

Nonlinear m -sectorial operators and time-dependent Ginzburg-Landau equations

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§0. Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. We consider the following problem:

$$(1) \quad \begin{aligned} \frac{\partial \Phi}{\partial t} - (\lambda + i\alpha)\Delta\Phi + (\kappa + i\beta)|\Phi|^{p-1}\Phi - \gamma\Phi &= 0, \\ \frac{\partial \Phi}{\partial \nu}(x, t) &= 0 \quad (x \in \partial\Omega, \quad t \geq 0), \\ \Phi(x, 0) &= \Phi_0(x). \end{aligned}$$

Here $\lambda > 0$, $\kappa > 0$, $p > 1$ and $\alpha, \beta, \gamma \in \mathbf{R}$ are constants; ν is unit outward normal of $\partial\Omega$, $i = \sqrt{-1}$ and Φ is $\mathbf{C}$ -valued. (1) is called the time-dependent Ginzburg-Landau equation when $p = 3$ (see Temam [5]). Introducing the new unknown $u = e^{-\gamma t}\Phi (= v + iw)$, the problem (1) is written as

$$(2) \quad \begin{aligned} \frac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)e^{(p-1)\gamma t}|u|^{p-1}u &= 0, \\ \frac{\partial u}{\partial \nu}(x, t) &= 0 \quad (x \in \partial\Omega, \quad t \geq 0), \\ u(x, 0) &= u_0(x) (= \Phi_0(x)). \end{aligned}$$

For the mathematical setting we introduce complex Hilbert space $X = L^2(\Omega; \mathbf{C})$ with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, and define operators A , B and time-dependent operators $A(t)$, $B(t)$ as follows:

$$\begin{aligned} D(A) &:= \{u \in H^2(\Omega; \mathbf{C}); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}, \\ Au &:= -\Delta v - i\Delta w \quad \text{for } u = v + iw \in D(A), \\ D(B) &:= \{u \in X; |u|^{p-1}u \in X\}, \end{aligned}$$

$$Bu := |u|^{p-1}u \text{ for } u \in D(B),$$

$$B(t)u := e^{(p-1)\gamma t}Bu \text{ for } u \in D(B(t)) := D(B), t \in [0, T],$$

$$D := D(A) \cap D(B),$$

$$A(t)u := (\lambda + i\alpha)Au + (\kappa + i\beta)B(t)u \text{ for } u \in D(A(t)) := D, t \in [0, T],$$

where $H^2(\Omega; \mathbb{C})$ is the usual Sobolev Hilbert space and $T > 0$ is arbitrary. Then the problem (2) is regarded as one of initial value problems for standard abstract evolution equations of the form

$$(3) \quad \begin{aligned} \frac{du}{dt} + A(t)u &= 0, \quad t \in [0, T], \\ u(0) &= u_0. \end{aligned}$$

To solve (3) we can apply the theory of nonlinear evolution equations developed by Kato [2]. In fact, under some conditions for $\lambda, \kappa, p, \alpha, \beta, \gamma$ we can show that $A(t)$ ($t \in [0, T]$) is m -accretive in X (see Lemma 10) and $A(\cdot)$ satisfies the smoothness condition:

$$\|A(t)u - A(s)u\| \leq C(T)|t - s|(1 + \|u\| + \|A(s)u\|), \text{ for } t, s \in [0, T], u \in D,$$

(see Lemma 11).

§1. The main theorem and its corollary

we obtain the following theorem.

Theorem. Let $\lambda > 0, \kappa > 0, p > 1, \frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1}, \lambda\kappa + \alpha\beta > |\lambda\beta - \alpha\kappa|$. Then for any $\Phi_0 \in D$, there exists a unique global strong solution $\Phi = \Phi(x, t)$, $(x, t) \in \Omega \times [0, \infty)$ to the problem (1) in X .

