

Analytic singularities of solutions to a radial p -Laplacian

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Abstract

Analytic local description of $W^{1,p}(I)$ solutions to a radial p -Laplace equation

$$r(|U_r|^{p-2}U_r)_r + (n-1)|U_r|^{p-2}U_r + r|U|^{q-2}U = 0$$

on $I = [a, b] \subset (0, \infty)$ is given near singular points by a Briot-Bouquet type theorem of two variables, where $1 < p, q < \infty$.

1 Introduction

An n -dimensional p -elliptic PDE for $u(x)$ is

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u(x)) + |u|^{q-2}u = 0, \quad (1)$$

where $x \in \mathbf{R}^n$ and $1 < p, q < \infty$.

If $x \in \mathbf{R}^1$, the equation reduces to

$$(|u_x|^{p-2}u_x)_x + |u|^{q-2}u = 0. \quad (2)$$

L. Paredes and the present author, making use of a Briot-Bouquet type theorem of one variable, gave analytic expression of solutions to the equation (2) near the singularities ([8]). Our analytic expression readily reproduces differentiability and analyticity obtained by M. Ôtani [6], [7] and by M. Ôtani and T. Idogawa in [4].

If $r = |x|$, $x \in \mathbf{R}^n$,
a radial solution $U(r) = u(x)$ satisfies

$$(r^{n-1}|U_r|^{p-2}U_r)_r + r^{n-1}|U|^{q-2}U = 0, \quad (3)$$

or

$$r(|U_r|^{p-2}U_r)_r + (n-1)|U_r|^{p-2}U_r + r|U|^{q-2}U = 0. \quad (4)$$

The aim of this report is to extend the results for (2) to the radial p Laplacian (3) by a Briot-Bouquet type theorem of two variables.

Remark 0.1. R. Gérard and H. Tahara studied $t \frac{\partial u}{\partial t} = \Phi(t, x, u, \frac{\partial u}{\partial x})$ and generalized it of many variables and of higher order case in [3]. Since our version of a Briot-Bouquet type theorem of two (or several) variables is not covered by theirs, a proof is given, inspired by their work.

2 A Briot-Bouquet type theorem

We recall a classical Briot-Bouquet type theorem of one variable in complex domain. We assume

- $\Phi(t, h)$ is holomorphic near $(0, 0) \in \mathbf{C}^2$,
- $\Phi(0, 0) = 0$,
- $\frac{\partial \Phi}{\partial h}(0, 0)$ is not any positive integers.

Theorem 1 (Briot-Bouquet).

$$t \frac{dh}{dt} = \Phi(t, h)$$

has a unique holomorphic solution near $t = 0$, satisfying $h(0) = 0$.

If $\Phi(t, h)$ is real analytic, so is $h(t)$, too.

Proof. Set $a_{\alpha, i} = \frac{1}{\alpha! i!} \frac{\partial^{\alpha+i} \Phi}{\partial t^\alpha \partial h^i}(0, 0)$. Notice we can rewrite the equation by

$$t \frac{dh}{dt} - a_{0,1} h = a_{1,0} t + \sum_{2 \leq \alpha+i} a_{\alpha, i} t^\alpha h^i.$$

Moreover, the left hand side satisfies the condition that there exists $\delta > 0$ such that

$$|\alpha - a_{0,1}| \geq \delta$$

for all $\alpha \in \mathbf{N}^* = \{1, 2, 3, \dots\}$.

Formal solution: Let $\hat{h}(t) = \sum_{\alpha=1}^{\infty} h_\alpha t^\alpha$ be a formal solution. Then, we have

$$\begin{aligned} (1 - a_{0,1}) h_1 &= a_{1,0}, \\ (\alpha - a_{0,1}) h_\alpha &= Q_\alpha(a_{\alpha', i}, h_{\alpha''}; \alpha' + i \leq \alpha, \alpha'' \leq \alpha - 1) \end{aligned}$$

for all $\alpha \geq 2$, where Q_α is a polynomial with nonnegative integer coefficients.

