

Seifert construction for nilpotent groups and Application to S^1 -fibred nilBott Tower

By

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Abstract

We shall introduce a notion of S^1 -fibred nilBott tower. It is an iterated S^1 -bundle whose top space is called an S^1 -fibred nilBott manifold. The nilBott tower is a generalization of *real Bott tower* from the viewpoint of fibration. We prove that any S^1 -fibred nilBott manifold is *diffeomorphic* to an infranilmanifold. An S^1 -fibred nilBott tower defines a sequence of group extensions. We study the group extension at each stage to apply Seifert rigidity for S^1 -fibred nilBott manifolds.

§ 1. Introduction

Let M be a closed aspherical manifold which is a top space of an iterated S^1 -bundles over a point:

$$(1.1) \quad M = M_n \rightarrow M_{n-1} \rightarrow \dots \rightarrow M_1 \rightarrow \{\text{pt}\}.$$

Suppose X is the universal covering of M and each X_i is the universal covering of M_i and put $\pi_1(M_i) = \pi_i$ ($i = 1, \dots, n-1$) and $\pi_1(M) = \pi$.

Definition 1.1. An S^1 -fibred nilBott tower is a sequence (1.1) which satisfies I, II and III below ($i = 1, \dots, n-1$). The top space M is said to be an S^1 -fibred nilBott manifold (of depth n).

I. M_i is a fiber space over M_{i-1} with fiber S^1 .

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II. For the group extension

$$(1.2) \quad 1 \rightarrow \mathbb{Z} \rightarrow \pi_i \longrightarrow \pi_{i-1} \rightarrow 1$$

associated to the fiber space (I), there is an equivariant principal bundle:

$$(1.3) \quad \mathbb{R} \rightarrow X_i \xrightarrow{p_i} X_{i-1}.$$

III. Each π_i normalizes $\mathbb{R}$.

The purpose of this paper is to announce the following result.

Theorem 1.2. *Suppose that M is an S^1 -fibred nilBott manifold.*

- (I) *If every cocycle of $H_\phi^2(\pi_{i-1}; \mathbb{Z})$ which represents a group extension (1.2) is of finite order, then M is diffeomorphic to a Riemannian flat manifold.*
- (II) *If there exists a cocycle of $H_\phi^2(\pi_{i-1}; \mathbb{Z})$ which represents a group extension (1.2) is of infinite order, then M is diffeomorphic to an infranilmanifold. In addition, M cannot be diffeomorphic to any Riemannian flat manifold.*

§ 2. Preliminaries

§ 2.1. Infrahomogeneous space

Let G be a (noncompact) simply connected Lie group, and $\text{Aut}(G)$ denote the group of automorphisms of G onto itself. Put $A(G) = G \rtimes \text{Aut}(G)$. $A(G)$ becomes a group;

$$(g, \alpha) \cdot (h, \beta) = (g \cdot \alpha(h), \alpha \cdot \beta)$$

($g, h \in G, \alpha, \beta \in \text{Aut}(G)$). $A(G)$ is called the affine group of G . Here, letting $X = G$, an affine action $(A(G), X)$ is obtained as follows:

$$((g, \alpha), x) = g \cdot \alpha(x).$$

Let $H \subset \text{Aut}(G)$ be a compact subgroup (for example, maximal compact subgroup, finite groups). Form a subgroup $E(G) = G \rtimes H \subset A(G)$. Consider the action $(E(G), X)$. We note that if H is compact, then it is easy to check the following.

Lemma 2.1 (Proper action). *$(E(G), X)$ is a proper action.*

By Lemma 2.1, if $\pi \subset E(G)$ is a discrete subgroup, we obtain a properly discontinuous action (π, X) .

Definition 2.2. The quotient space X/π is said to be an infrahomogeneous orbifold. When π has no elements of finite order, π is said to be torsionfree, and X/π is called an infrahomogeneous manifold.

Example 2.3.

- (1) Taking the vector space $\mathbb{R}^n$ as G it gives the usual affine group $A(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL(n, \mathbb{R})$. If H is a maximal compact subgroup $O(n)$ of $GL(n, \mathbb{R})$, we have the euclidean group $E(\mathbb{R}^n) = \mathbb{R}^n \rtimes O(n)$. A discrete uniform subgroup π of $E(\mathbb{R}^n)$ is called a crystallographic group. If $\pi \subset E(\mathbb{R}^n)$ is a torsionfree crystallographic group, π is called a Bieberbach group. Moreover, the infrahomogeneous space $\mathbb{R}^n/\pi$ is an Euclidean space form, i.e. a Riemannian flat manifold.
- (2) When G is a simply connected nilpotent Lie group $\mathcal{N}$, for any torsionfree discrete uniform subgroup $\pi \subset E(\mathcal{N})$, $\mathcal{N}/\pi$ is called an infranilmanifold.

We have the fundamental classical result for crystallographic groups.

Theorem 2.4 (Bieberbach first theorem). *Let $\pi \subset E(\mathbb{R}^n)$ be a crystallographic group, then $\mathbb{R}^n \cap \pi \cong \mathbb{Z}^n$ and $\pi/\mathbb{R}^n \cap \pi$ is a finite group.*

The above theorem is extended to the almost crystallographic groups. See [4] for instance.

Theorem 2.5 (Auslander-Bieberbach theorem). *Let π be a torsionfree discrete uniform subgroup of $E(\mathcal{N})$, then $\mathcal{N} \cap \pi$ is a maximal normal nilpotent subgroup of π and $\pi/\mathcal{N} \cap \pi$ is a finite group.*

§ 3. Nil Geometry

Let

$$(3.1) \quad 1 \rightarrow \Delta \rightarrow \pi \rightarrow F \rightarrow 1$$

be a group extension where π is a torsionfree group, Δ is a torsionfree finitely generated nilpotent group, and F is a finite group. By Mal'cev's *existence* theorem, there is a (simply connected) nilpotent Lie group $\mathcal{N}$ containing Δ as a discrete uniform subgroup. The rest of this section is to review the following realization theorem obtained in [5].