Put $\alpha = \beta = 0$ in the problem (1). Then we have

Corollary. If $\lambda > 0, \kappa > 0, p > 1$, then for any $\Phi_0 \in D$ the problem

$$(4) \quad \begin{aligned} \frac{\partial \Phi}{\partial t} - \lambda \Delta \Phi + \kappa |\Phi|^{p-1} \Phi - \gamma \Phi &= 0, & (x, t) \in \Omega \times [0, \infty), \\ \frac{\partial \Phi}{\partial \nu}(x, t) &= 0, & (x, t) \in \partial \Omega \times [0, \infty), \\ \Phi(x, 0) &= \Phi_0(x), & x \in \Omega. \end{aligned}$$

has a unique global strong solution $\Phi = \Phi(x, t)$.

§2. Proof of theorem

In this section we shall prove our main Theorem. The proof needs some lemmas. Throught this section, we assume that $\lambda > 0, \kappa > 0, p > 1$. It is well-known that the operator A is a nonnegative selfadjoint operator in X . So we can easily obtain

Lemma 1. $(\lambda + i\alpha)A$ is m -accretive in X .

In the next Lemma 2 which implies that B is a nonlinear sectorial operator, the constant $\frac{p-1}{2\sqrt{p}}$ is recently determined by [3].

Lemma 2([3]). For any $u_1, u_2 \in D(B)$ we have

$$|\operatorname{Im}(Bu_1 - Bu_2, u_1 - u_2)| \leq \frac{p-1}{2\sqrt{p}} \operatorname{Re}(Bu_1 - Bu_2, u_1 - u_2).$$

Since the operator B is sectorial like this, the accretiveness of B is preserved under a little rotation. So we can obtain

Lemma 3. Let $\frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1}$. Then $(\kappa + i\beta)B$ is accretive in X (We can replace B by $B(t)$).

Proof. Let $u_1, u_2 \in D(B)$. Then it follows from Lemma 2 that

$$\begin{aligned} & \operatorname{Re}((\kappa + i\beta)(Bu_1 - Bu_2), u_1 - u_2) \\ & \geq \kappa \operatorname{Re}(Bu_1 - Bu_2, u_1 - u_2) - |\beta| |\operatorname{Im}(Bu_1 - Bu_2, u_1 - u_2)| \\ & \geq \kappa \left(\frac{2\sqrt{p}}{p-1} - \frac{|\beta|}{\kappa} \right) |\operatorname{Im}(Bu_1 - Bu_2, u_1 - u_2)| \geq 0. \quad \square \end{aligned}$$

Let $f \in X$ then for almost every $x \in \Omega$ the equation

$$z + |z|^{p-1}z = f(x) \text{ in } \mathbf{C}$$

has a unique solution $z = u(x)$ such that $|u(x)| \leq |f(x)|$. Therefore $u \in D(B)$ and we obtain the following lemma.

Lemma 4. B is m -accretive in X (We can replace B by $B(t)$).

For every $\varepsilon > 0$ we set

$$J_\varepsilon = (I + \varepsilon B)^{-1}, \quad B_\varepsilon = \frac{1}{\varepsilon}(I - J_\varepsilon).$$

Lemma 5. Let $\frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1}$. Then $(\kappa + i\beta)B_\varepsilon$ is accretive in X .

Proof. Let $v_1, v_2 \in X$. Then it follows from Lemma 3 that

$$\begin{aligned}
& \operatorname{Re}((\kappa + i\beta)(B_\varepsilon v_1 - B_\varepsilon v_2), v_1 - v_2) \\
&= \operatorname{Re}((\kappa + i\beta)(B_\varepsilon v_1 - B_\varepsilon v_2), J_\varepsilon v_1 - J_\varepsilon v_2) \\
&+ \operatorname{Re}((\kappa + i\beta)(B_\varepsilon v_1 - B_\varepsilon v_2), (v_1 - J_\varepsilon v_1) - (v_2 - J_\varepsilon v_2)) \\
&= \operatorname{Re}((\kappa + i\beta)(B(J_\varepsilon v_1) - B(J_\varepsilon v_2)), J_\varepsilon v_1 - J_\varepsilon v_2) \\
&+ \operatorname{Re}((\kappa + i\beta)(B_\varepsilon v_1 - B_\varepsilon v_2), \varepsilon(B_\varepsilon v_1 - B_\varepsilon v_2)) \\
&\geq \varepsilon \kappa \|B_\varepsilon v_1 - B_\varepsilon v_2\|^2 \geq 0. \quad \square
\end{aligned}$$

Lemma 6. $C^1(\Omega; \mathbb{C}) \cap X$ is invariant under $(I + \varepsilon B)^{-1}$.