Convergence: Then, we prove convergence of $\hat{h}(t)$ through the implicit function theorem.

An auxiliary equation of $H(t)$ is given by

$$\delta H = |a_{1,0}|t + \sum_{2 \leq \alpha+i} |a_{\alpha,i}|t^\alpha H^i.$$

with $H(0) = 0$.

There exists a unique convergent series function $H(t) = \sum_{\alpha=1}^{\infty} H_\alpha t^\alpha$ by the implicit function theorem. Since $\delta H_\alpha = Q_\alpha(|a_{\alpha',i}|, H_{\alpha''}; \alpha' + i \leq \alpha, \alpha'' \leq \alpha - 1)$,

$$\begin{aligned} |h_1| &= |a_{1,0}|/|1 - a_{0,1}| \\ &\leq |a_{1,0}|/\delta = H_1, \\ |h_\alpha| &= |Q_\alpha(a_{\alpha'}, h_{\alpha''})/(\alpha - a_{0,1})| \\ &\leq Q_\alpha(|a_{\alpha'}|, H_{\alpha''})/\delta = H_\alpha \\ &\text{for } \alpha \geq 2 \text{ by induction.} \end{aligned}$$

□

We will make use of a Briot-Bouquet type theorem of two variables for our main results. We state it in a slightly more general form for convenience.

Let $\mathbf{N} = \{0, 1, 2, \dots\}$. $B = \{\beta\}$ is a fixed finite subset of $\mathbf{N}^d$, where $\beta = (\beta_1, \dots, \beta_d)$ is a d -dimensional multi-index with $|\beta| \geq 1$. Let $(t, h, \rho_B) = (t_1, \dots, t_d, h, \{\rho_\beta; \beta \in B\}) \in \mathbf{C}^{d+1+|B|}$ be local variables near the origin, where $|B|$ is the number of the elements in B . Let $\xi = (\xi_1, \dots, \xi_d) \in \mathbf{C}^d$ be global variables. $\alpha = (\alpha_1, \dots, \alpha_d)$, β and γ denote d -dimensional power indices in $\mathbf{N}^d$.

Theorem 2. *We assume that a holomorphic function $\phi(t, h, \rho_B)$ and a polynomial*

$$L(\xi) = \sum_{0 \leq |\gamma| \leq r} l_\gamma \xi^\gamma$$

satisfy

(i) $\phi(t, \rho_B)$ has a power series expansion near $(0, 0)$ without linear

parts:

$$\begin{aligned}\phi(t, \rho_B) &= \sum_{2 \leq |\alpha| + |i_B|} a_{\alpha, i_B} t^\alpha \rho_B^{i_B} \\ &= \sum_{2 \leq |\alpha| + |i_B|} a_{\alpha, i_B} t^\alpha \prod_{\beta \in B} \rho_\beta^{i_\beta},\end{aligned}$$

where $|i_B| = \sum_{\beta \in B} i_\beta$ for a multi-index $i_B = (i_\beta)_{\beta \in B}$ and

(ii) there exists a positive constant δ such that for all d -dimensional multi indices α with $|\alpha| \geq 1$,

$$\left| \sum_{0 \leq |\gamma| \leq r} l_\gamma \alpha^\gamma \right| \geq \delta \max\{1, \alpha^\beta; \beta \in B\}, \quad (5)$$

where α^β denotes the coefficient of $(t \frac{\partial}{\partial t})^\beta t^\alpha$.

Then, a nonlinear equation

$$\begin{aligned}& \sum_{0 \leq |\gamma| \leq r} l_\gamma \left(t \frac{\partial}{\partial t} \right)^\gamma h(t) = a \cdot t + \\ & + \sum_{2 \leq |\alpha| + |i_B|} a_{\alpha, i_B} t^\alpha \\ & \cdot \prod_{\beta \in B} \left(\left(t \frac{\partial}{\partial t} \right)^\beta h(t) \right)^{i_\beta}\end{aligned} \quad (6)$$

has a unique holomorphic solution $h(t)$ near the origin with $h(0) = 0$.