Theorem 3.1 (Realization). *There exists a discrete faithful representation $\rho : \pi \rightarrow E(\mathcal{N})$ such that $\rho|_{\Delta} = \text{id}$. In particular, $\mathcal{N}/\rho(\pi)$ is an infranilmanifold.*

In order to prove this theorem, we need several facts. So we shall prepare them in turn.

§ 3.1. 2-cocycle

We recall the group cohomology. (Compare [10], [2] for example.)

Let G, Q be groups and $\phi : Q \rightarrow \text{Aut}(G)$ a function. Suppose there is a function $f : Q \times Q \rightarrow G$ which satisfies that

- (i) $\phi(\alpha)(\phi(\beta)(n)) = f(\alpha, \beta)\phi(\alpha\beta)(n)f(\alpha, \beta)^{-1}$
- (ii) $f(\alpha, 1) = f(1, \alpha) = 1,$
- (iii) $\phi(\alpha)(f(\beta, \gamma))f(\alpha, \beta\gamma) = f(\alpha, \beta)f(\alpha\beta, \gamma),$

where $n \in G$ and $\alpha, \beta, \gamma \in Q$. Then f defines a group E which is the product $G \times Q$ with the group law:

$$(3.2) \quad (n, \alpha)(m, \beta) = (n \cdot \phi(\alpha)(m) \cdot f(\alpha, \beta), \alpha\beta).$$

Then there is a ϕ -group extension $1 \rightarrow G \rightarrow E \xrightarrow{\nu} Q \rightarrow 1$ where $\nu(n, \alpha) = \alpha$ and the group E is denoted by $G \times_{(f, \phi)} Q$.

Conversely, given a group extension $1 \rightarrow G \rightarrow E \xrightarrow{\nu} Q \rightarrow 1$, we can associate E with a ϕ -group extension. Choose a section $q : Q \rightarrow E$ ($\nu \circ q = \text{id}$), and $q(1) = 1$. A function $\phi : Q \rightarrow \text{Aut}(G)$ is defined to be

$$\phi(\alpha)(n) = q(\alpha)nq(\alpha)^{-1} \quad (\forall \alpha \in Q, \forall n \in G).$$

Both $q(\alpha\beta)$, $q(\alpha)q(\beta)$ are mapped to $\alpha\beta \in Q$, so there is an element $f(\alpha, \beta) \in G$ such that $f(\alpha, \beta) \cdot q(\alpha\beta) = q(\alpha)q(\beta)$. Then it is easily checked that $f : Q \times Q \rightarrow G$ satisfies the above (i) (ii) (iii).

Let $\text{Opext}(Q, G, \phi)$ be the set of all congruence classes of ϕ -group extensions. Then an element $[f] \in \text{Opext}(Q, G, \phi)$ is represented by an extension $1 \rightarrow G \rightarrow E \rightarrow Q \rightarrow 1$ with $E = G \times_{(f, \phi)} Q$. It is easy to check that $[f_1] = [f_2] \in \text{Opext}(Q, A, \phi)$ if and only if there is a function $\lambda : Q \rightarrow \mathcal{C}(G)$ such that

$$(3.3) \quad f_1(\alpha, \beta) = \delta^1 \lambda(\alpha, \beta) \cdot f_2(\alpha, \beta) \quad (\forall \alpha, \beta \in Q).$$

Here $\mathcal{C}(G)$ is the center of G and δ^1 is defined by $\delta^1 \lambda(\alpha, \beta) = \phi(\alpha)(\lambda(\beta))\lambda(\alpha)\lambda(\alpha\beta)^{-1}$. For simplicity, we write it as $f_1 = \delta^1 \lambda \cdot f_2$.

In particular, when G is an abelian group A , $\phi : Q \rightarrow \text{Aut}(A)$ is a homomorphism and hence A is a Q -module. So there is the group cohomology $H_\phi^2(Q, A)$ and f is a 2-cocycle by (iii), i.e. $[f] \in H_\phi^2(Q, A)$. Therefore any extension $1 \rightarrow A \rightarrow E \rightarrow Q \rightarrow 1$ corresponds to a cocycle $[f] \in H_\phi^2(Q, A)$. It is easy to check the following.

Proposition 3.2. *Suppose that A is an abelian group. Then there is a one-to-one correspondence between $H_\phi^2(Q, A)$ and $\text{Opext}(Q, A, \phi)$.*

Remark. Suppose $Q = F$ is a finite group and $f : F \times F \rightarrow \mathbb{R}^n$ is a 2-cocycle relative to $\phi : F \rightarrow \text{Aut}(\mathbb{R}^n)$. Put $h : F \rightarrow \mathbb{R}^n$;

$$(3.4) \quad h(\alpha) = \sum_{\tau \in F} f(\alpha, \tau).$$

Then

$$\begin{aligned} \delta^1 h(\alpha, \beta) &= \phi(\alpha)(h(\beta)) - h(\alpha\beta) + h(\alpha) \\ &= \sum_{\tau \in F} \phi(\alpha)(f(\beta, \tau)) - \sum_{\tau \in F} f(\alpha\beta, \tau) + \sum_{\tau \in F} f(\alpha, \tau) \\ &= \sum_{\tau \in F} (f(\alpha\beta, \tau) - f(\alpha, \beta\tau) + f(\alpha, \beta)) - \sum_{\tau \in F} f(\alpha\beta, \tau) + \sum_{\tau \in F} f(\alpha, \tau) \\ &= |F|f(\alpha, \beta) \end{aligned}$$

