

On the characters of Wenzl's $(3, l)$ -diagram representations of the Iwahori-Hecke algebras at $\sqrt[l]{1}$

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§ 1. Wenzl's (k, l) -diagram representations

(1.1) Throughout this note, we assume q to be a non-zero complex number. Let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ be the set of natural numbers, the additive group of integers, the field of real numbers, the field of complex numbers, respectively. For $n \in \mathbf{N}$, define $\mathbf{H}_n(q)$ to be the $\mathbf{C}$ -algebra (with 1) by the generators T_i , $1 \leq i \leq n-1$, and the relations:

$$(T_i - q)(T_i + 1) = 0, T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, T_i T_j = T_j T_i (|i - j| \geq 2).$$

The algebra $\mathbf{H}_n(q)$ is called the *Iwahori-Hecke algebra* of type A_n . Let $\mathbf{C}^\times = \mathbf{C} \setminus \{0\}$ and $\mathbf{N}' = \mathbf{N} \setminus \{1\}$. Let $\mathbf{l} : \mathbf{C}^\times \rightarrow \mathbf{N}' \cup \{+\infty\}$ be the map such that

$$\mathbf{l}(q) = \begin{cases} \min\{a \in \mathbf{N}' \mid q^a = 1\} & \text{if } \prod_{r=2}^s \left(\sum_{t=0}^{r-1} q^t \right) = 0 \text{ for some } s \in \mathbf{N}', \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\mathbf{Z}_+ = \{0\} \cup \mathbf{N}$. Put $\mathbf{Z}_+^\infty = \{(x_1, x_2, \dots) \in \mathbf{Z}^\infty \mid x_i \in \mathbf{Z}_+ (i \in \mathbf{N})\}$. For $i \in \mathbf{N}$, let $p_i : \mathbf{Z}_+^\infty \rightarrow \mathbf{Z}_+$ be the map such that $p_i(x_1, x_2, \dots) = x_i$. An element λ of $\mathbf{Z}_+^\infty$ is called a *partition* if $p_i(\lambda) \geq p_{i+1}(\lambda)$ for any $i \in \mathbf{N}$ and there exists $j \in \mathbf{N}$ such that $p_j(\lambda) = 0$. Let Λ be the set of partitions. Let $\mathbf{k} : \Lambda \rightarrow \mathbf{Z}_+$ be the map such that $\mathbf{k}(\lambda) = \min\{j \in \mathbf{N} \mid p_j(\lambda) = 0\} - 1$. Let $\mathbf{n} : \Lambda \rightarrow \mathbf{Z}_+$ be the map such that $\mathbf{n}(\lambda) = \sum_{i=1}^{+\infty} p_i(\lambda)$. For $n \in \mathbf{Z}_+$, let $\Lambda_n = \{\lambda \in \Lambda \mid \mathbf{n}(\lambda) = n\}$. For $k \in \mathbf{Z}_+$, let $\Lambda^k = \{\lambda \in \Lambda \mid \mathbf{k}(\lambda) = k\}$.

(1.2) Let $\phi = (0, 0, \dots) \in \Lambda$, and let $\Lambda' = \Lambda \setminus \{\phi\}$. For $l \in \mathbf{N}' \cup \{+\infty\}$, let

$$\Lambda^{(l)} = \{\phi\} \cup \{\lambda \in \Lambda' \mid \mathbf{k}(\lambda) \leq l-1, p_1(\lambda) - p_{\mathbf{k}(\lambda)}(\lambda) \leq l - \mathbf{k}(\lambda)\}.$$

For k , $1 \leq k \leq l-1$, let $\Lambda^{(k,l)} = \bigcup_{a=1}^k (\Lambda^a \cap \Lambda^{(l+a-k)})$. Note that $\Lambda^{(l)} = \bigcup_{1 \leq k \leq l-1} \Lambda^{(k,l)}$ and $\Lambda^{(+\infty)} = \Lambda$. Let $\Lambda_n^{(l)} = \Lambda^{(l)} \cap \Lambda_n$ and $\Lambda_n^{(k,l)} = \Lambda^{(k,l)} \cap \Lambda_n$. The element $\lambda \in \Lambda$ will also be denoted by

$$[1^{\{i \mid p_1(\lambda)=1\}} 2^{\{i \mid p_1(\lambda)=2\}} \dots]$$

if $\lambda \in \Lambda'$, and $[0]$ if $\lambda = \phi$. For example, we have $\Lambda_0^{(l)} = \{[0]\}$, $\Lambda_1^{(l)} = \{[1]\}$ and $\Lambda_2^{(l)} = \{[2], [1^2]\}$.