Proof. We will follow the previous proof.

Construction of $\hat{h}(t)$: We set

$$\hat{h}(t) = \sum_{|\alpha| \geq 1} h_\alpha t^\alpha. \quad (7)$$

Substituting (7) into (6), we have

$$L(\alpha) h_\alpha = a_\alpha \quad \text{when } |\alpha| = 1,$$

and

$$\begin{aligned}
L(\alpha)h_\alpha &= Q_\alpha(a_{\alpha', i_B}, h_{\alpha''}, (\alpha_\beta)^\beta h_{\alpha_\beta}; \beta \in B \\
&\text{the indices at least satisfy} \\
|\alpha'| + |i_B| &\leq |\alpha|, |\alpha''| \leq |\alpha| - 1 \\
|\alpha_\beta| &\leq |\alpha| - 1,
\end{aligned}$$

where $\alpha', \alpha'', \alpha_\beta$ are copies of α . Thus, h_α are determined succesively.

Convergence of $\hat{h}$: An auxiliary analytic equation (cf. Gérard-Tahara [3])

$$\begin{aligned}
\delta H &= |a_1|t_1 + |a_2|t_2 \\
&+ \sum_{2 \leq |\alpha| + |i_B|} |a_{\alpha, i_B}| t^\alpha (H(t))^{|i_B|}.
\end{aligned}$$

Solving this equation of H by the implicit function theorem, we have a unique holomorphic solution near the origin $t = 0$ with $H(0) = 0$. We claim

$$H(t) = \sum_{\alpha} H_{\alpha} t^{\alpha} \gg \hat{h}(t).$$

More strongly we claim,

$$H_{\alpha} \geq \max\{|h_{\alpha}|, |\alpha^{\beta} h_{\alpha}|; \beta \in B\}.$$

We notice

$$\begin{aligned}
H_{\alpha} &= \frac{1}{\delta} Q_{\alpha}(|a_{\alpha', i_B}|, H_{\alpha''}, H_{\alpha_\beta}; \text{the indices satisfy} \\
&\text{at least } |\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1, \\
|\alpha_\beta| &\leq |\alpha| - 1, \beta \in B).
\end{aligned} \tag{8}$$

We start with $|\alpha| = 1$:

$$|h_{\alpha}| = |a_{\alpha}/L(\alpha)| \leq |a_{\alpha}|/\delta = H_{\alpha}.$$

Then, by induction, we have

$$\begin{aligned}
\max\{1, \alpha^{\beta}; \beta \in B\} \cdot |h_{\alpha}| &\leq \frac{1}{\delta} Q_{\alpha}(|a_{\alpha', i_B}|, |h_{\alpha''}|, |(\alpha_{\beta})^{\beta} h_{\alpha_{\beta}}|; \\
&\text{indices satisfy at least} \\
|\alpha'| + |i_B| &\leq |\alpha|, |\alpha''| \leq |\alpha| - 1, \\
|\alpha_{\beta}| &\leq |\alpha| - 1)
\end{aligned}$$

$$\leq \frac{1}{\delta} Q_\alpha(|a_{\alpha', i, i_B}|, H_{\alpha'}, H_{\alpha_\beta};$$

the indices satisfy at least

$$\begin{aligned} |\alpha'| + |i_B| &\leq |\alpha|, |\alpha''| \leq |\alpha| - 1, \\ |\alpha_\beta| &\leq |\alpha| - 1 \end{aligned}$$

Therefore, we have obtained $\max\{|h_\alpha|, |\alpha^\beta h_\alpha|\} \leq H_\alpha$. $\square$

We need this Briot-Bouquet type theorem in case where $d = 2$, $r = 2$, $\max\{|\beta|; \beta \in B\} = 1$.

