

A variational approach to construction of weak solutions of semilinear hyperbolic systems

by

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1 Introduction

Let Ω be a bounded domain of $\mathbf{R}^k$ with Lipschitz boundary $\partial\Omega$. We consider the following system of hyperbolic equations for a map $u(x, t) : \Omega \times (0, +\infty) \rightarrow \mathbf{R}^l$:

$$(1.1) \quad \begin{aligned} a_{ij}(x) \frac{\partial^2 u^i(x, t)}{\partial t^2} - D_\alpha (b_{ij}^{\alpha\beta}(x) D_\beta u^i(x, t)) \\ + c_{ij}(x) \|u(x, t)\|_c^{m-2} u^i(x, t) = 0 \quad \text{in } \Omega, \quad j = 1, \dots, l, \end{aligned}$$

where $D_\alpha = \partial/\partial x^\alpha$, $\|u(x, t)\|_c = (c_{ij}(x) u^i(x, t) u^j(x, t))^{1/2}$ and $m > 1$. Here and in the sequel, summation over repeated indices is understood, the greek indices run from 1 to k , and the latin ones from 1 to l . We assume that the coefficients $a_{ij}(x)$, $b_{ij}^{\alpha\beta}(x)$ and $c_{ij}(x)$ are bounded functions defined on Ω and satisfy the conditions

$$(1.2) \quad \begin{aligned} a_{ij}(x) \xi^i \xi^j &\geq \lambda_0 |\xi|^2 \quad \forall \xi \in \mathbf{R}^l, \\ b_{ij}^{\alpha\beta}(x) \eta_\alpha^i \eta_\beta^j &\geq \lambda_1 |\eta|^2 \quad \forall \eta \in \mathbf{R}^{kl}, \end{aligned}$$

$$(1.3) \quad c_{ij}(x) \xi^i \xi^j \geq \lambda_2 |\xi|^2 \quad \forall \xi \in \mathbf{R}^l,$$

$$(1.3) \quad a_{ij}(x) = a_{ji}(x), \quad b_{ij}^{\alpha\beta}(x) = b_{ji}^{\beta\alpha}(x), \quad c_{ij}(x) = c_{ji}(x),$$

for some positive constants λ_0 , λ_1 and λ_2 . The initial and boundary conditions are

$$(1.4) \quad u(x, 0) = u_0(x), \quad \frac{\partial}{\partial t} u(x, 0) = v_0(x),$$

$$(1.5) \quad u(x, t) = w(x) \quad \text{on} \quad \partial\Omega,$$

where $u_0(x)$, $v_0(x)$ and $w(x)$ are given maps such that $u_0(x) = w(x)$ on $\partial\Omega$.

(1.1) can be considered as a generalization of a semilinear wave equation. About weak solutions of a semilinear wave equation see, for example, [5, 6, 14, 17, 18].

We define a weak solution of (1.1) satisfying the initial and boundary conditions (1.4) and (1.5) as follows.

DEFINITION 1.1. Let $\gamma_{\partial\Omega}$ and $\gamma_{t=0}$ denote the trace operators to $\partial\Omega$ and $\Omega \times \{0\}$, respectively. For $u_0, w \in H^{1,2} \cap L^m(\Omega)$ and $v_0 \in L^2(\Omega)$ satisfying $\gamma_{\partial\Omega} u_0 = \gamma_{\partial\Omega} w$, a map $u(x, t) : \Omega \times [0, T) \rightarrow \mathbf{R}^l$ is called a weak solution of (1.1) on $[0, T)$ satisfying the initial and boundary conditions (1.4) and (1.5) if the following conditions are satisfied:

- (i) $u \in L^\infty(0, T; L^m(\Omega)) \cap L^\infty(0, T; H^{1,2}(\Omega))$ with $u_t \in L^\infty(0, T; L^2(\Omega))$.
- (ii) $\gamma_{t=0} u(x, t) = u_0(x)$ and $\gamma_{\partial\Omega} u(x, t) = \gamma_{\partial\Omega} w(x)$ for $0 < t < T$.
- (iii) For any $\psi(x, t) \in C_0^1([0, T]; C_0(\Omega)) \cap C([0, T]; C^1(\Omega))$,

$$(1.6) \quad \begin{aligned} & \int_0^T \int_\Omega \left\{ -a_{ij}(x) \frac{\partial u^i}{\partial t}(x, t) \frac{\partial \psi^j}{\partial t}(x, t) + b_{ij}^{\alpha\beta}(x) D_\alpha u^i(x, t) D_\beta \psi^j(x, t) \right. \\ & \quad \left. + c_{ij}(x) \|u(x, t)\|_c^{m-2} u^i(x, t) \psi^j(x, t) \right\} dx dt \\ & = \int_\Omega a_{ij}(x) v_0^i(x) \psi^j(x, 0) dx. \end{aligned}$$

Then our main result can be stated as follows.

THEOREM 1.1. Let Ω be a bounded domain of $\mathbf{R}^k$ with Lipschitz boundary $\partial\Omega$. Suppose that (1.2) and (1.3) are satisfied. For any $v_0 \in L^2(\Omega)$ and $u_0, w \in H^{1,2} \cap L^m(\Omega)$ with $\gamma_{\partial\Omega} u_0 = \gamma_{\partial\Omega} w$, there exists a weak solution of (1.1) which satisfies the initial and boundary conditions (1.4) and (1.5) in any time interval $[0, T)$, $T < \infty$.

Though (1.1) can be solved by several well known methods (the Faedo-Galerkin method, semigroup theory, etc.), we introduce an approach which is not so familiar to construct solutions of hyperbolic systems. Since weak solutions are not uniquely determined in general, it would be fruitful to consider various constructions.

We prove Theorem 1.1 by using of Rothe's time-discretization method and the direct method of calculus of variations. Rothe's time-discretization method has been used to construct solutions of parabolic and hyperbolic equations. (cf. [3], [12], [13] and [16])

Moreover, in 1971, K.Rektorys [15] combined the time-discretization method and the direct method of calculus of variations to construct solutions of parabolic equations. Roughly speaking, his method is summarized as follows: For the equation

$$(1.7) \quad \frac{\partial u}{\partial t} - \left(\text{the Euler-Lagrange equation of } \int_{\Omega} F(x, u, Du) dx \right) = 0,$$

they consider the auxiliary variational functionals

$$(1.8) \quad G_n(u) = \int_{\Omega} \left\{ \frac{\|u\|^2 - 2u \cdot u_{n-1}}{2h} + F(x, u, Du) \right\} dx,$$

and define u_n successively as the minimizer of $G_n(u)$. Using the sequence $\{u_n\}$, they construct approximate solutions and prove that the approximate solutions converge to a solution of (1.7) as $h \rightarrow 0$. In [15] existence of weak solutions of linear parabolic equations was proved.

Recently, a similar method was rediscovered by N.Kikuchi [4]. Subsequently, F.Bethuel, J.-M.Coron, J.-M.Ghidaglia and A.Soyeur [1] showed the existence of the Morse semiflow associated to relaxed energies for harmonic maps into S^n in the same procedure. T.Nagasawa [8] gives a new approach to solving the Navier-Stokes equations based on the same idea. In this paper we apply the similar procedure in order to construct approximate solutions of (1.1). We consider the auxiliary variational functionals related to the equation (1.1) and construct approximate solutions by using the minimizers of the functionals. Using the same method T.Nagasawa and the author [9, 10] constructed weak solutions of semilinear hyperbolic system with damping or strong damping term and proved their exponential decay property. More recently, T.Nagasawa and the author [11] constructed a weak solution of a semilinear hyperbolic system on a time-dependent domain.

This note is an epitome of [19].

