frac{z - \delta_j}{1 - \bar{\delta}_j z}$ ($|\delta_j| < 1$) is a finite Blaschke product and $\alpha_j^* = g(\delta_j)$. In view of this we introduce the integral

$$(2.26) \quad V = |w/G|^2 + \frac{\lambda}{8} |M/G|^2 \quad (\text{in } D), \quad V = 1 \quad (\text{on } \partial D)$$

for the first relation of (2.4). From

$$\begin{aligned} Q(V) &= \frac{\lambda}{4} Q(|w/G|^2, |M/G|^2) = \frac{\lambda}{2} \left| \omega_z \left(\frac{w}{G}, \frac{M}{G} \right) \right|^2 \\ &= \frac{\lambda}{2} \left| \omega_z(w, M) / G^2 \right|^2 \end{aligned}$$

is deduced that

$$(2.27) \quad \omega_z(w, M) = g' G^2 \quad (\text{in } D) \quad \text{with} \quad |G|^2 = 1 + \frac{\lambda}{8} |M|^2 \quad (\text{on } \partial D)$$

Here, we introduce the Riemannian surface $\hat{D}$ so that the mapping $z \in D \mapsto w \in \hat{D}$ is a homeomorphism. It is an m -covering of D , and a similar formula to that of Schwarz holds on it. The holomorphic function $g(z)$ in D induces that of $\hat{g} = \hat{g}(w)$ of w in $\hat{D}$ through the relation $g(z) = \hat{g}(w(z))$.

We can show that $(\alpha_1^*, \dots, \alpha_m^*)$ is a non-degenerate critical point of K in (1.4) if and only if

$$(2.28) \quad \hat{g}''(0) = 0 \quad \text{and} \quad |\hat{g}'''(0)/\hat{g}'(0)| \neq 2.$$

Then we can solve the equation

$$(2.29) \quad w_w(w, \hat{M}) = \hat{g}' \hat{G}^2 \text{ (in } \hat{D}) \text{ with } |\hat{G}|^2 = 1 + \frac{\lambda}{8} |\hat{M}|^2 \text{ (on } \partial \hat{D})}$$

to obtain holomorphic functions $\hat{G}(w)$ and $\hat{M}(w)$ of w in $\hat{D}$, satisfying

$$(2.30) \quad \hat{G}(w) \rightarrow 1 \quad (\text{as } \lambda \downarrow 0)$$

The holomorphic functions $G(z) = \hat{G}(w(z))$ and $M(z) = \hat{M}(w(z))$ satisfy (2.27). Thus the multi-point blow-up solutions have been constructed.

References

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