

EXAMPLES OF SEMI-STABLE DEGENERATIONS OF KUNEV SURFACES

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0. This article consists of some examples of semi-stable degenerations of Kunev surfaces together with some remarks concerning about a compactification of their moduli space and the Torelli problem.

A Kunev surface X is defined as a canonical surface with $\chi(\mathcal{O}_X) = 2$ and $(\omega_X)^2 = 1$ which has an involution σ such that X/σ is a K3 surface with rational double points (RDP, for short). Let $\hat{X}$ be the minimal model of a Kunev surface. Then it is known that $\hat{X}$ is a simply connected surface of general type with $p_g = c_1^2 = 1$ and its bicanonical map is a Galois cover of $\mathbb{P}^2$ with $\mathcal{Gal} \simeq (\mathbb{Z}/2)^{\oplus 2}$ whose branch locus consists of two cubics and a line on $\mathbb{P}^2$, and that $\hat{X}$ has the following numerical invariants:

$$\begin{aligned} (0.1) \quad & h^{2,0}(\hat{X}) = h^{0,2}(\hat{X}) = 1, \quad h_{prim}^{1,1}(\hat{X}) = 18. \\ & H^0(T_{\hat{X}}) = H^2(T_{\hat{X}}) = 0, \quad h^1(T_{\hat{X}}) = 18. \\ & h^{2,0}(\hat{X})^\sigma = h^{0,2}(\hat{X})^\sigma = 1, \quad h_{prim}^{1,1}(\hat{X})^\sigma = 10. \\ & h^1(T_{\hat{X}})^\sigma = 12. \end{aligned}$$

It is also known that the moduli space $\mathfrak{M}$ (resp. $\mathfrak{R}$) of surfaces with $p_g = c_1^2 = 1$ (resp. Kunev surfaces) is irreducible and rational (resp. irreducible). (For the above facts, see [Ca.1], [Ca.2], [U.2], [T.2], [SSU], [M].)

On the Hodge theoretic view-point, surfaces with $p_g = c_1^2 = 1$ or, as its subfamily, Kunev surfaces are interesting materials. After Kunev constructed an example of a Kunev surface as a

counterexample to the infinitesimal Torelli theorem, the following results are known:

(0.2) *The generic infinitesimal Torelli theorem holds for surfaces in $\mathfrak{M}$ ([Ca.1]).*

(0.3) *The period map Φ_2 of surfaces in $\mathfrak{M}$ has positive dimensional fibers ([T.1], [U.1], [U.2]; [T.1] treats only Kunev surfaces).*

(0.4) $\mathfrak{R}$ in $\mathfrak{M}$ is characterized by $\dim \Phi_2^{-1} \Phi_2([X]) = 2$, which is the maximal dimension of the fibers of Φ_2 ([U.1]).

(0.5) *The infinitesimal mixed Torelli theorem holds for pairs (X, C) of surfaces X in $\mathfrak{M}$ and their smooth canonical curves C ([U.3]).*

(0.6) *The generic mixed Torelli theorem holds for Kunev surfaces ([L], [SSU]; there is a point about monodromy which is not clear in [L]).*

(0.7) *There exists a Zariski open subset U of $\mathfrak{R}$ such that $\Phi^{-1}\Phi(U) = U$, where $\Phi : \mathfrak{M} \longrightarrow \Gamma \backslash D$ is the mixed period map ([SSU]).* Hence, in order to solve the mixed Torelli problem for surfaces in $\mathfrak{M}$ via Kunev locus $\mathfrak{R}$, it is necessary to study the following:

(0.8) *A compactification of the mixed period map $\Phi : \mathfrak{M} \longrightarrow \Gamma \backslash D$.*

(0.9) *The monodromy Γ in (0.7), where we used a geometric one.*

(For a general reference of the above as well as for the terminology such as *mixed* period map, *mixed* Torelli etc., see [SSU].)

We shall report here some results in an experiment concerning about the problem (0.8). We shall construct some examples of semi-stable degenerations of pairs of Kunev surfaces and their canonical curves in the sequel. The examples in 1, 2 and 3 are of

type I with respect to the local monodromy of the pure second cohomology. The examples in 4 and 5 are of *type II* and *type III* respectively.

As for the problem of a compactification of the moduli space $\mathfrak{R}$ of Kunev surfaces, Horikawa and Shah constructed a compactification of the moduli space of K3 surfaces of degree 2 as one of the moduli space of sextic curves on $\mathbb{P}^2$ by the geometric invariant theory ([H], [Sh]). The latter contains a 10-dimensional subspace $\bar{\mathfrak{R}}$ which is a compactification of

$$\mathfrak{R} = \{ \Sigma C_j \in \text{Sym}^2 | \mathcal{O}_{\mathbb{P}^2}(3) \mid \mid \Sigma C_j \text{ has only simple singularities} \} / \text{SL}_3.$$

A "compactification" of $\mathfrak{R}$ sits over $\bar{\mathfrak{R}}$.

We use the terminology a *numerical K3 surface* for a minimal surface with $p_g = 1$, $q = 0$ and $c_1^2 = 0$ (for this terminology, cf. also [K]). Numerical K3 surfaces appeared in this article have an elliptic fibration with one double fiber.