Proof. For any $f = g + ih \in C^1(\Omega; \mathbb{C}) \cap X$ we know that the equation

$$u_\varepsilon(x) + \varepsilon |u_\varepsilon(x)|^{p-1} u_\varepsilon(x) = f(x)$$

has a unique solution $u_\varepsilon(x) = v_\varepsilon(x) + iw_\varepsilon(x) \in D(B)$. It remains to show that $u_\varepsilon(x) \in C^1(\Omega; \mathbb{C})$. This equation is rewritten in the form:

$$\begin{cases} v_\varepsilon(x) + \varepsilon(v_\varepsilon(x)^2 + w_\varepsilon(x)^2)^{\frac{p-1}{2}} v_\varepsilon(x) = g(x), \\ w_\varepsilon(x) + \varepsilon(v_\varepsilon(x)^2 + w_\varepsilon(x)^2)^{\frac{p-1}{2}} w_\varepsilon(x) = h(x). \end{cases}$$

Put

$$\begin{aligned} F(x, v_\varepsilon, w_\varepsilon) &:= v_\varepsilon + \varepsilon(v_\varepsilon^2 + w_\varepsilon^2)^{\frac{p-1}{2}} v_\varepsilon - g(x), \\ G(x, v_\varepsilon, w_\varepsilon) &:= w_\varepsilon + \varepsilon(v_\varepsilon^2 + w_\varepsilon^2)^{\frac{p-1}{2}} w_\varepsilon - h(x). \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial(F, G)}{\partial(v_\varepsilon, w_\varepsilon)} &:= \begin{vmatrix} \frac{\partial F}{\partial v_\varepsilon} & \frac{\partial F}{\partial w_\varepsilon} \\ \frac{\partial G}{\partial v_\varepsilon} & \frac{\partial G}{\partial w_\varepsilon} \end{vmatrix} \\ &= \{1 + \varepsilon(v_\varepsilon^2 + w_\varepsilon^2)^{\frac{p-1}{2}}\}^2 + \{1 + \varepsilon(v_\varepsilon^2 + w_\varepsilon^2)^{\frac{p-1}{2}}\} \\ &\quad \times \varepsilon(p-1)(v_\varepsilon^2 + w_\varepsilon^2)^{\frac{p-3}{2}}(v_\varepsilon^2 + w_\varepsilon^2) \geq 1. \end{aligned}$$

Therefore we can apply the implicit function theorem. $\square$

Lemma 7. $\operatorname{Re}(Au, B_\varepsilon u) \geq 0$ for $u \in D(A)$.

Proof. Put

$$\tilde{D}(A) := \{f = g + ih \in C^2(\Omega; \mathbf{C}) \cap H^2(\Omega; \mathbf{C}); \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial\Omega\}.$$

It suffices to show our lemma for $f = g + ih \in \tilde{D}(A)$. We set

$$v_\varepsilon + iw_\varepsilon = (I + \varepsilon B)^{-1}(g + ih).$$

Then we have

$$\begin{aligned} \frac{\partial v_\varepsilon}{\partial x_j} &= \frac{1}{q} \left\{ (1 + aw_\varepsilon^2 + b) \frac{\partial g}{\partial x_j} - aw_\varepsilon v_\varepsilon \frac{\partial h}{\partial x_j} \right\}, \\ \frac{\partial w_\varepsilon}{\partial x_j} &= \frac{1}{q} \left\{ -av_\varepsilon w_\varepsilon \frac{\partial g}{\partial x_j} + (1 + av_\varepsilon^2 + b) \frac{\partial h}{\partial x_j} \right\}, \end{aligned}$$

where

$$a = \varepsilon(p-1)(v_\varepsilon^2 + w_\varepsilon^2)^{\frac{p-3}{2}}, \quad b = \varepsilon(v_\varepsilon^2 + w_\varepsilon^2)^{\frac{p-1}{2}}, \quad q = (b+1)^2 + a(b+1)(v_\varepsilon^2 + w_\varepsilon^2).$$