(1.3) For $\mu, \lambda \in \Lambda$, we write $\mu \subset^{(k,l)} \lambda$ if $\mu, \lambda \in \Lambda^{(k,l)}$ and $p_i(\mu) \leq p_i(\lambda)$ for any i , $1 \leq i \leq k$. We write $\mu \subset_{+1}^{(k,l)} \lambda$ if $\mu \subset^{(k,l)} \lambda$ and $\mathbf{n}(\mu) = \mathbf{n}(\lambda) - 1$. For $\mu \subset_{+1}^{(k,l)} \lambda$, put

$$\mathbf{r}(\mu, \lambda) = \sum_{i=0}^{+\infty} i \cdot (p_i(\lambda) - p_i(\mu)), \quad \text{and} \quad \mathbf{c}(\mu, \lambda) = p_{\mathbf{r}(\mu \subset_{+1}^{(k,l)} \lambda)}(\lambda).$$

(1.4) For $\mu \subset^{(k,l)} \lambda$, we set $\mathbf{n}(\lambda/\mu) = \mathbf{n}(\lambda) - \mathbf{n}(\mu)$, and call an element $(\lambda_0, \lambda_1, \dots, \lambda_{\mathbf{n}(\lambda/\mu)})$ of $\Lambda_{\mathbf{n}(\lambda)} \times \Lambda_{\mathbf{n}(\lambda)+1} \times \dots \times \Lambda_{\mathbf{n}(\lambda)}$ a (k, l) -standard tabuleau of λ/μ if $\lambda_0 = \mu$, $\lambda_{\mathbf{n}(\lambda/\mu)} = \lambda$ and $\lambda_i \subset_{+1}^{(k,l)} \lambda_{i+1}$, $1 \leq i \leq \mathbf{n}(\lambda/\mu) - 1$. Let $\text{STab}^{(k,l)}(\lambda)$ denote the set of (k, l) -standard tableaux of λ/μ .

(1.5) A standard tabuleau $(\lambda_0, \lambda_1, \dots, \lambda_{\mathbf{n}(\lambda/\mu)}) \in \text{STab}^{(k,l)}(\lambda/\mu)$ will also be denoted by the table such that the i -th Arabic figure is put on $(\mathbf{r}(\lambda_{i-1}, \lambda_i), \mathbf{c}(\lambda_{i-1}, \lambda_i))$ -position.

(1.6) Example.

$$\text{STab}^{(2,l)}([2^1 3^1]/[1]) = \begin{cases} \left\{ \begin{array}{ccccc} 12 & 13 & 23 & 14 & 24 \\ 34 & 24 & 14 & 23 & 13 \end{array} \right\} & \text{if } l \geq 5, \\ \left\{ \begin{array}{ccccc} 13 & 23 & 14 & 24 \\ 24 & 14 & 23 & 13 \end{array} \right\} & \text{if } l = 4, \\ \left\{ \begin{array}{c} 24 \\ 13 \end{array} \right\} & \text{if } l = 3 \end{cases}$$

(1.7) For $\mu \subset^{(k,l)} \lambda$, and for i , $0 \leq i \leq \mathbf{n}(\lambda/\mu)$, let $h_i : \text{STab}^{(k,l)}(\lambda/\mu) \rightarrow \Lambda_{\mathbf{n}(\mu)+i}^{(k,l)}$ be the map such that $h_i(x)$ is the i -th component of x , and let $f_i : \text{STab}^{(k,l)}(\lambda/\mu) \rightarrow \text{STab}^{(k,l)}(\lambda/\mu)$ be the map such that $f_i(\mathbf{t}) = (h_0(\mathbf{t}), \dots, h_{i-1}(\mathbf{t}), \nu, h_{i+1}(\mathbf{t}), \dots, h_{\mathbf{n}(\lambda/\mu)}(\mathbf{t}))$ if there exists an element ν of $\Lambda^{(k,l)}$ with $h_{i-1}(\mathbf{t}) \subset_{+1}^{(k,l)} \nu \subset_{+1}^{(k,l)} h_{i+1}(\mathbf{t})$ and $\nu \neq h_i(\mathbf{t})$, and $f_i(\mathbf{t}) = \mathbf{t}$ otherwise. For $\alpha \subset_{+1}^{(k,l)} \beta \subset_{+1}^{(k,l)} \gamma$, put