3 Local uniqueness

To assure that weak solutions in $W^{1,p}(I)$ are filled locally by our analytical method, we need local uniqueness of solutions to the Cauchy problem to (3). We adapt proofs of uniqueness in J. Benedikt [1] and P. Drábek and M. Ôtani [2] for forth order p -elliptic equations, completing them with an energy inequality.

Let $I = [a, b]$ be a compact interval in $(0, \infty)$.

Proposition 1 (Local uniqueness). *Let r_0 be an arbitrary positive constant in I . Local solutions on I are uniquely determined near r_0 by initial data $U(r_0)$ and $U_r(r_0)$.*

Proof. Set $V(r) = r^{n-1}|U_r(r)|^{p-2}U_r(r)$, $p' = p/(p-1)$ and $f_p(X) = |X|^{p-2}X$. Notice $f_{p,X}(X) = (p-1)|X|^{p-2}$. Then, we have from the equation,

$$\begin{cases} V_r(r) &= -r^{n-1}|U(r)|^{q-2}U(r) = -r^{n-1}f_q(U(r)), \\ U_r(r) &= r^{\frac{1-n}{p-1}}|V(r)|^{p'-2}V(r) = f_{p'}(r^{1-n}V(r)). \end{cases} \quad (9)$$

Suppose

$$\begin{aligned} V_1(r_0) &= V_2(r_0), \quad U_1(r_0) = U_2(r_0) \\ |U_0| + |V_0| &> 0. \end{aligned}$$

We will show that there exists a positive constant ϵ such that $U_1(r) = U_2(r)$ on $J(\epsilon) = [r_0 - \epsilon, r_0 + \epsilon]$, as proved in Benedikt [1] in 4th order p elliptic ordinary differential equation.

Case (i): $1 < p \leq 2$ and $2 \leq q$.

$$\begin{aligned} V_1(r) - V_2(r) &= r^{n-1} \{f_p(U_{1,r}(r)) - f_p(U_{2,r}(r))\} \\ &= r^{n-1} \int_{U_{2,r}(r)}^{U_{1,r}(r)} f_{p,X}(\tau) d\tau \end{aligned}$$

We set $K_1 = \max\{|U_{i,r}(r)|; r \in I, i = 1, 2\}$. Noticing $f_{p,X}(X)$ is positive decreasing on $(0, \infty)$, when $1 < p < 2$, we have

$$\begin{aligned} |r^{n-1} \{f_p(U_{1,r}(r)) - f_p(U_{2,r}(r))\}| \\ \geq (r_0 - \epsilon)^{n-1} (p-1) K_1^{p-2} |U_{1,r}(r) - U_{2,r}(r)|. \end{aligned}$$

We set $K_0 = \max\{|U_i(r)|; r \in I, i = 1, 2\}$.

$$V_1(r) - V_2(r) = - \int_{r_0}^r \tau^{n-1} \{f_q(U_1(\tau)) - f_q(U_2(\tau))\} d\tau.$$

On the other hand, we have

$$\begin{aligned} |f_q(U_1(\tau)) - f_q(U_2(\tau))| &= \left| \int_{U_2(\tau)}^{U_1(\tau)} f_{q,X}(\sigma) d\sigma \right| \\ &\leq (q-1) K_2^{q-2} |U_1(\tau) - U_2(\tau)| \\ &\leq (q-1) K_2^{q-2} \left| \int_{r_0}^r \{U_{1,r}(\sigma) - U_{2,r}(\sigma)\} d\sigma \right| \\ &\leq (q-1) K_2^{q-2} \epsilon \|U_{1,r} - U_{2,r}\|_{J(\epsilon)}, \end{aligned}$$

where $\|U\|_{J(\epsilon)} = \max_{|r-r_0| \leq \epsilon} |U(r)|$.

Choosing ϵ sufficiently small, we conclude

$$\|U_{1,r} - U_{2,r}\|_{J(\epsilon)} = 0.$$

Hence, $V_1 = V_2$ on $J(\epsilon)$, therefore, $V_{1,r} = V_{2,r}$, which gives $U_1(r) = U_2(r)$ on $J(\epsilon)$

Since we can proceed the rest as in [1], we show only classification of cases.