1. Let C_1 and C_2 be general cubics on $\mathbb{P}^2$. Denote by $\check{C}_j \subset \check{\mathbb{P}}^2$ the dual curve of $C_j \subset \mathbb{P}^2$, i.e., the image of the Gauss map. Then each $\check{C}_j$ has nine cusps corresponding to nine inflexes on C_j , $\Sigma \check{C}_j$ has nine bitangents $\check{D}_i$ with tangent points P_{i1} and P_{i2} ($1 \leq i \leq 9$) subjected to nine nodes of ΣC_j , and we have two stratifications of $\check{\mathbb{P}}^2$ determined by $\Sigma \check{C}_j$ and $\Sigma \check{D}_i$:

$$\begin{aligned} (1.1) \quad \check{\mathbb{P}}^2 &= (\check{\mathbb{P}}^2 - \Sigma \check{C}_j) \amalg (\Sigma \check{C}_j - (\Sigma \check{P}_{ji} + \text{Sing}(\Sigma \check{C}_j))) \\ &\quad \amalg (\Sigma \text{Sing}(\check{C}_j)) \amalg (\cap \check{C}_j) \amalg (\Sigma \check{P}_{ji}) \\ &=: R_0 \amalg R_1 \amalg R'_1 \amalg R_2 \amalg R'_0. \end{aligned}$$

$$(1.2) \quad \check{\mathbb{P}}^2 = (\check{\mathbb{P}}^2 - \Sigma \check{D}_i) \amalg (\Sigma \check{D}_i - \text{Sing}(\Sigma \check{D}_i)) \amalg \text{Sing}(\Sigma \check{D}_i)$$

$$=: S_0 \amalg S_1 \amalg S_2.$$

We denote by Y the minimal K3 surface which is obtained as the minimal resolution of the double cover of $\mathbb{P}^2$ branched along ΣC_j . Let $\alpha_1 : Y \rightarrow \mathbb{P}^2$ be the projection and E_i ($1 \leq i \leq 9$) be the exceptional curves for α_1 , i.e., (-2) -curves.

(1.3) Case $t_0 \in S_1 \cap R_0$: We may assume $t_0 \in \check{D}_1$. Let Δ be a small disc with center $0 = t_0$ intersecting transversely with $\check{D}_1$ such that $\Delta^* := \Delta - \{0\} \subset S_0$. We denote by $\mathcal{L} \subset \Delta \times \mathbb{P}^2$ the total space of the family of lines $\{L_t\}_{t \in \Delta}$. We can construct a semi-stable degeneration of pairs of Kunev surfaces and the canonical curves over Δ in the following way: (0) Set $\alpha = 1 \times \alpha_1 : \Delta \times Y \rightarrow \Delta \times \mathbb{P}^2$ and $\delta_i = \Delta \times E_i$ ($1 \leq i \leq 9$). (i) Let $\beta : \mathcal{Y} \rightarrow \Delta \times Y$ be the blowing-up along $\alpha^{-1}\mathcal{L} \cap \delta_1$. Denote by $W_{\mathcal{Y}}$ the exceptional divisor. (ii) Take the double cover $\gamma : \mathcal{X}_1 \rightarrow \mathcal{Y}$ branched along $(\alpha\beta)^{-1}\mathcal{L} + \beta^{-1}(\Sigma \delta_i)$. (iii) Let $\delta : \mathcal{X}_1 \rightarrow \mathcal{X}$ be the contraction of $(\beta\gamma)^{-1}(\Sigma \delta_i)$. (iv) Let $\mathcal{X} \rightarrow \bar{\mathcal{X}}$ be the contraction of $\delta(\beta\gamma)^{-1}W_{\mathcal{Y}}$. (In the above notation, we use $\alpha^{-1}\mathcal{L}$ etc. as the proper transforms.)

Set $\mathcal{L}_{\mathcal{X}} = (\delta(\alpha\beta\gamma)^{-1}\mathcal{L}$ with reduced structure) and $W_{\mathcal{X}} = \delta\gamma^{-1}W_{\mathcal{Y}}$. Then we can prove the following: (a) The projection $f : (\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \rightarrow \Delta$ is a semi-stable degeneration of pairs (for the terminology, see [SSU]). $K_{\mathcal{X}} = \mathcal{L}_{\mathcal{X}} + W_{\mathcal{X}}$. (b) The pair $(X_t, L_{X,t}) := f^{-1}(t)$ consists of a minimal Kunev surfaces and its canonical divisor which is an ample, smooth curve of genus 2 for $t \in \Delta^*$. The central fiber of f is as Figure 1. (c) $\bar{\mathcal{X}}$ has an isolated singular point, raised from $C_1 \cap C_2 \cap L_0$, which is analytically isomorphic to the cone over the Veronese embedding of $\mathbb{P}^2 \subset \mathbb{P}^5$ by $|O_{\mathbb{P}^2}(2)|$. $K_{\bar{\mathcal{X}}}$ is

nef.

(1.4) Case $t_0 \in R'_0$: We may assume $t_0 = \check{P}_{11}$. We use the notation Δ , $\mathcal{L}$ etc. in the same sense as in (1.3). The construction (0)-(iv) in (1.3) yields a family of pairs $f : (\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \longrightarrow \Delta$ and $\bar{\mathcal{X}}$. We can prove the following: (a) *The central fiber X_0 of f is not a divisor with normal crossings (see Figure 2). The total space $\mathcal{X}$ is smooth and $K_{\mathcal{X}} = \mathcal{L}_{\mathcal{X}} + W_{\mathcal{X}}$.* (b) *After a base extension $\Delta_2 \longrightarrow \Delta$, $s \mapsto t = s^2$, we can get a semi-stable reduction $f' : (\mathcal{X}', \mathcal{L}_{\mathcal{X}'}) \longrightarrow \Delta_2$ of f whose central fiber is as Figure 3. (Because of the limit of pages, we omit the details of the process of reduction and contraction.)* (c) *The same statement as (c) in (1.3) holds.*

(1.5) Case $t_0 \in S_2$: We may assume $t_0 \in \check{D}_1 \cap \check{D}_2$. We use the notation Δ , $\mathcal{L}$ etc. in a similar sense as in (1.3). An analogous construction (0)-(iv) as in (1.3) yields a family of pairs $f : (\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \longrightarrow \Delta$ and $\bar{\mathcal{X}}$ with the following properties: (a) *The same statement as (a) in (1.3) holds.* (b) *The central fiber of f is as Figure 4.* (c) *$\bar{\mathcal{X}}$ has two Veronese cone singularities raised from two points $C_1 \cap C_2 \cap L_0$. $K_{\bar{\mathcal{X}}}$ is nef.*

Remark. For fixed general cubics C_1 and C_2 , we constructed a complete family of degenerations of Kunev surfaces over $\check{\mathbb{P}}^2$ and described their fibers X_t , $t \in \check{\mathbb{P}}^2$, in [U.4, I], [U.5]. We point out here only that the stratification (1.1) controls RDP on the main components V_t of X_t , whereas the stratification (1.2) controls their Kodaira dimensions. In fact, we know that *the main component V_t is a (singular) Kunev surface, numerical K3 surface with one double fiber, or K3 surface according to $t \in S_0$, S_1 , or S_2 (for*

more general statement, see Remark in 3).