$$d(\alpha, \beta, \gamma) = \mathbf{c}(\alpha, \beta) - \mathbf{c}(\beta, \gamma) + \mathbf{r}(\beta, \gamma) - \mathbf{r}(\alpha, \beta).$$

For $\mathbf{t} \in \text{STab}^{(k,l)}(\lambda/\mu)$ and i , $1 \leq i \leq \mathbf{n}(\lambda/\mu) - 1$, put $\mathbf{d}(\mathbf{t}; i) = d(h_{i-1}(\mathbf{t}), h_i(\mathbf{t}), h_{i+1}(\mathbf{t}))$.

(1.8) For $q \in \mathbb{C}^\times$, and for $d \in \mathbb{Z}$, $1 \leq |d| \leq \mathbf{l}(q) - 1$, put

$$b_d(q) = -\lim_{z \rightarrow q} \frac{1-z}{1-z^d},$$

and let $c_d(q) \in \{w \in \mathbb{C} | \text{Im } w > 0\} \cup \{x \in \mathbb{R} | x \geq 0\}$ be such that

$$c_d(q)^2 = \lim_{z \rightarrow q} \frac{z(1-z^{d-1})(1-z^{d+1})}{(1-z^d)^2}.$$

Note that $b_1(q) = -1$, $b_{-1}(q) = q$.

Theorem (1.9) ([Wenzl]). (i) Let $q \in \mathbf{C}^\times$, and let $k \in \mathbf{N}$ be such that $k \leq l(q) - 1$. Let $\mu, \lambda \in \Lambda^{(k, l(q))}$ be such that $\mu \subset^{(k, l(q))} \lambda$. Let $V_{\lambda/\mu}^{(k, l(q))}$ be a $\mathbf{C}$ -vector space with a basis $\{v_t \mid t \in \text{STab}^{(k, l(q))}(\lambda/\mu)\}$. Then there exists a representation $\pi_{\lambda/\mu}^{(k, l(q))} : \mathbf{H}_{\mathbf{n}(\lambda/\mu)}(q) \rightarrow \text{End}(V_{\lambda/\mu}^{(k, l(q))})$ of $\mathbf{H}_{\mathbf{n}(\lambda/\mu)}(q)$ such that

$$\pi_{\lambda/\mu}^{(k, l(q))}(T_i)v_t = \begin{cases} b_{d(t; i)}(q)v_t + c_{d(t; i)}(q)v_{f_i(t)} & \text{if } f_i(t) \neq t, \\ b_{d(t; i)}(q)v_t & \text{otherwise} \end{cases}$$

($1 \leq i \leq \mathbf{n}(\lambda/\mu) - 1$).

(ii) For $\lambda \in \Lambda^{(l(q))}$, let $\pi_\lambda^{(l(q))} = \pi_{\lambda/\phi}^{(k(\lambda), l(q))}$. Then $\pi_\lambda^{(l(q))}$ is irreducible. For $\mu, \lambda \in \Lambda_n^{(l(q))}$, $\mu \neq \lambda$, $\pi_\mu^{(l(q))}$ and $\pi_\lambda^{(l(q))}$ are not equivalent.

(iii) Let $l(q) > n$. Then $\mathbf{H}_n(q)$ is semisimple, $\Lambda_n^{(l(q))} = \Lambda_n$, and $\{\pi_\lambda^{(l(q))} \mid \lambda \in \Lambda_n\}$ is a complete set of irreducible representations of $\mathbf{H}_n(q)$.