It follows from this relation that

$$\begin{aligned} &\operatorname{Re}(Af, B_\varepsilon f) \\ &= \operatorname{Re} \left(-\Delta g - i\Delta h, \frac{1}{\varepsilon} [(g - v_\varepsilon) + i(h - w_\varepsilon)] \right) \\ &= \frac{1}{\varepsilon} \left\{ \int_{\Omega} \nabla g \cdot \nabla (g - v_\varepsilon) \, dx + \int_{\Omega} \nabla h \cdot \nabla (h - w_\varepsilon) \, dx \right\} \\ &= \frac{1}{\varepsilon} \sum_{j=1}^N \int_{\Omega} \frac{\partial g}{\partial x_j} \cdot \frac{1}{q} \left\{ q \frac{\partial g}{\partial x_j} - (1 + aw_\varepsilon^2 + b) \frac{\partial g}{\partial x_j} + aw_\varepsilon v_\varepsilon \frac{\partial h}{\partial x_j} \right\} \, dx \\ &\quad + \frac{1}{\varepsilon} \sum_{j=1}^N \int_{\Omega} \frac{\partial h}{\partial x_j} \cdot \frac{1}{q} \left\{ q \frac{\partial h}{\partial x_j} + av_\varepsilon w_\varepsilon \frac{\partial g}{\partial x_j} - (1 + av_\varepsilon^2 + b) \frac{\partial h}{\partial x_j} \right\} \, dx \\ &\geq \frac{1}{\varepsilon} \int_{\Omega} \frac{1}{q} \left[\{b^2 + b + ab(v_\varepsilon^2 + w_\varepsilon^2)\} (|\nabla g|^2 + |\nabla h|^2) + a(v_\varepsilon |\nabla g| - w_\varepsilon |\nabla h|)^2 \right] \, dx \\ &\geq 0. \quad \square \end{aligned}$$

Remark 8. Since X is a (complex) Hilbert space in our case, $B_\varepsilon u$ converges Bu ($u \in D(B)$) in X as $\varepsilon \downarrow 0$. Therefore we also have from Lemma 7 that

$$\operatorname{Re}(Au, Bu) \geq 0 \quad \text{for } u \in D = D(A) \cap D(B).$$

Now we shall prove that the operator

$$A(t) = (\lambda + i\alpha)A + (\kappa + i\beta)B(t)$$

is m -accretive for every $t \in [0, T]$. Following the idea of T. Kato (see Brezis, Crandall and Pazy [1]), for every $f \in X$ we consider the approximate equations:

$$(5) \quad Au_\varepsilon + \frac{\kappa + i\beta}{\lambda + i\alpha} B_\varepsilon u_\varepsilon + u_\varepsilon = f, \quad \varepsilon > 0.$$

Since $(\lambda + i\alpha)A + (\kappa + i\beta)B_\varepsilon$ is m -accretive in X (for $\frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1}$), (5) has a unique solution $u_\varepsilon \in D(A)$.

Lemma 9. *Let u_ε be the solution of (5). If $\lambda\kappa + \alpha\beta > 0$, then $\|B_\varepsilon u_\varepsilon\|$ is bounded for any $\varepsilon > 0$.*

Proof. It follows from (5) that

$$\operatorname{Re}(Au_\varepsilon, B_\varepsilon u_\varepsilon) + \frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2} \|B_\varepsilon u_\varepsilon\|^2 + \operatorname{Re}(u_\varepsilon, B_\varepsilon u_\varepsilon) = \operatorname{Re}(f, B_\varepsilon u_\varepsilon)$$