i.e. $\delta^1 \frac{1}{|F|}h = f$. It implies that

$$(3.5) \quad H_\phi^2(F; \mathbb{R}^n) = 0.$$

§ 3.2. Pushout

Let π , Δ and $\mathcal{N}$ be as before and $1 \rightarrow \Delta \rightarrow \pi \rightarrow Q \rightarrow 1$ a group extension which is represented by $[f] \in \text{Opext}(Q, \Delta, \phi)$. Given a function $\phi : Q \rightarrow \text{Aut}(\Delta)$, Mal'cev's unique extension theorem implies that each automorphism $\phi(\alpha) : \Delta \rightarrow \Delta$ extends uniquely to an automorphism $\bar{\phi}(\alpha) : \mathcal{N} \rightarrow \mathcal{N}$. In particular, this gives a correspondence $\bar{\phi} : Q \rightarrow \text{Aut}(\mathcal{N})$. Note that it is not necessarily a homomorphism. In general it satisfies

$$(3.6) \quad \bar{\phi}(\alpha)(\bar{\phi}(\beta)(x)) = f(\alpha, \beta)\bar{\phi}(\alpha\beta)(x)f(\alpha, \beta)^{-1} \quad (x \in \mathcal{N}).$$

Then the “pushout” $\pi\mathcal{N} = \{(x, \alpha) \mid x \in \mathcal{N}, \alpha \in Q\}$ can be constructed. Its group law is defined by $(x, \alpha) \cdot (y, \beta) = (x\bar{\phi}(\alpha)(y)f(\alpha, \beta), \alpha\beta)$;

$$(3.7) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{N} & \longrightarrow & \pi\mathcal{N} & \longrightarrow & Q \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \parallel \\ 1 & \longrightarrow & \Delta & \longrightarrow & \pi & \longrightarrow & Q \longrightarrow 1. \end{array}$$

This group (extension) $\pi\mathcal{N}$ is also represented by $[f] \in \text{Opext}(Q, \mathcal{N}, \bar{\phi})$.

§ 3.3. Existence of the Seifert construction

Let W be a contractible smooth manifold. Suppose that a group Q acts properly discontinuously on W such that the quotient space W/Q is compact. Given a group extension:

$$(3.8) \quad 1 \longrightarrow \Delta \longrightarrow \pi \xrightarrow{\nu} Q \longrightarrow 1,$$

we shall show that there is an action of π on $\mathcal{N} \times W$ which is compatible with the left translations of $\mathcal{N}$. Let $\text{Diff}(\mathcal{N} \times W)$ be the group of all diffeomorphisms of $\mathcal{N} \times W$ onto itself. $\mathcal{N}$ is a subgroup of $\text{Diff}(\mathcal{N} \times W)$ via an embedding: $l(n)(m, \alpha) = (nm, \alpha)$.

We denote $\text{Diff}^F(\mathcal{N} \times W)$ the normalizer of $l(\mathcal{N})$ in $\text{Diff}(\mathcal{N} \times W)$. Let $\text{Map}(W, \mathcal{N})$ be the set of smooth maps from W into $\mathcal{N}$. Then $\text{Diff}^F(\mathcal{N} \times W)$ coincides with the group $\text{Map}(W, \mathcal{N}) \rtimes (\text{Aut}(\mathcal{N}) \times \text{Diff}(W))$ with the group law:

$$(\lambda_1, g_1, h_1)(\lambda, g, h) = ((g_1 \circ \lambda \circ h_1^{-1}) \cdot \lambda_1, g_1 g, h_1 h)$$

and

$$(\lambda, g, h)(x, w) = (g(x) \cdot \lambda(hw), hw)$$

for $(x, w) \in \mathcal{N} \times W$, defines an action on $\mathcal{N} \times W$. See [5].

We call the set (Δ, π, Q, W) a smooth data for the group extension (3.8). The following theorem is obtained in [5].

Theorem 3.3. *For any smooth data (Δ, π, Q, W) , there exists a continuous homomorphism $\Psi : \pi \rightarrow \text{Diff}^F(\mathcal{N} \times W)$ such that $\Psi|_{\Delta} = l$.*

Ψ is called the Seifert construction of the smooth data (Δ, π, Q, W) . We shall review the proof of [5].

Proof. Using the pushout (3.6) in § 3.2, if we show that there exists a continuous homomorphism $\bar{\Psi} : \pi\mathcal{N} \rightarrow \text{Diff}^F(\mathcal{N} \times W)$ such that $\bar{\Psi}|_{\mathcal{N}} = l$, then a Seifert construction $\Psi : \pi \rightarrow \text{Diff}^F(\mathcal{N} \times W)$ is obtained as a restriction. Suppose there exists a $\bar{\Psi}$. For $(n, \alpha) \in \pi\mathcal{N}$, if we put $\bar{\Psi}(1, \alpha) = (\lambda, g, h) \in \text{Map}(W, \mathcal{N}) \rtimes (\text{Aut}(\mathcal{N}) \times \text{Diff}(W))$, then $\bar{\Psi}(n, \alpha) = \ell(n)\bar{\Psi}(1, \alpha) = (n \cdot \lambda, g, h)$. Then it is easy to check that

$$\bar{\Psi}(n, \alpha) = (n \cdot \lambda(\alpha), \mu(n) \circ \bar{\phi}(\alpha), \alpha)$$

where $\lambda : Q \rightarrow \text{Map}(W, \mathcal{N})$ satisfies

$$(3.9) \quad f(\alpha, \beta) = (\bar{\phi}(\alpha) \circ \lambda(\beta) \circ \alpha^{-1}) \cdot \lambda(\alpha) \cdot \lambda(\alpha\beta)^{-1} \quad (\alpha, \beta \in Q),$$

where f be a function representing the group extension (3.8). Therefore to guarantee the existence of such $\bar{\Psi}$, we have only to find a map λ satisfying the condition (3.9).