2. Next we consider the case that both cubics C_1 and C_2 degenerate into two pairs of cocurrent three lines. Denote by Q_j the triple point of C_j ($j = 1, 2$). We deal with the general situation in this case, i.e., we assume moreover that ΣC_j consists of six different lines and that $Q_1 \neq Q_2$. Denote by $\check{Q}_j$ the line on $\check{\mathbb{P}}^2$ corresponding to the pencil of lines through Q_j on $\mathbb{P}^2$ ($j = 1, 2$). Then we have a stratification of $\check{\mathbb{P}}^2$:

$$(2.1) \quad \check{\mathbb{P}}^2 = (\check{\mathbb{P}}^2 - \Sigma \check{Q}_j) \amalg (\Sigma \check{Q}_j - \check{Q}_1 \cap \check{Q}_2) \amalg (\check{Q}_1 \cap \check{Q}_2) \\ =: T_0 \amalg T_1 \amalg T_2.$$

Let Y be the minimal K3 surface which is obtained as the minimal resolution of the double cover of $\mathbb{P}^2$ branched along ΣC_j and let $\alpha_1 : Y \rightarrow \mathbb{P}^2$ be the projection. We denote by E_i ($1 \leq i \leq 9$) the (-2) -curves on Y which are mapped by α_1 to the nine points $C_1 \cap C_2$. Notice that, beside the E_i , Y carries two D_4 -configurations of (-2) -curves over the triple points Q_j of C_j .

(2.2) Case $t_0 \in T_1$ and the line $L_{t_0} \not\subset \Sigma C_j$: In this case,

the line L_{t_0} passes one and only one of the two triple points, say Q_1 . Let Δ be a small disc with center $0 = t_0$ intersecting transversely with $\check{Q}_1$ such that $\Delta^* \subset T_0 - (\Sigma C_j)^\vee$. Let $\mathcal{L} \subset \Delta \times \mathbb{P}^2$ be the total space of the family of lines as before. Similarly as in (1.3), we can construct a family of pairs $f : (\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \rightarrow \Delta$ which is a degeneration of pairs of Kunev surfaces and the canonical curves in the following way: (0) Set $\alpha = 1 \times \alpha_1 : \Delta \times Y \rightarrow \Delta \times \mathbb{P}^2$, $\mathcal{E}_i = \Delta \times E_i$ ($1 \leq i \leq 9$) and $\mathcal{Q}_j = \Delta \times \alpha_1^{-1}(Q_j)$ ($j = 1, 2$).

- (i) Take the double cover $\beta : \mathcal{X}_1 \longrightarrow \Delta \times Y$ branched along $\alpha^{-1}\mathcal{L} + \sum \mathcal{E}_i$. (ii) Let $\gamma : \mathcal{X}_1 \longrightarrow \mathcal{X}$ be the contraction of $\beta^{-1}(\sum \mathcal{E}_i)$. (iii) Let $\delta : \mathcal{X} \longrightarrow \bar{\mathcal{X}}$ be the contraction of $\gamma\beta^{-1}(\sum \mathcal{D}_j)$.

Set $\mathcal{L}_{\mathcal{X}} = (\gamma(\alpha\beta))^{-1}\mathcal{L}$ with reduced structure) and $f : (\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \longrightarrow \Delta$, the projection. Then we can prove: (a) X_0 of the central fiber $f^{-1}(0) = (X_0, L_{X,0})$ is irreducible. Rising from the cocurrent four lines $C_1 + L$ on $\mathbb{P}^2$, X_0 has the singular locus which is a rational curve consisting of double points, on which there are four cuspidal points. The total space $\mathcal{X}$ is smooth and $K_{\mathcal{X}} = \mathcal{L}_{\mathcal{X}}$ which is nef. (b) After a base extension $\Delta_2 \longrightarrow \Delta$, $s \mapsto t = s^2$, we get a semi-stable degeneration $f' : (\mathcal{X}', \mathcal{L}_{\mathcal{X}'}) \longrightarrow \Delta_2$ of pairs of K3 surfaces and the canonical curves, whose central fiber is as Figure 6. (We omit the details of the process.) (c) Rising from the two triple points Q_1 and Q_2 , $\bar{\mathcal{X}}$ has four compounds R.D.P. of type D_4 , two of which coming from Q_1 clash to make up a simple elliptic singularity of type $\tilde{E}_8$ in the sense of K. Saito on the central fiber $\bar{X}_0$ with a local equation

$$z^2 + y(x^4 + y^2) = 0.$$

(2.3) Case $t_0 \in T_2$: We use the notation Δ , $\mathcal{L}$, $\mathcal{E}_i$, $\mathcal{D}_j$ etc in (2.2). An analogous construction (0)-(iii) in (2.2) yields a family of pairs $f : (\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \longrightarrow \Delta$ and $\bar{\mathcal{X}}$ with the following properties: (a) A similar statement of (a) in (2.2) holds but now the singular locus of X_0 consists of two copies of the rational curves as before. (b) The same statement as (b) in (2.2) holds and the central fiber of the semi-stable degeneration $f' : (\mathcal{X}', \mathcal{L}_{\mathcal{X}'}) \longrightarrow \Delta_2$ is as Figure 6. (c) A similar statement of (c) in (2.2) holds but now each pair of compounds R.D.P. of type D_4 on $\bar{\mathcal{X}}$, coming from

Q_j ($j = 1, 2$), clashes to make up a simple elliptic singularity of type E_8 .