Case (ii): $1 < p \leq 2$ and $1 < q < 2$

Subcase (ii-1): We assume also $U_0 \neq 0$.

Subcase (ii-2): $U_0 = 0, V_0 \neq 0$.

Case (iii): $2 < p$, and $2 \leq q$.

Subcase (iii-1): $V_1(r_0) = V_2(r_0) \neq 0$.

Subcase (iii-2): $V_1(r_0) = V_2(r_0) = 0$ and $U_1(r_0) = U_2(r_0) \neq 0$.

Case (iv): $2 < p$ and $1 < q < 2$.

Subcase (iv-1): $U_0 = 0$, and $V_0 \neq 0$.

Subcase (iv-2): $U_0 \neq 0$, and $V_0 = 0$.

We need different arguments, when $U_0 = V_0 = 0$. We assume $U_0 = V_0 = 0$.

To complete the proof, we use an energy inequality.

Proposition 2. (i) *Every nonzero $W^{1,p}(I)$ solution U on I has $C^1(\bar{I})$ regularity.*

(ii) *Then, it satisfies the energy equality*

$$\begin{aligned} & \frac{p-1}{p}|U_r(r)|^p + \frac{1}{q}|U(r)|^q \\ &= \frac{p-1}{p}|U_r(c)|^p + \frac{1}{q}|U(c)|^q \\ & - (n-1) \int_c^r \frac{1}{\sigma}|U_r(\sigma)|^p d\sigma \end{aligned} \tag{10}$$

for all $r, c \in I$.

(iii) *If $U(r_0) = U_r(r_0) = 0$ for some $r_0 \in I$, then, $U(r) = 0$ on I .*

Remark 2.1. (i) is due to M. Ôtani [6].

When $n = 1$, (ii) is the energy equality.

When $n = 1$, (iii) is trivial (for any $p, q > 1$) in virtue of the energy equality. When $n > 1$, this completes the proof of local uniqueness.

Proof of (iii). $U(r) = 0$ for all $r \in [r_0, b]$ in virtue of the energy inequality.

For all $r \in [a, r_0]$ we have

$$\frac{p-1}{p}|U_r(r)|^p + \frac{1}{q}|U(r)|^q = (n-1) \int_r^{r_0} \frac{1}{\sigma}|U_r(\sigma)|^p d\sigma,$$

therefore,

$$\frac{p-1}{p}|U_r(r)|^p \leq (n-1) \int_r^{r_0} \frac{1}{\sigma}|U_r(\sigma)|^p d\sigma.$$

By Gronwall's lemma, we have $U_r(r) = 0$ on I . Since $U(r_0) = 0$, it implies $U(r) = 0$ on I . $\square$

Remark 2.2. We note a different proof, when $q \geq p$ as in [2].

We note

$$r^{n-1} f_p(U_r(r)) = V(r) = \int_{r_0}^r V_r(\tau) d\tau = - \int_{r_0}^r \tau^{n-1} f_q(U(\tau)) d\tau.$$

We have $r_0^{n-1} \|U_r\|_{J(\epsilon)}^{p-1} \leq \epsilon \|V_r\|_{J(\epsilon)} \leq \epsilon(r_0 + \epsilon)^{n-1} \|U\|^{q-1} \leq \epsilon^q(r_0 + \epsilon)^{n-1} \|U_r\|_{J(\epsilon)}^{q-1}$. If we assume $\|U\|_{J(\epsilon)} > 0$, we have $r_0^{n-1} \leq \epsilon^q \|U_r\|_{J(\epsilon)}^{q-p}$ for any small positive ϵ . This gives contradiction, hence $\|U\|_{J(\epsilon)} = 0$.

4 Analytic singularities

We shall now describe local analytic singularities of the solution $U(r)$ to (4).