§2. Note on irreducible characters

(2.1) Denote by $\mathbf{H}_n(q)^*$ the $\mathbf{C}$ -vector space of $\mathbf{C}$ -linear maps of $\mathbf{H}_n(q)$ into $\mathbf{C}$. If an element $f \in \mathbf{H}_n(q)^*$ satisfies the condition such that

$$\forall X, \forall Y \in \mathbf{H}_n(q) \quad f(XY - YX) = 0,$$

we call f a *class function* of $\mathbf{H}_n(q)$. Denote by $\mathbf{CF}(\mathbf{H}_n(q))$ the set of the class functions of $\mathbf{H}_n(q)$. For $\lambda \in \Lambda_n^{(l(q))}$, denote by $\chi_\lambda^{(l(q))}$ the character $\text{Trace} \circ \pi_\lambda^{(l(q))}$. It is clear that $\chi_\lambda^{(l(q))} \in \mathbf{CF}(\mathbf{H}_n(q))$.

For $k, m \in \mathbf{N}$, $1 \leq m < k \leq n$, put

$$\begin{aligned} S_{m,k} &= (T_{k-m} \cdots T_1)(T_{k-m+1} \cdots T_2) \cdots (T_{k-1} \cdots T_m) \\ &= (T_{k-m} \cdots T_{k-1})(T_{k-m-1} \cdots T_{k-2}) \cdots (T_1 \cdots T_m). \end{aligned}$$

Then we have $S_{m,k}T_j = T_{j-m}S_{m,k}$ ($m+1 \leq j \leq k-1$) and $S_{m,k}T_i = T_{k-m+i}S_{m,k}$ ($1 \leq i \leq m-1$).

For $a, n \in \mathbf{N}$, $1 < a < n$, denote by $\iota_{a,n}$ the monomorphism of $\mathbf{H}_a(q)$ into $\mathbf{H}_n(q)$ such that $\iota_{a,n}(T_i) = T_i$ ($1 \leq i \leq a-1$). For $X \in \mathbf{H}_a(q)$, $\iota_{a,n}(X)$ will also be denoted by X . For $k, m \in \mathbf{N}$, $1 \leq m < k \leq n$, and for $X \in \mathbf{H}_a(q)$, $Y \in \mathbf{H}_{k-m}(q)$, put

$$X \# Y = XS_{k-m,k}YS_{k-m,k}^{-1} \in \mathbf{H}_n(q).$$

For $f \in \mathbf{CF}(\mathbf{H}_n(q))$, we have

$$f(X \# Y) = f(YS_{k-m,k}^{-1}XS_{k-m,k}) = f(YS_{m,k}XS_{m,k}^{-1}) = f(Y \# X).$$

We also have $f((X \# Y) \# Z) = f(X \# (Y \# Z))$.

Let $\chi_{\lambda/\mu}^{(\mathbf{k}(\lambda), \mathbf{l}(q))} = \text{Trace} \circ \pi_{\lambda/\mu}^{(\mathbf{k}(\lambda), \mathbf{l}(q))}$. The following lemma is easily obtained by the definition of $\pi^{(l)}$.

Lemma (2.2) Let $k, m \in \mathbf{N}$, $1 < m < n$, and $X \in \mathbf{H}_m(q)$, $Y \in \mathbf{H}_{k-n}(q)$. Let $\lambda \in \Lambda^{(l(q))}$. Then

$$\pi_{\lambda}^{(l(q))}(X \# Y) \cong \bigoplus_{\mu \in \Lambda_m^{(\mathbf{k}(\lambda), \mathbf{l}(q))}} \pi_{\mu}^{(l(q))}(X) \otimes \pi_{\lambda/\mu}^{(\mathbf{k}(\lambda), \mathbf{l}(q))}(Y).$$

In particular,

$$\chi_{\lambda}^{(l(q))}(X \# Y) = \sum_{\mu \in \Lambda_m^{(\mathbf{k}(\lambda), \mathbf{l}(q))}} \chi_{\mu}^{(l(q))}(X) \chi_{\lambda/\mu}^{(\mathbf{k}(\lambda), \mathbf{l}(q))}(Y).$$