Remark that if $\mathcal{N}$ is a vector space V then $\text{Map}(W, V)$ is a topological group with Q -action by

$$(3.10) \quad \alpha \cdot \lambda(w) = \bar{\phi}(\alpha)(\lambda(\alpha^{-1}w)).$$

So we have a group cohomology $H_{\bar{\phi}}^2(Q, \text{Map}(W, V))$. First note that $H_{\bar{\phi}}^2(Q, \text{Map}(W, V)) = 0$ for any vector space V . This vanishing is obtained by using Shapiro's lemma. (See [3], page 251, Lemma 8.4.)

By induction, we suppose that the statement is true for any nilpotent Lie group whose dimension is less than $\dim \mathcal{N}$. Let $\mathcal{C}$ be the center of $\mathcal{N}$ and put $\mathcal{N}_1 = \mathcal{N}/\mathcal{C}$, $\pi\mathcal{N}_1 = \pi\mathcal{N}/\mathcal{C}$. Consider the group extension

$$(3.11) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{N} & \longrightarrow & \pi\mathcal{N} & \xrightarrow{\nu} & Q \longrightarrow 1 \\ & & \downarrow p & & \downarrow p & & \parallel \\ 1 & \longrightarrow & \mathcal{N}_1 & \longrightarrow & \pi\mathcal{N}_1 & \xrightarrow{\nu_1} & Q \longrightarrow 1, \end{array}$$

with a section $q_1 = p \circ q$ of ν_1 where q is a section to ν . The section q_1 determines $f_1 : Q \times Q \rightarrow \mathcal{N}_1$ and $\bar{\phi}_1 : Q \rightarrow \text{Aut}(\mathcal{N}_1)$ as in §3.1. We suppose by induction on the dimension of $\mathcal{N}$ that there exists $\lambda_1 : Q \rightarrow \text{Map}(W, \mathcal{N}_1)$ such that

$$f_1(\alpha, \beta) = (\bar{\phi}_1(\alpha) \circ \lambda_1(\beta) \circ \alpha^{-1}) \cdot \lambda_1(\alpha) \cdot \lambda_1(\alpha\beta)^{-1}$$

Choose any lift $\lambda' : Q \rightarrow \text{Map}(W, \mathcal{N})$ of λ_1 so that $\lambda_1 = p \circ \lambda'$. Put

$$g(\alpha, \beta) = (\bar{\phi}(\alpha) \circ \lambda'(\beta) \circ \alpha^{-1}) \cdot \lambda'(\alpha) \cdot \lambda'(\alpha\beta)^{-1},$$

then there exists an element $c(\alpha, \beta) \in \text{Map}(W, \mathcal{C})$ such that

$$f(\alpha, \beta) = c(\alpha, \beta) \cdot g(\alpha, \beta).$$

Since both f and g satisfy (iii) in §3.1, c is also a 2-cocycle i.e. $[c] \in H_{\bar{\phi}}^2(Q, \text{Map}(W, \mathcal{C}))$ which vanishes because $\mathcal{C}$ is a vector space. So there is a function $\eta : Q \rightarrow \text{Map}(W, \mathcal{C})$ such that

$$c(\alpha, \beta) = (\bar{\phi}_1(\alpha) \circ \eta(\beta) \circ \alpha^{-1}) \cdot \eta(\alpha) \cdot \eta(\alpha\beta)^{-1}.$$

Put $\lambda = \eta \cdot \lambda' : Q \rightarrow \text{Map}(W, \mathcal{N})$, then λ satisfies (3.9). □

Remark. Let $1 \rightarrow \mathbb{Z} \rightarrow \pi_i \rightarrow \pi_{i-1} \rightarrow 1$ be a group extension as in (1.2). Then π_i acts on the universal cover X_i of M_i as freely. Assume that $\Psi_i : \pi_i \rightarrow \text{Diff}(X_i)$ is the representation homomorphism for this action (π_i, X_i) , then $\Psi_i : \pi_i \rightarrow \Psi_i(\pi_i)$ is the Seifert construction of the smooth data $(\mathbb{Z}, \pi_i, \pi_{i-1}, X_{i-1})$.

§ 3.4. Infranilmanifold

Let $(\Delta, \pi, F, \{pt\})$ be a smooth data with finite group F and f a function representing the given group extension $1 \rightarrow \Delta \rightarrow \pi \rightarrow F \rightarrow 1$. In the same way as the proof of Theorem 3.3, we can obtain a 1-chain $\chi : F \rightarrow \mathcal{N}$ such that $f = \delta^1 \chi$;

$$(3.12) \quad f(\alpha, \beta) = \bar{\phi}(\alpha)(\chi(\beta))\chi(\alpha)\chi(\alpha\beta)^{-1} \quad (\alpha, \beta \in F).$$

We shall repeat the construction of χ for our use. Let $\bar{f} : F \times F \rightarrow \mathcal{N}/\mathcal{C}$ be a function which represents $1 \rightarrow \mathcal{N}_1 \rightarrow \pi\mathcal{N}_1 \rightarrow F \rightarrow 1$, then we suppose $\bar{f} = \delta^1 \bar{\lambda}$ for some function $\bar{\lambda} : F \rightarrow \mathcal{N}/\mathcal{C}$ by induction. Choose a lift $\lambda : F \rightarrow \mathcal{N}$ of $\bar{\lambda}$. It is easy to see the function $g = f \cdot (\delta^1 \lambda)^{-1}$ is a cocycle lying in $\mathcal{C}$, that is $[g] \in H_{\bar{\phi}}^2(F, \mathcal{C})$. As $H_{\bar{\phi}}^2(F, \mathcal{C}) = 0$ from (3.5), there is a map $\mu : F \rightarrow \mathcal{C}$ such that $\delta^1 \mu = g$. Then $f = \delta^1(\mu \cdot \lambda)$ and the 1-chain χ denoted by $\mu \cdot \lambda$.