Noting that $B0 = 0$ and

$$\begin{aligned} \operatorname{Re}(B_\varepsilon u_\varepsilon, u_\varepsilon) &= \operatorname{Re}(B(J_\varepsilon u_\varepsilon) - B0, J_\varepsilon u_\varepsilon - 0) + \operatorname{Re}(B_\varepsilon u_\varepsilon, \varepsilon B_\varepsilon u_\varepsilon) \\ &\geq \varepsilon \|B_\varepsilon u_\varepsilon\|^2 \geq 0, \end{aligned}$$

we have from lemma 7 that

$$\frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2} \|B_\varepsilon u_\varepsilon\|^2 \leq \|f\| \|B_\varepsilon u_\varepsilon\|. \quad \square$$

In (5), now it is routine work to prove that there exists a unique $u \in D$ such that

$u_\varepsilon \longrightarrow u$ strongly in X , $Au_\varepsilon \longrightarrow Au$ weakly in X , $Bu_\varepsilon \longrightarrow Bu$ weakly in X as $\varepsilon \downarrow 0$. Hence we obtain

Lemma 10. Let $\frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1}$ and $\lambda\kappa + \alpha\beta > 0$. Then $(\lambda + i\alpha)A + (\kappa + i\beta)B$ is m -accretive in X . The same is true for $A(t) = (\lambda + i\alpha)A + (\kappa + i\beta)B(t)$ for every $t \in [0, T]$.

Lemma 11. Let $\lambda\kappa + \alpha\beta > |\lambda\beta - \alpha\kappa|$. Then there exists a constant $C = C(T) > 0$ such that

$$\|A(t)u - A(s)u\| \leq C|t - s|\|A(s)u\| \quad \text{for } t, s \in [0, T], u \in D.$$

Proof. Let $t, s \in [0, T]$ and $u \in D$. By the mean value theorem there exists a $C_1(T) > 0$ such that

$$\begin{aligned} \|A(t)u - A(s)u\| &= \|(\kappa + i\beta)(e^{(p-1)\gamma t} - e^{(p-1)\gamma s})Bu\| \\ &\leq C_1(T)|t - s|\|B(s)u\|. \end{aligned}$$

On the other hand we know from Remark 8 that

$$\operatorname{Re}(Au, Bu) \geq 0 \quad \text{for } u \in D.$$

From this inequality we see that

$$\begin{aligned} \left\| \frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2} B(s)u \right\|^2 &\leq \operatorname{Re}(Au, \frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2} B(s)u) + \left\| \frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2} B(s)u \right\|^2 \\ &\leq \|Au + \frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2} B(s)u\| \cdot \left\| \frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2} B(s)u \right\|, \end{aligned}$$

and hence

$$\begin{aligned} \frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2} \|B(s)u\| &\leq \|Au + \frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2} B(s)u\| \\ &\leq \|Au + \frac{\kappa + i\beta}{\lambda + i\alpha} B(s)u\| + \frac{|\lambda\beta - \alpha\kappa|}{\lambda^2 + \alpha^2} \|B(s)u\|. \end{aligned}$$

So we have

$$\|B(s)u\| \leq \frac{|\lambda - i\alpha|}{(\lambda\kappa + \alpha\beta) - |\lambda\beta - \alpha\kappa|} \|(\lambda + i\alpha)Au + (\kappa + i\beta)B(s)u\|$$

Thus we obtain

$$\|A(t)u - A(s)u\| \leq \frac{C_1(T)|\lambda - i\alpha|}{(\lambda\kappa + \alpha\beta) - |\lambda\beta - \alpha\kappa|} |t - s| \|A(s)u\|. \quad \square$$

Now we are in a position to prove our Theorem.

Proof of Theorem (completed). Since the domain D of $A(t)$ is independent of $t \in [0, T]$ and $A(t)$ is m -accretive in X , by Lemma 11 we can apply Kato's Theorem ([2]). Noting that $T > 0$ is arbitrary, the solution $\Phi(x, t)$ to (1) exists for $(x, t) \in \Omega \times [0, \infty)$.

Added after the conference. In our Theorem, we can weaken the assumption. Namely our Theorem is still true when the condition $\lambda\kappa + \alpha\beta > |\lambda\beta - \alpha\kappa|$ is replaced by $\lambda\kappa + \alpha\beta > 0$ (see [6]).

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