2 A Variational Approach and Energy Estimates

To construct a weak solution of (1.1), we determine a family $\{u_n\}$ as follows:

(I) ($n = 1$.) Let $v_0(x) = (v_0^1(x), \dots, v_0^l(x))$ be a given map of class $L^2(\Omega)$ as in Theorem 1.1. Take $v(x, t) \in L^\infty(\mathbf{R}; H^{1,2}(\Omega)) \cap L^\infty(\mathbf{R}; L^m(\Omega))$ such that

$$(2.1) \quad \begin{cases} v(x, 0) = 0, \quad v_t(x, 0) = v_0(x) & \text{in } \Omega, \quad v(x, t) = 0 \quad \text{on } \partial\Omega, \\ v_t(x, t) \text{ is weak continuous with respect to } t. \end{cases}$$

Let us define $u_1(x) = u_0(x) + v(x, h)$.

To get a map $v(x, t)$ satisfying (2.1), for example, we solve the initial-boundary value problems

$$(2.2) \quad \begin{cases} v_{tt}^i(x, t) - \Delta v^i(x, t) + |v^i|^{m-2}v^i = 0, & \text{for } x \in \Omega, t \in \mathbf{R}, \\ v^i(x, 0) = 0, \quad v_t^i(x, 0) = v_0^i(x), \\ v^i(x, t) = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 2 of [18] guarantees the existence of weak solutions $\{v^i(x, t)\}$ of (2.2) in the class $L^\infty(\mathbf{R}; H^{1,2}(\Omega)) \cap L^\infty(\mathbf{R}; L^m(\Omega))$ with the weak continuous time derivatives $\{v_t^i(x, t)\}$. Moreover, they satisfy the following energy estimates for all t .

$$(2.3) \quad \int_{\Omega} \left\{ \frac{1}{2} |v_t^i|^2 + \frac{1}{2} \|Dv^i\|^2 + \frac{1}{m} |v^i|^m \right\} dx \leq \int_{\Omega} \frac{1}{2} |v_0^i|^2 dx.$$

(II) ($n \geq 2$.) Given $u_{n-2}, u_{n-1} \in H^{1,2} \cap L^m(\Omega)$ and $h > 0$, we consider the following functional for $u(x) \in H^{1,2}(\Omega) \cap L^m(\Omega)$.

$$(2.4) \quad \mathcal{F}_n(u) = \int_{\Omega} \left\{ \frac{1}{2} \frac{\|u - 2u_{n-1} + u_{n-2}\|_a^2}{h^2} + \frac{1}{2} \|Du\|_b^2 + \frac{1}{m} \|u\|_c^m \right\} dx,$$

where $\|u\|_a^2 = a_{ij}(x)u^i u^j$, $\|\eta\|_b^2 = b_{ij}^{\alpha\beta}(x)\eta_\alpha^i \eta_\beta^j$. For $n \geq 2$, let $u_n(x)$ be a minimizer of $\mathcal{F}_n$ in the class $\{u \in H^{1,2} \cap L^m : u = w \text{ on } \partial\Omega\}$.

The Euler-Lagrange equation of $\mathcal{F}_n(u)$ is

$$(2.5) \quad \begin{aligned} 0 &= \frac{d}{d\varepsilon} \mathcal{F}_n(u + \varepsilon\varphi) \Big|_{\varepsilon=0} \\ &= \int_{\Omega} \left\{ \frac{1}{h^2} a_{ij}(x) (u^i - 2u_{n-1}^i + u_{n-2}^i) \varphi^j + b_{ij}^{\alpha\beta}(x) D_\alpha u^i D_\beta \varphi^j \right. \\ &\quad \left. + c_{ij}(x) \|u\|_c^{m-2} u^i \varphi^j \right\} dx \quad \forall \varphi \in H_0^{1,2} \cap L^m(\Omega, \mathbf{R}^l). \end{aligned}$$

The lower semicontinuity of L^p -norms guarantees the existence of a minimizer of $\mathcal{F}_n(u)$. Moreover one can see that a minimizer satisfies (2.5) by means of differentiability of the integrand of $\mathcal{F}_n$ with respect to Du and u . (About general theory of the direct method of calculus of variations see Chapter I of [2].)

Thus u_n ($n \geq 2$) satisfies (2.5) and we get the following lemma.