3. Consider now the case that C_1 is degenerating into a smooth conic Q and a line L . Assume that Q , L and a smooth cubic C_2 are in general position on $\mathbb{P}^2$. Let $\{C_t\}$ be a family of cubics on $\mathbb{P}^2$ over a disc Δ such that $C_0 = Q + L$ and that C_t is smooth and intersects transversely with C_2 for $t \in \Delta^*$. Denote by $\mathcal{C}_1$ the total space of the family $\{C_t\}_{t \in \Delta}$, $\mathcal{C}_2 = \Delta \times C_2$ and $\mathcal{L} = \Delta \times L$. Assume that $\mathcal{C}_1$ is smooth. Then we can construct a degeneration of pairs $f : (\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \rightarrow \Delta$ of Kunev surfaces and their canonical curves in an analogous way as (0)-(iv) in (1.3). On this stage, X_0 of the central fiber $f^{-1}(0)$ is not reduced, hence we need a semi-stable reduction by extending the base $\Delta_2 \rightarrow \Delta$, $s \mapsto t = s^2$. Figure 7 illustrates the central fiber $(X'_0, L_{X'_0}) = f'^{-1}(0)$ of the resulting family $f' : (\mathcal{X}', \mathcal{L}_{\mathcal{X}'}) \rightarrow \Delta_2$. Contracting W'_i ($3 \leq i \leq 5$, $9 \leq i \leq 11$), W'_2 and W'_i ($6 \leq i \leq 8$) in this order, we get a simpler semi-stable degeneration $f'' : (\mathcal{X}'', \mathcal{L}_{\mathcal{X}''}) \rightarrow \Delta_2$ of Kunev surfaces. The central fiber $(X''_0, L_{X''_0}) = f''^{-1}(0)$ is as Figure 8. $K_{\mathcal{X}''}$ is not nef. Let $\mathcal{X}'' \rightarrow \bar{\mathcal{X}}$ be the contraction of W''_i ($12 \leq i \leq 17$). Then $K_{\bar{\mathcal{X}}}$ becomes nef and $\bar{\mathcal{X}}$ has six Veronese cone singularities.

Remark (cf. [U.4, III], [U.5]). Recall the notation $\mathfrak{X}$ in 0. For any fixed $\{\Sigma C_j\} \in \mathfrak{X}$, we define functions in $t \in \check{\mathbb{P}}^2$ by

$$m(t) = \sum_{P \in \mathbb{P}^2} \min\{I(P, L_t \cap C_j) \mid j = 1, 2\}, \text{ and}$$

$$n(t) = \#\{\text{triple points of } C_j \text{ on } L_t, j = 1, 2\}.$$

Notice that if C_j has a triple point then C_j consists of three distinct lines with a common point. These functions define two stratifications of $\check{\mathbb{P}}^2$:

$$\check{\mathbb{P}}^2 = S_0 \amalg S_1 \amalg S_2, \text{ where } S_m = \{t \in \check{\mathbb{P}}^2 \mid m = \min\{2, m(t)\}\}.$$

$$\check{\mathbb{P}}^2 = T_0 \amalg T_1 \amalg T_2, \text{ where } T_n = \{t \in \check{\mathbb{P}}^2 \mid n = n(t)\}.$$

Notice that $\text{codim } S_m = m$, $\text{codim } T_0 = 0$, and $\text{codim } T_n = n$ if T_n is non-empty ($n = 1, 2$). We can construct a complete family of degenerations of Kunev surfaces $\mathcal{F} : \mathcal{X} \longrightarrow \check{\mathbb{P}}^2$ and we can prove:

(3.1) The main component V_t of the fiber $\mathcal{F}^{-1}(t)$ is a (singular) Kunev surface, numerical K3 surface with one double fiber, K3 surface, elliptic surface with $p_g = q = 1$, or abelian surface according to $t \in S_0 \cap T_0$, S_1 , S_2 , $S_0 \cap T_1$, or T_2 .

(3.2) If $t \notin S_0 \cap T_0$, the main component V_t is an elliptic surface. V_t has constant J -invariant if and only if $t \in T_1 \cap T_2$. If this is the case, the K3 surface Y is a Kummer surface associated to a decomposable abelian surface $D_1 \times D_2$, where D_j is an elliptic curve ($j = 1, 2$).

Combining with the Clemens-Schmid exact sequence ([Cl]), (3.1) yields a uniform explanation of the appearance of positive dimensional fibers of the period map of Kunev surfaces and elliptic surfaces with $p_g = 1$ and $q = 0, 1$:

(3.3) $S_0 \cap T_0$, S_1 , and $S_0 \cap T_1$ appear as positive dimensional fibers of the period map of the pure second cohomology of Kunev surfaces, numerical K3 surfaces with one double fiber, and elliptic surfaces with $p_g = q = 1$ respectively. (For Kunev surfaces and elliptic surfaces with $p_g = q = 1$, these phenomena were pointed out separately in [T.1], [U.1], [U.2] and [Sa].)

4. Next we consider the case that C_1 is approaching C_2 . Let C_2 be a smooth cubic and L a line on $\mathbb{P}^2$. Assume that C_2 and L intersect transversely. Let $\{C_t\}$ be a smooth family of cubics on $\mathbb{P}^2$ over a disc Δ such that $C_0 = C_2$ and that C_t intersects transversely with C_2 for $t \in \Delta^*$. Denote by $\mathcal{C}_1$ the total space of the family $\{C_t\}_{t \in \Delta}$, $\mathcal{C}_2 = \Delta \times C_2$ and $\mathcal{L} = \Delta \times L$. Then, as before, we can construct a degeneration of pairs $f : (\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \longrightarrow \Delta$ of Kunev surfaces and their canonical curves. On this stage, X_0 of the central fiber $f^{-1}(0)$ is not reduced, hence we need a semi-stable reduction by extending the base $\Delta_2 \longrightarrow \Delta$, $s \mapsto t = s^2$. The resulting family $f' : (\mathcal{X}', \mathcal{L}_{\mathcal{X}'}) \longrightarrow \Delta_2$ has the central fiber illustrated as Figure 9. Contracting W'_i ($2 \leq i \leq 7$) we get a nef terminal model $\bar{\mathcal{X}}$ which has six Veronese cone singularities.