(2.3) For $\lambda \in \Lambda_n^{(l(q))}$, denote by $e_{\lambda}^{(l(q))}$ a primitive idempotent of $\mathbf{H}_n(q)$ corresponding to the irreducible representation $\pi_{\lambda}^{(l(q))}$. For $k, m, n \in \mathbf{N}$, $1 < m < n$, and for $\mu \in \Lambda_m^{(k, l(q))}$, $\nu \in \Lambda_{n-m}^{(k, l(q))}$, $\lambda \in \Lambda_n^{(k, l(q))}$, $\mu, \nu \subset^{(k, l(q))} \lambda$, define the integer $d^{(k, l(q))}(\lambda; \mu, \nu)$ to be $\chi_{\lambda}^{(k, l(q))}(e_{\mu}^{(l(q))} \# e_{\nu}^{(l(q))}) = \chi_{\lambda/\mu}^{(k, l(q))}(e_{\nu}^{(l(q))})$. Denote by $\mathbf{S}(\mathbf{N})$ the group of the bijective maps of $\mathbf{N}$ onto itself. Let $s_i \in \mathbf{S}(\mathbf{N})$ ($i \in \mathbf{N}$) be the map such that $p_i(s_i(\alpha)) = p_{i+1}(\alpha)$, $p_{i+1}(s_i(\alpha)) = p_i(\alpha)$ and $p_j(s_i(\alpha)) = p_j(\alpha)$ ($j \neq i, i+1$). Let $s_0^{(k, l)} \in \mathbf{S}(\mathbf{N})$ ($i \in \mathbf{N}$) be the map such that $p_1(s_0^{(k, l)}(\alpha)) = p_k(\alpha) + l$, $p_k(s_0^{(k, l)}(\alpha)) = p_1(\alpha) - l$ and $p_j(s_0^{(k, l)}(\alpha)) = p_j(\alpha)$ ($j \neq 1, k$). Let $W^{(k, l)}$ be the subgroup of $\mathbf{S}(\mathbf{N})$ generated by $s_0^{(k, l)}$ and s_i ($1 \leq i \leq k$). Define $\delta(k) \in \lambda$ to be $(k-1, k-2, \dots, 1, 0, \dots)$. Goodman-Wenzl [GW] proved:

Theorem (2.4) ([GW])

$$d^{(k, l)}(\lambda; \mu, \nu) = \sum_{w \in W^{(k, l)}} \text{sgn}(w) d^{(k, l)}(w(\lambda + \delta(k)) - \delta(k); \mu, \nu).$$

Since this identity coincides with the Kac-Walton one, $d^{(k, l)}(\lambda; \mu, \nu)$ is the same as the $\text{SU}(k)$ -fusion coefficient with level $l - k$ (see [GW]).

(2.5) An element α of $\mathbf{Z}_+^{\infty}$ is called a *composition* if there exists $j \in \mathbf{N}$ such that $p_i(\alpha) > 0$ for $i < j$ and $p_k(\alpha) = 0$ for $k \geq j$. Denote by Ω the set of the compositions. Let $\omega : \Omega \rightarrow \Lambda$ be the map such that $\omega(\alpha) = \sigma(\alpha)$ for some $\sigma \in \mathbf{S}(\mathbf{N})$. Let $\Omega_n = \omega^{-1}(\Lambda_n)$ and $\Omega' = \omega^{-1}(\Lambda')$.

(2.6) The maps $\mathbf{k} \circ \omega$, $\mathbf{n} \circ \omega$ of Ω into $\mathbf{Z}_+$ will also be denoted by $\mathbf{k}$, $\mathbf{n}$ respectively. For $\alpha \in \Omega'$ and i , $1 \leq i \leq \mathbf{k}(\alpha)$, let $x_i(\alpha) = \sum_{1 \leq j < i} p_j(\alpha) \in \mathbf{Z}_+$, and let $X(\alpha; i)$ be the element of $\mathbf{H}_{n(\alpha)}(q)$ such that $X(\alpha; i) = T_1 T_2 \cdots T_{p_i(\alpha)-1}$ if $p_i(\alpha) \geq 2$, and $X(\alpha; i) = 1$ if $p_i(\alpha) = 1$. Put

$$X(\alpha) = X(\alpha; 1) \# \cdots \# X(\alpha; \mathbf{k}(\alpha)).$$

By the following theorem proved by Ram [Ram], we see that any class function $f : \mathbf{H}_n(q) \rightarrow \mathbf{C}$ is determined only by the values $f(X(\lambda))$ ($\lambda \in \Lambda_n$).