When $n = 1$, a classical Briot-Bouquet type theorem of one variable was sufficient to obtain the unique existence of analytic solution to the nonlinear ordinary differential equation ([8]).

When $U(r_0) \neq 0$ and $U_r(r_0) \neq 0$, r_0 is an analytic point of solution $U(r)$. Hence, we consider two types of singularities:

- $r_0 = \sigma$ where $U(\sigma) = 0$ and $U_r(\sigma) = A \neq 0$,
- $r_0 = \tau$ where $U(\tau) = A \neq 0$ and $U_r(\tau) = 0$.

CASE 1. σ where $U(\sigma) = 0$ and $U_r(\sigma) = A \neq 0$. We assume $A > 0$ without loss of generality. We treat the case where $\sigma > 0$.

Theorem 3. *For $1 < p, q < \infty$, there exists a unique analytic function $F(t, s)$ in a neighborhood of the origin such that we have near $r = \sigma$*

$$U(r) = (r - \sigma)F(r - \sigma, |r - \sigma|^q). \quad (11)$$

$F(t, s)$ is a holomorphic solution to

$$\begin{aligned} & (p-1)(\sigma+t)\{F(t,s) + tF_t(r,s) + qsF_s(t,s)\}^{p-2} \\ & \{tF_t(t,s) + qsF_s(t,s) \\ & + t(tF_t(t,s))_t + 2qtsF_{t,s}(t,s) + q^2s(sF_s(t,s))_s\} \\ & + (\sigma+t)s(F(t,s))^{q-1} \quad (\text{continued}) \end{aligned} \quad (12)$$

$$\begin{aligned}
& + (n-1)t\{F(t, s) + tF_t(t, s) + qsF_s(t, s)\}^{p-1} \\
& = 0
\end{aligned} \tag{13}$$

with

$$F(0, 0) = A. \tag{14}$$

Consequently, we can compute the expansion of $U(r)$ at $r = \sigma$:

$$\begin{aligned}
U(r) = (r - \sigma) \left\{ A - \frac{n-1}{2\sigma(p-1)} A(r - \sigma) \right. \\
\left. - \frac{A^{q-p+1}}{(p-1)q(q+1)} |r - \sigma|^q + \dots \right\}.
\end{aligned} \tag{15}$$

Proof. We reduce equation (13) by change of the unknown function

$$F(t, s) = A + h(t, s)$$

into

$$\begin{aligned}
L \left(t \frac{\partial}{\partial t}, s \frac{\partial}{\partial s} \right) h(t, s) &= - \frac{1}{(p-1)(\sigma + t)} \\
&\times \{A + h(t, s) + th_t(t, s) + qsh_s(t, s)\}^{2-p} \\
&\times [(\sigma + t)s(A + h(t, s))^{q-1} + (n-1)t \\
&\{A + h(t, s) + th_t(t, s) + qsh_s(t, s)\}^{p-1}]
\end{aligned} \tag{16}$$

$$\begin{aligned}
&= a_{1,0}t + a_{0,1}s \\
&+ \sum_{2 \leq p+q+i+j+k} a_{p,q,i,j,k} t^p s^q \\
&\times (h(t, s))^i \left(t \frac{\partial h}{\partial t}(t, s) \right)^j \left(s \frac{\partial h}{\partial s}(t, s) \right)^k,
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
a_{1,0} &= - \frac{(n-1)A}{(p-1)\sigma} \\
a_{0,1} &= - \frac{1}{p-1} A^{q-p+1}.
\end{aligned} \tag{18}$$

We have $L(\alpha, \beta) = \alpha + q\beta + \alpha^2 + 2q\alpha\beta + q^2\beta^2$.

Since $L(1, 0) = 2$ and $L(0, 1) = q(q + 1)$, $F_t(0, 0)$ and $F_s(0, 0)$ are determined and so on. Thus, the unique existence of the solution is obtained by the B-B type theorem of two variables.