5. We consider here the case that C_1 and C_2 are degenerating into the same set of three distinct lines ΣM_i . Let L be a line on $\mathbb{P}^2$. Assume that $\Sigma M_i + L$ has no triple points. Let $\{C_{1,t}\}$ and $\{C_{2,t}\}$ be two families of cubics on $\mathbb{P}^2$ over a disc Δ such that $C_{1,0} = C_{2,0} = \Sigma M_i$ and that $C_{1,t}$ and $C_{2,t}$ are smooth and intersect transversely for $t \in \Delta^*$. Denote by $\mathcal{C}_1$ and by $\mathcal{C}_2$ the total space of the families $\{C_{1,t}\}_{t \in \Delta}$ and $\{C_{2,t}\}_{t \in \Delta}$ respectively and $\mathcal{L} = \Delta \times L$. We also assume that $\mathcal{C}_j$ ($j = 1, 2$) are smooth. Then, as before, we can construct a degeneration of pairs $f : (\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \longrightarrow \Delta$ of Kunev surfaces and their canonical curves. Since X_0 of the central fiber $f^{-1}(0)$ is not

reduced, we should perform a semi-stable reduction by extending the base $\Delta_2 \longrightarrow \Delta$, $s \longmapsto t = s^2$. The resulting family $f' : (\mathcal{X}', \mathcal{L}_{\mathcal{X}'}) \longrightarrow \Delta_2$ has the central fiber illustrated as Figure 10. Let $\bar{\mathcal{X}} \longrightarrow \bar{\Delta}_2$ be the contraction of W'_i ($19 \leq i \leq 24$). Then $K_{\bar{\mathcal{X}}} = \mathcal{L}_{\bar{\mathcal{X}}}$, which is nef, and $\bar{\mathcal{X}}$ has six Veronese cone singularities.

Figure 1

$$f^{-1}(0) = (X_0, L_{X,0}), \quad X_0 = V + W_{\mathfrak{A}}$$

$$K_{\mathfrak{A}} = \mathcal{L}_{\mathfrak{A}} + W_{\mathfrak{A}}$$

V : a homotopic K3 surface

$$K_V = \mathcal{L}_{\mathfrak{A}}|_V : \text{an elliptic curve}$$

$$W := W_{\mathfrak{A}} \simeq \mathbb{P}^2, \quad \mathcal{L}_{\mathfrak{A}}|_{W_{\mathfrak{A}}} : \text{a line}$$