Theorem (2.7)([Ram]). *Let $n \in \mathbf{N}$. Then*

$$\forall X \in \mathbf{H}_n(q) \exists x_\lambda \in \mathbf{C} (\lambda \in \Lambda_n) \forall f \in \mathbf{CF}(\mathbf{H}_n(q)) \quad f(X) = \sum_{\lambda \in \Lambda_n} x_\lambda f(C(\lambda))$$

We can calculate the coefficients x_λ 's via an inductive process.

(2.8) For $\mu, \lambda \in \Lambda^{(k, l(q))}$, $\mu \subset^{(k, l(q))} \lambda$, put $\Delta^{(k, l(q))}(\lambda/\mu) = \chi^{(k, l(q))}(\lambda/\mu)(X([\mathbf{n}(\lambda/\mu)^1])$. It turns out that

$$\Delta^{(k, l(q))}(\lambda/\mu) = \sum_{\mathbf{t} \in \text{STab}^{(k, l(q))}(\lambda/\mu)} \prod_{1 \leq i \leq \mathbf{n}(\lambda/\mu)} b_{\mathbf{d}(\mathbf{t}; i)}(q).$$

As an immediate consequence of Lemma (2.2), we have:

Theorem (2.9) *Let $\lambda \in \Lambda_n^{(k, l)}$. Let $\alpha \in \Omega_n$. Then*

$$\chi_\lambda^{(k, l)}(X(\alpha)) = \sum_{\substack{\phi = \mu_0 \subset^{(k, l(q))} \dots \subset^{(k, l(q))} \mu_r = \lambda \\ p_i(\alpha) = \mathbf{n}(\mu_i/\mu_{i-1})}} \prod_{i=1}^r \Delta^{(k, l(q))}(\mu_i/\mu_{i-1}).$$

Ram[Ram] gave an explicit formula of $\Delta^{(k, l(q))}(\mu_i/\mu_{i-1})$ for $l(q) > \mathbf{n}(\lambda)$.

§3. Main result

(3.1) Let $l = l(q)$. In §3, $a \equiv b$ means $a - b \in l\mathbf{Z}$. For $\lambda \in \Lambda^{(k, l(q))}$, $\Delta(\lambda/\phi)$ will also be denoted by $\{p_1(\lambda), \dots, p_{\mathbf{k}(\lambda)}(\lambda)\}$. If $\mathbf{k}(\lambda) = 2$, we have:

$$\{r, r - a\} = \begin{cases} -q^{-a-3} & r \equiv -1 \\ q^{-a-2} & a \geq 1, r \equiv 0 \\ q^{-a-2} + q^{a-1} & a = 0, r \equiv a \\ -q^a & r \equiv a + 1 \\ 0 & \text{otherwise} \end{cases}$$

(3.2) If $\mathbf{k}(\lambda) = 3$, we have:

$\{r, r, r\}$	$l \geq 5$	$q^{-4}(r \equiv -1), q^{-3} + q^{-2} + q^{-1}(r \equiv 0), 1(r \equiv 1), 0(\text{oth.})$
—	$l = 4$	$1(r \equiv 1, 3), -1(r \equiv 0, 2)$
$\{r, r, r-1\}$	$l \geq 6$	$q^{-5}(r \equiv -1), q^{-4}(r \equiv 0), -1(r \equiv 1), 0(\text{oth.})$
—	$l = 5$	$1(r \equiv -1), q(r \equiv 0), -1(r \equiv 1), 0(r \equiv 2), -q(r \equiv -2)$
—	$l = 4$	$1(r \equiv 0, 2), -1(r \equiv 1, 3)$
$\{r, r, r-a\} (a \geq 2)$	$l \geq a+5$	$q^{-(a+4)}(r \equiv -1), q^{-(a+3)}(r \equiv 0), 0(\text{oth.})$
—	$l = a+4$	$q(r \equiv 0), -q^a(r \equiv -2), 1(r \equiv -1), 0(\text{oth.})$
—	$l = a+3$	$1(r \equiv 0), q^{a-1}(r \equiv -2), q^a + q^{a+1} + q^{a+2}(r \equiv -1), 0(\text{oth.})$
$\{r, r-1, r-1\}$	$l \geq 6$	$-q^{-4}(r \equiv 0), 1(r \equiv 1), q(r \equiv 2), 0(\text{oth.})$
—	$l = 5$	$-q(r \equiv 0), 1(r \equiv 1), q(r \equiv 2), -1(r \equiv 3), 0(r \equiv 4)$
—	$l = 4$	$1(r \equiv 1, 3), -1(r \equiv 0, 2)$
$\{r, r-a, r-a\} (a \geq 2)$	$l \geq a+5$	$q^{a-1}(r \equiv a), q^a(r \equiv a+1), 0(\text{oth.})$
—	$l = a+4$	$q^{a-1}(r \equiv a), q^a(r \equiv a+1), -1(r \equiv a+2), 0(\text{oth.})$
—	$l = a+3$	$q^{a-1}(r \equiv a), q^a + q^{a+1} + q^{a+2}(r \equiv a+1), 1(r \equiv a+2), 0(\text{oth.})$
$\{r, r-1, r-2\}$	$l \geq 7$	$-q^{-5}(r \equiv 0), -q(r \equiv 2), 0(\text{oth.})$
—	$l = 6$	$-q(r \equiv 0, 2, 4), 0(\text{oth.})$
—	$l = 5$	$-1(r \equiv 0), 0(r \equiv 1), -q(r \equiv 2), 1(r \equiv -2), q(r \equiv -1)$
$\{r, r-1, r-a\} (a \geq 3)$	$l \geq a+5$	$-q^{-(a+3)}(r \equiv 0), 0(\text{oth.})$
—	$l = a+4$	$-q(r \equiv 0), -q^{a-1}(r \equiv -2), 0(\text{oth.})$
—	$l = a+3$	$-1(r \equiv 0), q^{a-2}(r \equiv -2), q^{a-1}(r \equiv -1), 0(\text{oth.})$
$\{r, r-a+1, r-a\} (a \geq 3)$	$l \geq a+5$	$-q^{a-1}(r \equiv a), 0(\text{oth.})$
—	$l = a+4$	$-q^{a-1}(r \equiv a), -q(r \equiv a+2), 0(\text{oth.})$
—	$l = a+3$	$-q^{a-1}(r \equiv a), 1(r \equiv a+1), q(r \equiv a+2), 0(\text{oth.})$
$\{r, r-b, r-a\} (a-2 \geq b \geq 2)$	$l \geq a+5$	0
—	$l = a+4$	$-q^{a-b}(r \equiv -2), 0(\text{oth.})$
—	$l = a+3$	$q^{a-b-1}(r \equiv -2), q^{a-b}(r \equiv -1), 0(\text{oth.})$

§4. On Littlewood-Richardson rule

(4.1) In this section, we use terminology in [Macdonald]. Let $\mu \subset \lambda$. Denote by $\mathbf{LP}(\lambda/\mu)$ the set of the lattice permutations of shape λ/μ . For $\mathbf{z} \in \mathbf{LP}(\lambda/\mu)$ and $\nu \in \Lambda_{\mathbf{n}(\lambda/\mu)}$, let $\mathbf{LP}(\lambda/\mu; \nu)$ be the set of the lattice permutations of shape λ/μ and weight ν . It is well-known that the cardinality of $\mathbf{LP}(\lambda/\mu; \nu)$ is equal to $d^{(k, l(q))}(\lambda; \mu, \nu)$ for $l(q) > \mathbf{n}(\lambda)$. See [Macdonald].

(4.2) The set $\mathbf{LP}([2^1 3^1 4^1]/[2])$ consists of the lattice permutations

$$\begin{array}{cccc} & 1 & 1 & & 1 & 1 & & 1 & 1 & & 1 & 1 \\ 1 & 1 & 2 & , & 1 & 1 & 2 & , & 1 & 2 & 2 & , & 1 & 2 & 2 \\ 2 & 2 & & & 2 & 3 & & & 2 & 3 & & & 3 & 3 \end{array}$$

The weights of the first, the second, the third, and the fourth lattice permutations are $[3^1 4^1]$, $[1^1 2^1 4^1]$, $[1^1 3^2]$, $[2^2 3^1]$, respectively.