Next, $(r - \sigma)F(r - \sigma, |r - \sigma|^q)$ is a C^2 function near σ . It satisfies (1) with the prescribed Cauchy data. By Proposition 1, it is equal to the unique solution $U(r)$ with the same Cauchy data. $\square$

Corollary 1 ([4], [7],[8]). (i) When q is an even integer more than 1, the solution $U(r)$ is real analytic near σ .

(ii) When q is not an even integer, the solution $U(r)$ is of class $C^{<q>}$ at σ , where $<x>$ is the least integer greater than or equal to x .

CASE 2. τ where $U(\tau) = A$ and $U_r(\tau) = 0$.

As in the case 1, we can assume without loss of generality that $A > 0$.

Theorem 4. For any p and q satisfying $1 < p, q < \infty$, there exists a unique analytic function $G(t, s)$ in a neighborhood of the origin such that we have near $r = \tau$

$$U(r) = A + |r - \tau|^{\frac{p}{p-1}} G\left(r - \tau, |r - \tau|^{\frac{p}{p-1}}\right), \quad (19)$$

where $G(t, s)$ is a holomorphic solution to the nonlinear equation:

$$\begin{aligned} & (p-1)(t+\tau)\left\{-\left(\frac{p}{p-1}G(t,s)\right.\right. \\ & \quad \left.+tG_t(t,s)+\frac{p}{p-1}sG_s(t,s)\right\}^{p-2} \\ & \quad \times \left\{\frac{p}{(p-1)^2}G(t,s)+\frac{p+1}{p-1}tG_t(t,s)\right. \\ & \quad \left.+\frac{p(p+1)}{(p-1)^2}sG_s(t,s)+t(tG_t)_t(t,s)\right. \\ & \quad \left.+\frac{2p}{p-1}tsG_{t,s}(t,s)+\frac{p^2}{(p-1)^2}s(sG_s)_s(t,s)\right\} \\ & \quad +(\tau+t)(A+sG(t,s))^{q-1} \\ & \quad - (n-1)t\left\{-\left(\frac{p}{p-1}G(t,s)\right.\right. \\ & \quad \left.+tG_t(t,s)+\frac{p}{p-1}sG_s(t,s)\right\}^{p-1} = 0 \end{aligned} \quad (20)$$

$$\quad \quad \quad (21)$$

with

$$G(0, 0) = -\frac{p-1}{p} A^{\frac{q-1}{p-1}}.$$

Consequently, we have a convergent expansion near $r = \tau$:

$$\begin{aligned} U(r) = & A + B|r - \tau|^{\frac{p}{p-1}} + C(r - \tau)|r - \tau|^{\frac{p}{p-1}} \\ & + D|r - \tau|^{\frac{2p}{p-1}} + \dots, \end{aligned} \quad (22)$$

where

$$B = -\frac{p-1}{p} A^{\frac{q-1}{p-1}} \text{ and} \quad (23)$$

$$C = \frac{(n-1)}{2(2p-1)} A^{\frac{q-1}{p-1}} \quad (24)$$

$$D = \frac{q-1}{2(2p-1)} \left(\frac{p-1}{p} \right)^2 A^{1+\frac{2(q-p)}{p-1}}. \quad (25)$$

Proof. We show, at first, unique existence of the solution $G(t, s)$.

We reduce the equation by change of unknown function by

$$G(t, s) = B + h(t, s),$$

into

$$\begin{aligned} & \frac{p}{(p-1)^2} h(t, s) + \frac{p+1}{p-1} t h_t(t, s) + \frac{p(p+1)}{(p-1)^2} s h_s(t, s) \\ & + t(t h_t)_t(t, s) + \frac{2p}{p-1} t s h_{t,s}(t, s) \\ & + \frac{p^2}{(p-1)^2} s(s h_s)_s(t, s) \end{aligned} \quad (26)$$