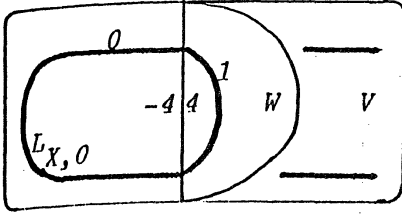


Figure 2

$$f^{-1}(0) = (X_0, L_{X,0}), \quad X_0 = V + W_{\mathfrak{A}}$$

$$K_{\mathfrak{A}} = \mathcal{L}_{\mathfrak{A}} + W_{\mathfrak{A}}$$

V : a homotopic K3 surface

$$W := W_{\mathfrak{A}} \simeq \mathbb{P}^2$$

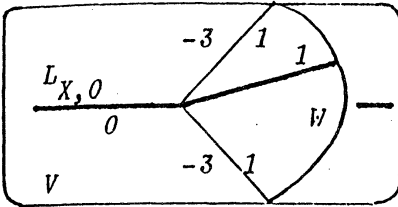


Figure 3

$$f'^{-1}(0) = (X'_0, L_{X',0}), \quad X'_0 = V' + \sum_1^3 W'_k$$

$$K_{X'} = \mathcal{L}_{X'} + \sum W'_k$$

V' : a numerical K3 surface with one double fiber

$$W'_k \simeq F_1 : \text{a rational ruled surface } (1 \leq k \leq 3)$$

$$\mathcal{L}_{X'} \cap V' : \text{an elliptic curve}$$

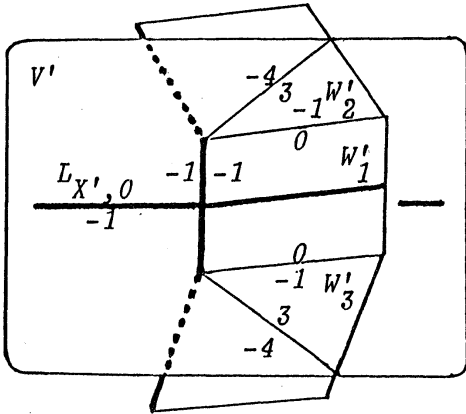


Figure 4

$$f^{-1}(0) = (X_0, L_{X,0}), \quad X_0 = V + \sum W_i$$

$$K_{\mathfrak{A}} = \mathcal{L}_{\mathfrak{A}} + \sum W_i$$

V : a K3 surface

$$K_V = \mathcal{L}_{\mathfrak{A}}|_V : \text{a } (-1)\text{-curve}$$

$$W_i \simeq \mathbb{P}^2 \quad (i = 1, 2)$$

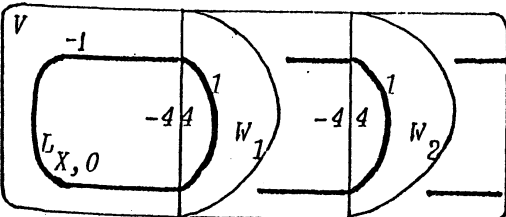
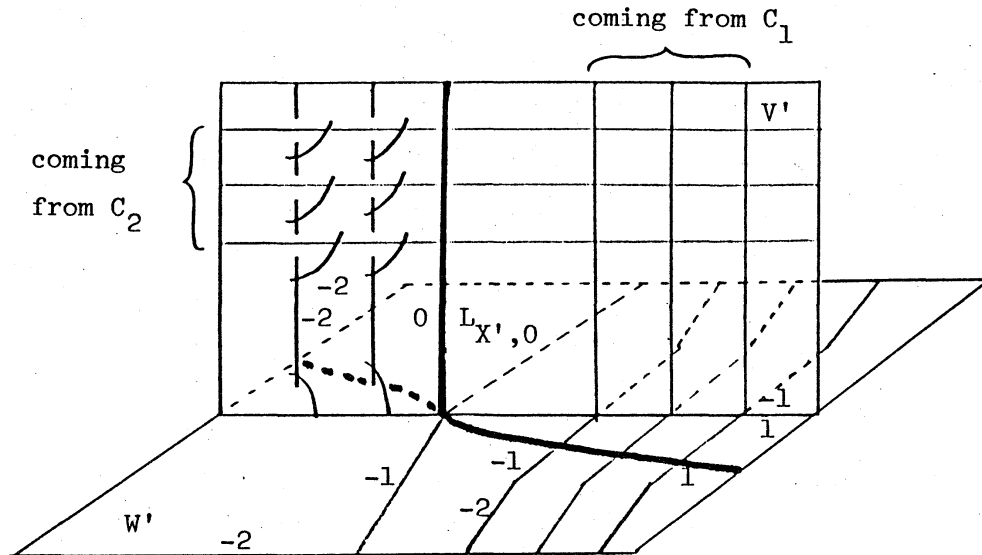


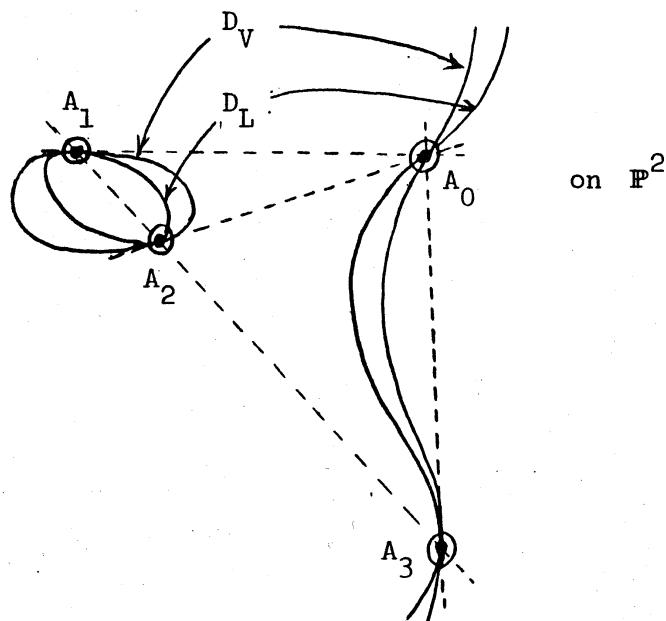
Figure 5



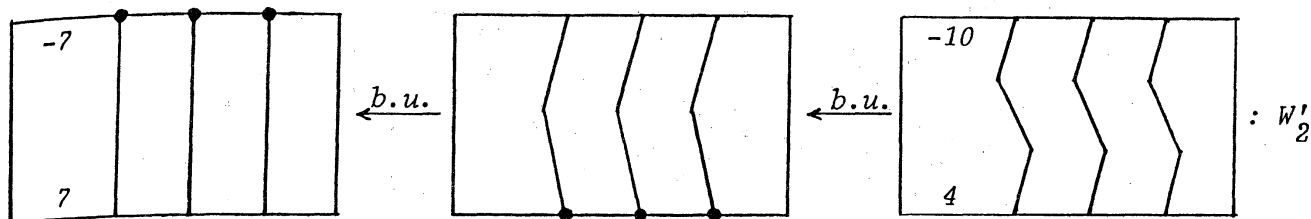
$$f'^{-1}(0) = (X'_0, L_{X'_0,0}), \quad X'_0 = V' + W', \quad K_{X'} = \mathcal{L}_{X'} + W'$$

V' : an elliptic surface with $p_g = q = 1$, with two singular fibers of type I_0^* and with four sections each of which is an elliptic curve with self-intersection -1

W' : One description is as follows. We start with a configuration below of two cubics D_V and D_L on $\mathbb{P}^2$ such that $D_V \simeq V' \cap W'$ and $D_L \simeq \mathcal{L}_{X'} \cap W'$.

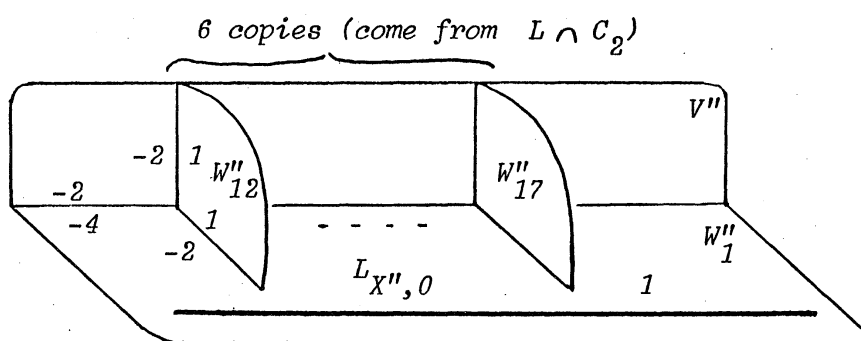


W'_2 : One description is six times blown-ups of $\mathbb{P}^1$ -bundle of degree 7 over a curve of genus 3 in the following way.