Now define an automorphism of $\mathcal{N}$ $h(\alpha) : \mathcal{N} \rightarrow \mathcal{N}$ for each $\alpha \in F$ to be

$$h(\alpha)(x) = \chi(\alpha)^{-1} \cdot \bar{\phi}(\alpha)(x) \cdot \chi(\alpha) \quad (x \in \mathcal{N}).$$

Using (3.6), we can prove that $h(\alpha\beta) = h(\alpha)h(\beta)$ for $\alpha, \beta \in F$. Therefore $h : F \rightarrow \text{Aut}(\mathcal{N})$ is a homomorphism. Since $\text{Aut}(\mathcal{N})$ is a noncompact Lie group, it has a maximal compact group $\mathcal{K}$. Then the finite subgroup $h(F)$ is conjugate to a subgroup of $\mathcal{K}$. We can assume that $h(F) \subset \mathcal{K}$.

Define $\rho : \pi \rightarrow E(\mathcal{N})$ to be

$$(3.13) \quad \rho((n, \alpha)) = (n\chi(\alpha), h(\alpha)) \quad (n \in \Delta, \alpha \in F).$$

It is easy to check that ρ is a homomorphism. We define an action of π on $\mathcal{N}$ to be

$$(3.14) \quad ((n, \alpha), x) = \rho(n, \alpha)(x) = n\bar{\phi}(\alpha)(x)\chi(\alpha) \quad ((n, \alpha) \in \pi).$$

Theorem 3.1 is obtained by the following proposition.

Proposition 3.4. *The action $(\pi, \mathcal{N})$ is a properly discontinuous free action. In particular, ρ is a faithful representation.*

Proof. First note that $\rho|_{\Delta} = id$, so Δ is contained in $\rho(\pi)$. Since Δ acts as left translations of $\mathcal{N}$ from (3.13), it acts properly discontinuously and freely. Moreover since Δ is a finite index subgroup of $\rho(\pi)$ from (3.1), $\rho(\pi)$ acts properly discontinuously on $\mathcal{N}$.

Let $(n, \alpha) \in \text{Ker } \rho$ be an element of π . Then $((n, \alpha), x) = x \quad (\forall x \in \mathcal{N})$ by (3.14). As π acts properly discontinuously, (n, α) is of finite order. On the other hand, π is torsionfree, we obtain $(n, \alpha) = 1$ and so ρ is faithful. $\square$

The following remark shows that ρ is a Seifert construction (cf. Theorem 3.3).

Remark. Let $A(\mathcal{N})^*$ be a group which is the product $\mathcal{N} \times \text{Aut}(\mathcal{N})$ with the group law:

$$(n, \alpha) \cdot (m, \beta) = (\alpha(m) \cdot n, \alpha \cdot \beta)$$

for $n, m \in \mathcal{N}$, and $\alpha, \beta \in \text{Aut}(\mathcal{N})$. The action $(A(\mathcal{N})^*, \mathcal{N})$ is obtained as follows:

$$((n, \alpha), x) = \alpha(x) \cdot n$$

for $x \in \mathcal{N}$. Then there is an isomorphism $\delta : A(\mathcal{N})^* \rightarrow A(\mathcal{N})$ defined by $\delta(n, \alpha) = (n, \mu(n^{-1})(\alpha))$. Here $\mu : \mathcal{N} \rightarrow \text{Aut}(\mathcal{N})$ is the conjugation homomorphism: $\mu(n)(x) = nxn^{-1}$. It is easily checked that

$$((n, \alpha), x) = (\delta(n, \alpha), x)$$

This shows that the affine action $(A(\mathcal{N}), \mathcal{N})$ coincides with the above action $(A(\mathcal{N})^*, \mathcal{N})$.

Remark. There is a commutative diagram.

$$(3.15) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{N} & \longrightarrow & E(\mathcal{N}) & \longrightarrow & \mathcal{K} \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \cup \\ 1 & \longrightarrow & \mathcal{N} \cap \rho(\pi) & \longrightarrow & \rho(\pi) & \longrightarrow & H \longrightarrow 1. \end{array}$$

By the theorem of Auslander-Bieberbach, $\mathcal{N} \cap \rho(\pi)$ is a maximal normal nilpotent subgroup of $\rho(\pi)$. Note that $\Delta \subset \mathcal{N} \cap \rho(\pi)$, so if Δ is maximal, then $\Delta = \mathcal{N} \cap \rho(\pi)$.

§ 3.5. Seifert rigidity

Let Δ_i be a discrete uniform subgroup of a simply connected nilpotent Lie group $\mathcal{N}_i$ ($i = 1, 2$) respectively. Let Ψ_1, Ψ_2 be Seifert constructions for smooth data $(\Delta_1, \pi_1, Q_1, W_1), (\Delta_2, \pi_2, Q_2, W_2)$ respectively. Suppose there exists an isomorphism $\theta : \pi_1 \rightarrow \pi_2$ inducing isomorphisms $\bar{\theta} : \Delta_1 \rightarrow \Delta_2, \hat{\theta} : Q_1 \rightarrow Q_2$. Furthermore (Q_1, W_1) is equivariantly diffeomorphic to (Q_2, W_2) with respect to $\hat{\theta}$. Then *Seifert rigidity* shows that $(\Psi_2(\pi_1), \mathcal{N}_1 \times W_1)$ is equivariantly diffeomorphic to $(\Psi_1(\pi_2), \mathcal{N}_2 \times W_2)$. See [5], page 441.