Theorem (4.3) Let $\mu, \lambda \in \Lambda^{(3,l)}$ be such that $\mu \subset^{(3,l)} \lambda$. Assume $p_1(\mu) - p_2(\mu) \leq 1$. For $\mathbf{z} \in \mathbf{LP}(\lambda/\mu)$, let z_{ij} be the Arabic figure at (i, j) -position, and let $\mathbf{bot}(\lambda/\mu; a)$ be the number of j 's such that $z_{3j} = a$. Denote by $\mathbf{Y}^{(3,l(q))}(\lambda/\mu)$ the set of the lattice permutations $\mathbf{z} \in \mathbf{LP}(\lambda/\mu)$ such that

- (1) $\mathbf{bot}(\lambda/\mu; 1) + 1 + p_1(\lambda) - p_3(\lambda) \leq l(q) - 3$ if $p_1(\mu) = p_2(\mu) + 1$, $z_{2,p_1(\mu)} = 1$, $z_{3,p_1(\mu)} = 2$,
- (2) $\mathbf{bot}(\lambda/\mu; 1) + 1 + p_1(\lambda) - p_3(\lambda) \leq l(q) - 3$ if $p_1(\mu) = p_2(\mu) + 1$, $z_{2,p_1(\mu)} = 1$, $z_{3,p_1(\mu)} \neq 2$, $\mathbf{bot}(\lambda/\mu; 2) = p_1(\lambda) - p_2(\lambda) + 1$,
- (3) $\mathbf{bot}(\lambda/\mu; 1) + 1 + p_1(\lambda) - p_3(\lambda) \leq l(q) - 3$ otherwise.

Then $d^{(3,l(q))}(\lambda; \mu, \nu)$ is equal to the number of the elements of $\mathbf{Y}^{(3,l(q))}(\lambda/\mu)$ whose weights are ν .

Example (10.3). On $\mathbf{Y}^{(3,l(q))}([7^1 10^1 12^1]/[5^2])$. We shall only write $a b c d e$ for

$$\mathbf{z} = \begin{array}{ccccccccc} & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & & & & 2 & 2 & 2 & 2 & 2 & & \\ a & b & c & d & e & 3 & 3 & & & & & & \end{array}.$$

(x, y, z) denotes weight. Then it consists of:

$$l(q) \geq 8, (7, 7, 5) 22333,$$

$$l(q) \geq 9, (8, 6, 5) 12333, (8, 7, 4) 12233,$$

$$l(q) \geq 10, (9, 5, 5) 11333, (9, 6, 4) 11233, (9, 7, 3) 11223,$$

$$l(q) \geq 11, (10, 5, 4) 11133, (10, 6, 3) 11123, (10, 7, 2) 11122,$$

$$l(q) \geq 12, (11, 5, 3) 11113, (11, 6, 2) 11112,$$

$$l(q) \geq 13, (12, 5, 2) 11111,$$

Example (10.4). On $\mathbf{Y}^{(3,l(q))}([6^1 9^1 11^1]/[3^1 4^1])$.

$$\mathbf{z} = \begin{array}{cccccccc} & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & & & & a & 2 & 2 & 2 & 2 & 2 & \\ b & c & d & e & 3 & 3 & & & & & & & \end{array}$$

$$l(q) \geq 8, (7, 6, 6)_{3333}^2, (7, 7, 5)_{2333}^2, (8, 6, 5)_{2333}^1, (8, 7, 4)_{2233}^1,$$

$$l(q) \geq 9, (8, 6, 5)_{1333}^2, (8, 7, 4)_{1233}^2, (8, 8, 3)_{2223}^1, (9, 5, 5)_{1333}^1, (9, 6, 4)_{1233}^1, (9, 7, 3)_{1223}^1,$$

$$l(q) \geq 10, (9, 6, 4)_{1133}^2, (9, 7, 3)_{1123}^2, (9, 8, 2)_{1222}^1, (10, 5, 4)_{1133}^1, (10, 6, 3)_{1123}^1,$$

$$l(q) \geq 11, (10, 6, 3)_{1113}^2, (10, 7, 2)_{1122}^1, (11, 5, 3)_{1113}^1,$$

$$l(q) \geq 12, (11, 6, 2)_{1112}^1$$

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