$W'_i \simeq F_2$ ($3 \leq i \leq 5$), $W'_i \simeq F_1$ ($6 \leq i \leq 8, 12 \leq i \leq 17$), $W'_i \simeq \mathbb{P}^2$ ($9 \leq i \leq 11$)

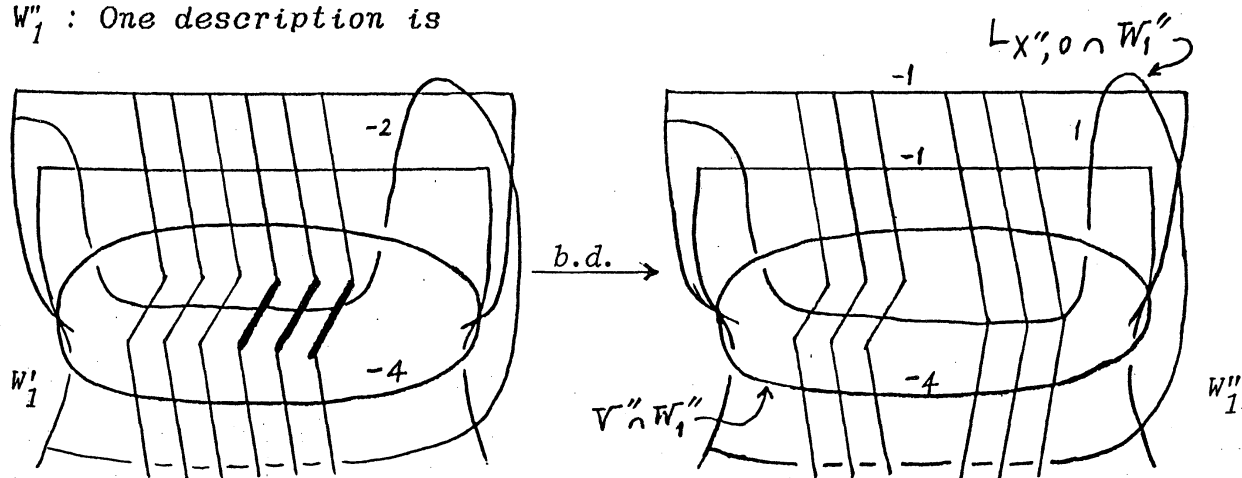
Figure 8



$$X''_0 = V'' + (\Sigma_1 + \Sigma_{12}^{17})W''_i, \quad K_{X''} = \mathcal{L}_{X''} + (\Sigma_1 + \Sigma_{12}^{17})W''_i$$

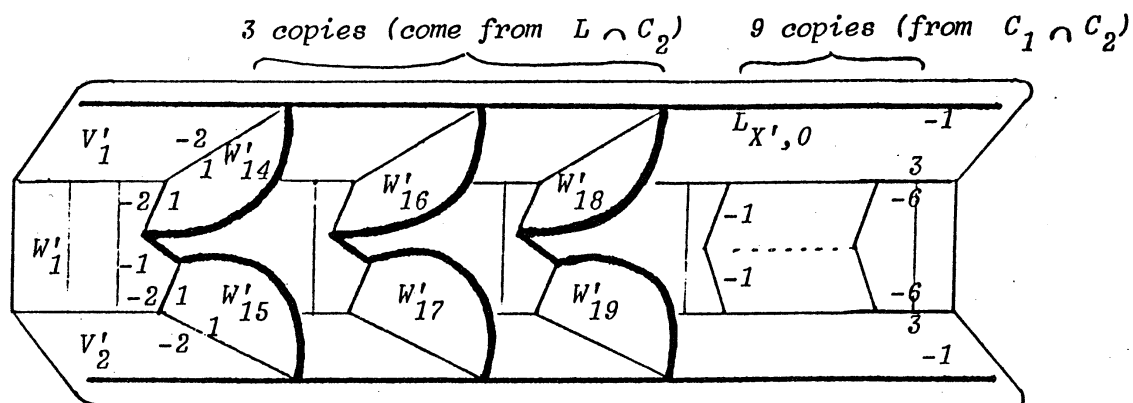
V'' : a minimal K3 surface

W''_1 : One description is



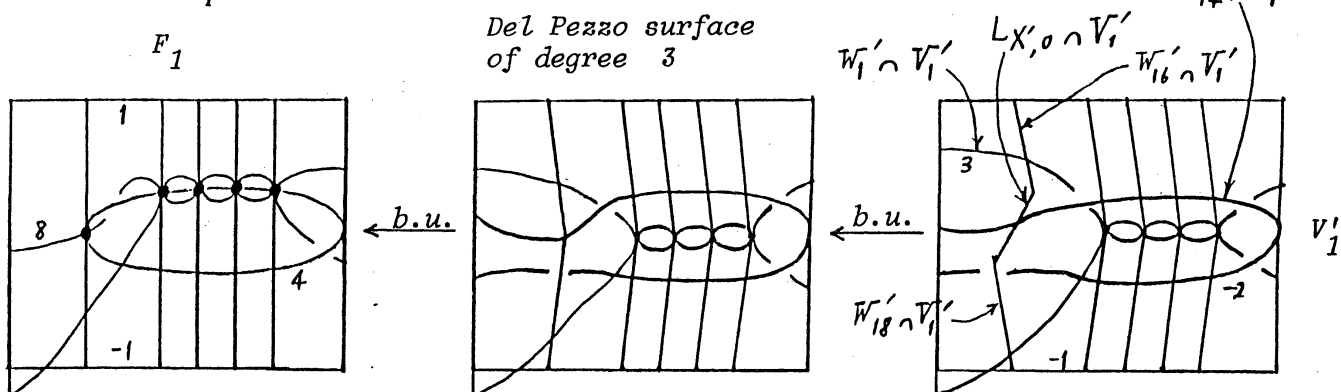
$W''_i \simeq \mathbb{P}^2$ ($12 \leq i \leq 17$), $L_{X'',0}$: a curve of genus 2