§ 4. S^1 -fibred nilBott tower

This section is to give an idea of proof of Theorem 1.2. The details will appear in [13]. (See also [6].) Let M_i be an S^1 -fibred nilBott manifold ($i = 1, \dots, n$). Let

$$(4.1) \quad 1 \rightarrow \mathbb{Z} \rightarrow \pi_i \longrightarrow \pi_{i-1} \rightarrow 1,$$

be a group extension associated with a fiber space:

$$(4.2) \quad S^1 \rightarrow M_i \rightarrow M_{i-1}.$$

The conjugate by each element of π_i defines a homomorphism $\phi : \pi_{i-1} \rightarrow \text{Aut}(\mathbb{Z}) = \{\pm 1\}$, so that the above group extension represents a 2-cocycle in $H_\phi^2(\pi_{i-1}; \mathbb{Z})$. Then we can find the commutative diagram of central extensions for each i :

$$(4.3) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \Delta_{i-1} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{R} & \longrightarrow & \tilde{\mathcal{N}}_i & \longrightarrow & \mathcal{N}_{i-1} \longrightarrow 1, \end{array}$$

where $\tilde{\Delta}_i$ and Δ_{i-1} are torsionfree finitely generated normal nilpotent subgroups of finite index in π_i and π_{i-1} respectively. And $\tilde{\mathcal{N}}_i, \mathcal{N}_{i-1}$ are simply connected nilpotent Lie groups containing $\tilde{\Delta}_i$ and Δ_{i-1} as a discrete cocompact subgroup, respectively:

$$(4.4) \quad \begin{array}{ccccccc} & & & & 1 & & 1 \\ & & & & \uparrow & & \uparrow \\ & & & & F & & F \\ & & & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \longrightarrow & \pi_{i-1} \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \Delta_{i-1} \longrightarrow 1 \\ & & & & \uparrow & & \uparrow \\ & & & & 1 & & 1 \end{array}$$

From Theorem 3.1 (see [5]) there exists a faithful representation

$$(4.5) \quad \rho_i : \pi_i \longrightarrow E(\tilde{\mathcal{N}}_i)$$

for which $\rho_i|_{\tilde{\Delta}_i} = \text{id}$ and the quotient $\tilde{\mathcal{N}}_i/\rho_i(\pi_i)$ is an infranilmanifold. On the other hand, (4.5) induces the following group extension:

$$(4.6) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \longrightarrow & \pi_{i-1} \longrightarrow 1 \\ & & \parallel & & \rho_i \downarrow & & \hat{\rho}_i \downarrow \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \rho_i(\pi_i) & \longrightarrow & \hat{\rho}_i(\pi_{i-1}) \longrightarrow 1. \end{array}$$

Since $\tilde{\Delta}_i$ centralizes $\mathbb{Z}$, $\tilde{\mathcal{N}}_i$ centralizes $\mathbf{R}$ in (4.3). And $\hat{\rho}_i$ is a monomorphism from π_{i-1} into $E(\mathcal{N}_{i-1})$. Thus we have two Seifert fibrations

$$(\mathbb{Z}, \mathbf{R}) \rightarrow (\rho_i(\pi_i), \tilde{\mathcal{N}}_i) \xrightarrow{\nu_i} (\hat{\rho}_i(\pi_{i-1}), \mathcal{N}_{i-1})$$

and

$$(\mathbb{Z}, \mathbb{R}) \rightarrow (\pi_i, X_i) \xrightarrow{p_i} (\pi_{i-1}, X_{i-1})$$

(cf. (1.3)).

By induction, assume that the isomorphism $\hat{\rho}$ induces an equivariant diffeomorphism of (π_{i-1}, X_{i-1}) onto $(\hat{\rho}_i(\pi_{i-1}), \mathcal{N}_{i-1})$. Then Seifert rigidity implies that (π_i, X_i) is equivariantly diffeomorphic to $(\rho_i(\pi_i), \tilde{\mathcal{N}}_i)$. Let $M = X_n/\pi_n$. As a consequence, M is diffeomorphic to an infranilmanifold $\tilde{\mathcal{N}}_n/\rho_n(\pi_n)$.

We conclude that any S^1 -fibred nilBott manifold M is diffeomorphic to an infranilmanifold. According to Cases I, II stated in Theorem 1.2, we prove that $\tilde{\mathcal{N}}_n$ is isomorphic to a vector space or $\tilde{\mathcal{N}}_n$ is a nilpotent Lie group but not a vector space respectively (cf. [13]).

In order to study S^1 -fibred nilBott manifolds further, we introduce the following definition:

Definition 4.1. If an S^1 -fibred nilBott manifold M satisfies Case I (respectively Case II) of Theorem 1.2, then M is said to be an S^1 -fibred nilBott manifold of finite type (respectively of infinite type). Apparently there is no intersection between finite type and infinite type. And S^1 -fibred nilBott manifolds are of finite type until dimension 2.

Remark. Let M be an S^1 -fibred nilBott manifold of finite type, then $\rho(\pi)$ is a Bieberbach group (cf. Theorem 1.2). By the Bieberbach Theorem, $\rho(\pi)$ satisfies a group extension

$$(4.7) \quad 1 \rightarrow \mathbb{Z}^n \rightarrow \rho(\pi) \longrightarrow H \rightarrow 1$$

where $\mathbb{Z}^n = \rho(\pi) \cap \mathbb{R}^n$, and H is the holonomy group of $\rho(\pi)$. By Proposition 3.4, we may identify $\rho(\pi)$ with π whenever π is torsionfree.

The following Proposition 4.2 and Corollary 4.3 have been proved. See [13] for details.