$$\begin{aligned} & = -\frac{p}{(p-1)^2} B - \frac{1}{(p-1)(t+\tau)} \\ & \times \left\{ -\frac{pB}{p-1} - \frac{p}{p-1} h(t, s) \right. \\ & \left. - t h_t(t, s) - \frac{p}{p-1} s h_s(t, s) \right\}^{2-p} \\ & \times \{(\tau+t)(A + sB + s h(t, s))\}^{q-1} \end{aligned} \quad (27)$$

$$\left. \begin{aligned} & - (n-1)t \left(-\frac{p}{p-1}B - \frac{p}{p-1}h(t,s) \right. \\ & \left. - th_t(t,s) - \frac{p}{p-1}sh_s(t,s) \right)^{p-1} \end{aligned} \right\}. \quad (28)$$

Developing the right hand side with respect ot (t, s, h, ρ, θ) , where $\rho = sh_s$ and $\theta = sh_s$, we, at first, obtain

$$B = -\frac{p-1}{p}A^{\frac{q-1}{p-1}}$$

by the condition that the constant term vanishes:

$$-\frac{p}{(p-1)^2}B - \frac{1}{p-1} \left\{ -\frac{p}{p-1}B \right\}^{2-p} A^{q-1} = 0.$$

Then, we have the development of the R.H.S. is

$$\begin{aligned} R.H.S. &= a_{1,0,0,0,0}t + a_{0,1,0,0,0}s \\ &+ \frac{2p-p^2}{(p-1)^2}h(t,s) + \frac{2-p}{p-1}th_t(t,s) \end{aligned} \quad (29)$$

$$\begin{aligned} &+ \frac{2p-p^2}{(p-1)^2}sh_s(t,s) \\ &+ \sum_{2 \leq \alpha + \beta + i + j + k} a_{\alpha, \beta, i, j, k} t^\alpha s^\beta \\ &\times (h(t,s))^i \left(t \frac{\partial h}{\partial t}(t,s) \right)^j \left(s \frac{\partial h}{\partial s}(t,s) \right)^k. \end{aligned} \quad (30)$$

Set

$$\begin{aligned} l_{0,0} &= \frac{p}{(p-1)^2} - \frac{2p-p^2}{(p-1)^2} = \frac{p}{p-1}, \\ l_{1,0} &= \frac{p+1}{p-1} - \frac{2-p}{p-1} = \frac{2p-1}{p-1}, \\ l_{0,1} &= \frac{p^2+p}{(p-1)^2} - \frac{2p-p^2}{(p-1)^2} = \frac{p(2p-1)}{(p-1)^2}, \\ l_{2,0} &= 1, l_{1,1} = \frac{2p}{p-1}, l_{0,2} = \frac{p^2}{(p-1)^2}. \end{aligned}$$

Thus, we have the reduced equation

$$\begin{aligned} & L \left(t \frac{\partial}{\partial t}, s \frac{\partial}{\partial s} \right) h(t, s) \\ &= a_{1,0,0,0,0} t + a_{0,1,0,0,0} s \\ &+ \sum_{2 \leq \alpha + \beta + i + j + k} a_{\alpha, \beta, i, j, k} t^\alpha s^\beta \end{aligned} \quad (31)$$

$$\times (h(t, s))^i \left(t \frac{\partial h}{\partial t}(t, s) \right)^j \left(s \frac{\partial h}{\partial s}(t, s) \right)^k. \quad (32)$$

Since L satisfies the condition (5), the unique existence of the solution to (21) is obtained by our Briot-Bouquet type theorem. $\square$

Corollary 2 ([4], [7],[8]). (i) If $p/(p-1)$ is an even integer, i.e. $p = (2m+2)/(2m+1)$ ($m = 0, 1, 2, \dots$), $u(x)$ is real analytic at τ . (ii) If $p/(p-1)$ is not an even integer, the solution $U(r)$ is of class $C^{<\frac{2-p}{p-1}>+1}$ at τ , where $<x>$ is the least integer greater than or equal to x . Especially, when $1 < p \leq 2$, $U(r)$ is of class C^2 at τ . When $2 < p$, $U(r)$ is not of class C^2 at τ .

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