Figure 9

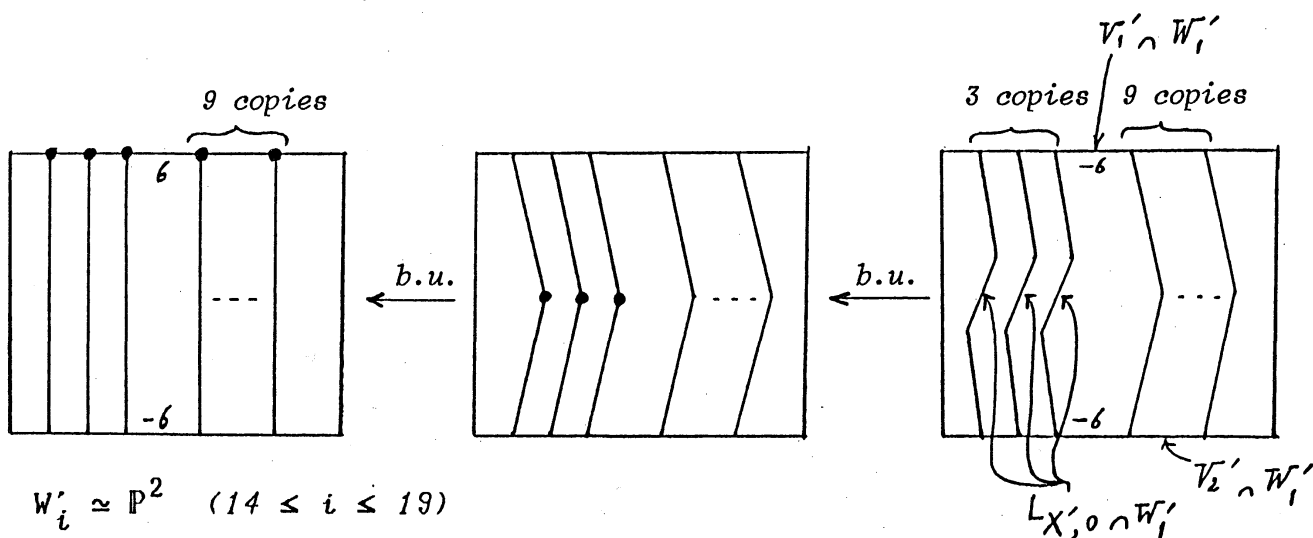


$$X'_0 = \sum_1^2 V'_i + (\sum_1 + \sum_{14}^{19}) W'_i, \quad K_{X'} = \mathcal{L}_{X'} + \sum_{14}^{19} W'_i$$

V'_i : a rational surface. One description is six times blown-ups of F_1 in the following way ($i = 1, 2$).



W'_i : One description is 15 times blown-ups of an elliptic ruled surface of degree 6 in the following way.



REFERENCES

- [Ca.1] Catanese, F., Surfaces with $K^2 = p_g = 1$ and their period mapping, Proc. Summer Meeting, Copenhagen 1978, Lect. Notes in Math. No 732, Springer-Verlag (1979) 1-29.
- [Ca.2] Catanese, F., The moduli and the global period mapping of the surfaces with $K^2 = p_g = 1$: a counter-example to the global Torelli problem, Comp. Math. 41-3 (1980) 401-414.
- [Cl] Clemens, C. H., Degenerations of Kähler manifolds, Duke Math. J. 44 (1977) 215-290.
- [H] Horikawa, E., Surjectivity of period map of K3 surfaces of degree 2, Math. Ann. 228 (1977) 113-146.
- [K] Kodaira, K., On homotopy K3 surface, Essays on topology and related topics, Mémoires dédiés à George de Rham, Springer, 1970, pp 58-69.
- [L] Letizia, M., Intersections of a plane curve with a moving line and a generic global Torelli-type theorem for Kunev surfaces, Amer. J. Math. 106-5 (1984) 1135-1146.
- [M] Morrison, D., On the moduli of Todorov surfaces, to appear.
- [Sa] Saito, M.-H., On the infinitesimal Torelli problem of elliptic surfaces, J. Math. Kyoto Univ. 23-3 (1983) 441-460.
- [Sh] Shah, J., A complete moduli space for K3 surfaces of degree 2, Ann. Math. 112 (1980) 485-510.
- [SSU] Saito, M.-H., Shimizu, Y. & Usui, S., Variation of mixed Hodge structure and Torelli problem, to appear in Advanced Study of Pure Math., North-Holland Publ. C. & Kinokuniya C. LTD.
- [T.1] Todorov, A. N., Surfaces of general type with $p_g = 1$ and (K, K)

- = 1: I, Ann. scient. Éc. Norm. Sup. 4-13 (1980) 1-21.
- [T.2] Todorov, A. N., A construction of surfaces with $p_g = 1$, $q = 0$ and $2 \leq (K^2) \leq 8$: Counterexamples of the global Torelli theorem, Invent. Math. 63 (1981) 287-304.
- [U.1] Usui, S., Period map of surfaces with $p_g = c_1^2 = 1$ and K ample, Mem. Fac. Sci. Kochi Univ. (Math.) 3 (1981) 37-73.
- [U.2] Usui, S., Effect of automorphisms on variation of Hodge structure, J. Math. Kyoto Univ. 21-4 (1981) 645-672.
- [U.3] Usui, S., Variation of mixed Hodge structure arising from family of logarithmic deformations, Ann. scient. Éc. Norm. Sup. 4-16 (1983) 91-107; Id II: Classifying space, Duke Math. J. 51-4 (1984) 851-875.
- [U.4] Usui, S., Degeneration of Kunev surfaces I; II, Proc. Japan Acad. 63-A-4 (1987) 110-113; 63-A-5 (1987) 167-169.
- [U.5] Usui, S., Degeneration of Kunev surfaces, to appear in Astérisque.

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