Proposition 4.2. *Suppose M is an S^1 -fibred nilBott manifold of finite type. Then the holonomy group of π is isomorphic to the power of cyclic group of order two $(\mathbb{Z}_2)^s$ in $(0 \leq s \leq n)$.*

Corollary 4.3. *Each S^1 -fibred nilBott manifold of finite type M admits a homologically injective T^k -action where $k = \text{Rank } H_1(M)$. Moreover, the action is maximal, i.e. $k = \text{Rank } C(\pi)$.*

§ 4.1. S^1 -fibred nilBott manifolds of depth 3 (Case I).

By the definition of S^1 -fibred nilBott manifold M_n of depth n , M_2 is either a torus or a Klein bottle. In particular, M_2 is a Riemannian flat manifold. A 3-dimensional S^1 -fibred nilBott manifold M_3 is either a Riemannian flat manifold or an infranil-Heisenberg manifold in accordance with the cases I (finite type) or II (infinite type) of Theorem 1.2.

On the other hand, there are 10-isomorphism classes $\mathcal{G}_1, \dots, \mathcal{G}_6, \mathcal{B}_1, \dots, \mathcal{B}_4$ of 3-dimensional Riemannian flat manifolds. (Refer to Wolf [18] for the classification of 3-dimensional Riemannian flat manifolds.) Among these, real Bott manifolds consist of 4; $\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_3$. (See [15].) We shall show that $\mathcal{B}_2, \mathcal{B}_4$ are S^1 -fibred nilBott manifolds.

$\mathcal{B}_2$: $T^3/\mathbb{Z}_2$ whose holonomy group $\mathbb{Z}_2 = \langle \alpha \rangle$ acts on T^3 ;

$$\alpha(z_1, z_2, z_3) = (-z_1 z_3, z_2 z_3, \bar{z}_3).$$

Define an S^1 -action on T^3 by

$$t(z_1, z_2, z_3) = (tz_1, tz_2, z_3).$$

Then it is easy to see that the S^1 -action induces an S^1 -action on $T^3/\mathbb{Z}_2$ naturally. This gives a principal bundle

$$S^1 \rightarrow T^3/\mathbb{Z}_2 \rightarrow K$$

where K is a Klein bottle. So the tower of S^1 -fiber bundles

$$T^3/\mathbb{Z}_2 \rightarrow K \rightarrow S^1 \rightarrow \{\text{pt}\}$$

is an S^1 -fibred nilBott tower.

$\mathcal{B}_4$: $T^3/(\mathbb{Z}_2)^2$ whose holonomy group $(\mathbb{Z}_2)^2 = \langle \alpha, \beta \rangle$ acts on T^3 ;

$$\begin{aligned} \alpha(z_1, z_2, z_3) &= (-z_1, \bar{z}_2, \bar{z}_3), \\ \beta(z_1, z_2, z_3) &= (z_1, -z_2, -\bar{z}_3). \end{aligned}$$

Denote an action of $(\mathbb{Z}_2)^2$ on T^2 by

$$\begin{aligned} \hat{\alpha}(z_1, z_2) &= (-z_1, \bar{z}_2), \\ \hat{\beta}(z_1, z_2) &= (z_1, -z_2). \end{aligned}$$

The quotient manifold is the Klein bottle $T^2/(\mathbb{Z}_2)^2 = (S^1 \times \mathbb{RP}^1)/\mathbb{Z}_2 = K$. The projection $P(z_1, z_2, z_3) = (z_1, z_2)$ is equivariant with respect to the $(\mathbb{Z}_2)^2$ -action on T^2 . So the tower

$$T^3/(\mathbb{Z}_2)^2 \rightarrow K \rightarrow S^1 \rightarrow \{\text{pt}\}$$

is an S^1 -fibred nilBott tower.

Proposition 4.4. *The 3-dimensional S^1 -fibred nilBott manifold of finite type are those of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$.*

Proof. As any real Bott manifold is an S^1 -fibred nilBott manifold of finite type (cf. [7]), consider Riemannian flat manifolds $\mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_6, \mathcal{B}_2, \mathcal{B}_4$ which are not real Bott manifolds. Since holonomy groups are the product of $\mathbb{Z}_2$ by Proposition 4.2, the remaining cases are either $\mathcal{G}_6, \mathcal{B}_2$, or $\mathcal{B}_4$ from the list [18]. Moreover, by Corollary 4.3, an S^1 -fibred nilBott manifold M of finite type admits a homologically injective T^k -action for $k = \text{Rank } H_1(M)$ ($k \geq 1$). In particular, $\mathbb{Z}^k$ is a direct summand of $H_1(M)$. By the classification of the first homology (cf. [18]), $H_1(M; \mathbb{Z}) = \mathbb{Z}_4 + \mathbb{Z}_4$ for $\mathcal{G}_6$. So it cannot admit a structure of S^1 -fibred nilBott manifold. For Riemannian flat 3-manifolds corresponding to $\mathcal{B}_2$ and $\mathcal{B}_4$, we have shown that they admit S^1 -fibred nilBott tower. $\square$

§ 4.2. S^1 -fibred nilBott manifolds of depth 3 (CaseII).

The 3-dimensional simply connected nilpotent Lie group $\mathcal{N}$ is isomorphic to the Heisenberg Lie group N_3 which is the product $R \times \mathbb{C}$ with group law:

$$(x, z) \cdot (y, w) = (x + y - \text{Im} \bar{z}w, z + w).$$

Then the maximal compact Lie subgroup of $\text{Aut}(N_3)$ is $U(1) \rtimes \langle \tau \rangle$ which acts on N_3

$$(4.8) \quad \begin{aligned} e^{i\theta}(x, z) &= (x, e^{i\theta}z) \quad (e^{i\theta} \in U(1)), \\ \tau(x, z) &= (-x, \bar{z}). \end{aligned}$$

A 3-dimensional compact infranilmanifold is obtained as a quotient N_3/Γ where Γ is a torsionfree discrete uniform subgroup of $E(N_3) = N_3 \rtimes (U(1) \rtimes \langle \tau \rangle)$. (See [4].)

Let

$$S^1 \rightarrow M_3 \rightarrow M_2$$

be an S^1 -fibred nilBott manifold of infinite type which has a group extension $1 \rightarrow \mathbb{Z} \rightarrow \pi_3 \rightarrow \pi_2 \rightarrow 1$. Since $R \subset N_3$ is the center of N_3 , there is a commutative diagram of central extensions:

$$(4.9) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_3 & \longrightarrow & \Delta_2 \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & R & \longrightarrow & N_3 & \longrightarrow & \mathbb{C} \longrightarrow 1 \end{array}$$

(cf. (4.3)). Using this, we obtain an embedding:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 & \longrightarrow & 1 \\
(4.10) & & \downarrow \iota & & \downarrow \rho & & \downarrow \hat{\rho} & & \\
1 & \longrightarrow & \mathbf{R} & \longrightarrow & \mathbf{E}(N_3) & \longrightarrow & \mathbb{C} \rtimes (\mathbf{U}(1) \rtimes \langle \tau \rangle) & \longrightarrow & 1.
\end{array}$$

Note that $\mathbb{C} \rtimes (\mathbf{U}(1) \rtimes \langle \tau \rangle) = \mathbb{R}^2 \rtimes \mathbf{O}(2) = \mathbf{E}(2)$. Since $\mathbf{R} \cap \pi_3 = \mathbb{Z}$ from (4.10), $\hat{\rho}(\pi_2)$ is a Bieberbach group in $\mathbf{E}(2)$ so that $\mathbb{R}^2/\hat{\rho}(\pi_2)$ is either T^2 or K .

We shall consider the following two cases.

Case (i): The holonomy group of π_3 is trivial.

Let $k \in \mathbb{Z}$ and define $\Delta(k)$ to be a subgroup of N_3 generated by

$$c = (2k, 0), a = (0, k), b = (0, k\mathbf{i}).$$

Put $Z = \langle c \rangle$ which is a central subgroup of $\Delta(k)$. It is easy to see that

$$(4.11) \quad [a, b] = c^{-k}.$$

Since $\mathbf{R}$ is the center of N_3 , we have a principal bundle

$$S^1 = \mathbf{R}/Z \rightarrow N_3/\Delta(k) \longrightarrow \mathbb{C}/\mathbb{Z}^2.$$

Then the euler number of the fibration is $\pm k$. (See [12] for example.)

Case (ii): The holonomy group is nontrivial.

Let $\Gamma(k)$ be a subgroup of $\mathbf{E}(N_3)$ generated by

$$n = ((k, 0), I), \alpha = \left((0, \frac{k}{2}), \tau \right), \beta = ((0, k\mathbf{i}), I).$$

Note that $\alpha^2 = ((0, k), I)$. Then it is easy to check that

$$(4.12) \quad \alpha n \alpha^{-1} = n^{-1}, \alpha \beta \alpha^{-1} = n^k \beta^{-1}, \beta n \beta^{-1} = n.$$

Then $M_3 = N_3/\Gamma(k)$ is an S^1 -fibred nilBott manifold:

$$S^1 \rightarrow N_3/\Gamma(k) \rightarrow K$$

where $S^1 = \mathbf{R}/\langle n \rangle$ is the fiber (but not an action).

Proposition 4.5. *A 3-dimensional S^1 -fibred nilBott manifold M_3 of infinite type is either a Heisenberg nilmanifold $N_3/\Delta(k)$ or an infranilmanifold $N_3/\Gamma(k)$.*

The details of Proposition 4.5 will appear in [13]. Propositions 4.4 and 4.5 are also obtained independently by Lee and Masuda (cf. [11]).

Remark. Originally, the 3×3 -unipotent upper triangular matrices N is called the Heisenberg nilpotent Lie group. Of course N_3 is isomorphic to N . We use N_3 because it is easy to see the automorphism group $E(N_3)$.

§ 5. Further remarks

Let $Q = \pi_1(K)$ be the fundamental group of the Klein bottle K . Q has a presentation:

$$(5.1) \quad \{g, h \mid ghg^{-1} = h^{-1}\}.$$

A group extension $1 \rightarrow \mathbb{Z} \rightarrow \pi \rightarrow G \rightarrow 1$ for any 3-dimensional S^1 -fibred nilBott manifold over K represents a 2-cocycle in $H_\phi^2(Q, \mathbb{Z})$ for some representation ϕ . Conversely, given a representation ϕ , we prove in [13] that any elements of $H_\phi^2(Q, \mathbb{Z})$ can be realized as an S^1 -fibred nilBott manifold, we have obtained the following table.

		Case 1	Case2	Case3	Case4
	$H_\phi^2(Q, \mathbb{Z})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
$\pi_1(M)$	$[f] = 0$	$\pi_1(\mathcal{B}_1)$	$\pi_1(\mathcal{B}_3)$	$\pi_1(\mathcal{G}_2)$	$\pi_1(\mathcal{B}_3)$
	$[f] \neq 0 : \text{torsion}$	$\pi_1(\mathcal{B}_2)$	$\pi_1(\mathcal{B}_4)$	-	$\pi_1(\mathcal{B}_4)$
	$[f] \neq 0 : \text{torsionfree}$	-	-	$\Gamma(k)$	-

Here

Case1. : $\phi(g) = 1, \phi(h) = 1,$

Case2. : $\phi(g) = 1, \phi(h) = -1,$

Case3. : $\phi(g) = -1, \phi(h) = 1,$

Case4. : $\phi(g) = -1, \phi(h) = -1.$

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