LEMMA 2.1. Let $\{u_n\}$ be as above. Then we have the following energy estimates:

$$(2.6) \quad \int_{\Omega} \frac{\|u_n - u_{n-1}\|_a^2}{2h^2} dx + \mathcal{E}(u_n) \leq K$$

for some positive constant K depending on u_0 and v_0 , where

$$\mathcal{E}(u) = \int_{\Omega} \left(\frac{1}{2} \|Du\|_b^2 + \frac{1}{m} \|u\|_c^m \right) dx.$$

Proof. Since u_n and u_{n-1} coincide on $\partial\Omega$, $u_n - u_{n-1}$ ($n \geq 1$) is an admissible test function for (2.5). Thus, using Young's inequality, we get

$$(2.7) \quad \begin{aligned} 0 &= \frac{d}{d\varepsilon} \mathcal{F}_n(u_n + \varepsilon(u_n - u_{n-1})) \Big|_{\varepsilon=0} \\ &\geq \int_{\Omega} \left\{ \left(\frac{\|u_n - u_{n-1}\|_a^2}{2h^2} + \frac{1}{2} \|Du_n\|_b^2 + \frac{1}{m} \|u_n\|_c^m \right) \right. \\ &\quad \left. - \left(\frac{\|u_{n-1} - u_{n-2}\|_a^2}{2h^2} + \frac{1}{2} \|Du_{n-1}\|_b^2 + \frac{1}{m} \|u_{n-1}\|_c^m \right) \right\} dx. \end{aligned}$$

This implies

$$\int_{\Omega} \frac{\|u_n - u_{n-1}\|_a^2}{2h^2} dx + \mathcal{E}(u_n) \leq \int_{\Omega} \frac{\|u_1 - u_0\|_a^2}{2h^2} dx + \mathcal{E}(u_1).$$

The definition of u_1 and (2.3) imply that

$$\begin{aligned} \int_{\Omega} \frac{\|u_1 - u_0\|_a^2}{h^2} dx &= \frac{1}{h^2} \int_{\Omega} \|v(x, h)\|_a^2 dx \leq \frac{c}{h^2} \int_{\Omega} \left\{ h \int_0^h \|v_t(x, t)\|^2 dt \right\} dx \\ &\leq \frac{c}{h} \int_0^h \int_{\Omega} \|v_0(x)\|^2 dx dt \leq c \int_{\Omega} \|v_0(x)\|^2 dx, \end{aligned}$$

where c is a constant depending only on (a_{ij}) , and $\| \cdot \|$ denotes the Euclidean norm. From the above estimates, remarking (2.3) again, we get

$$\int_{\Omega} \frac{\|u_n - u_{n-1}\|_a^2}{2h^2} dx + \mathcal{E}(u_n) \leq K$$

for some positive constant K depending only on u_0 and v_0 . □

3 Construction of Weak Solutions

Let $u_n(x)$ ($n \geq 1$) and $v(x, t)$ be as in the previous section. Using $u_n(x)$, we construct two maps $u_h(x, t)$ and $\bar{u}_h(x, t)$ which approximate to a weak solution of (1.1). Let us define

$$\bar{u}_h(x, t) = u_n(x) \quad \text{for } (n-1)h < t \leq nh, \quad n \geq 1,$$

$$u_h(x, t) = \frac{t - (n-1)h}{h} u_n(x) + \frac{nh - t}{h} u_{n-1}(x) \quad \text{for } (n-1)h < t \leq nh, \quad n \geq 2,$$

moreover, for $-1 \leq t \leq h$, put

$$u_h(x, t) = u_0(x) + v(x, t).$$

Then, from (2.5), we can see that

$$(3.1) \quad \int_h^T \int_{\Omega} \left[a_{ij}(x) \frac{1}{h} \left\{ \frac{\partial}{\partial t} u_h^i(x, t) - \frac{\partial}{\partial t} u_h^i(x, t-h) \right\} \varphi^j(x) \right. \\ \left. + b_{ij}^{\alpha\beta}(x) D_{\alpha} \bar{u}_h^i D_{\beta} \varphi^j + c_{ij}(x) \|\bar{u}_h\|_c^{m-2} \bar{u}_h^i \varphi^j \right] \eta(t) dx dt = 0$$

for any $T > 0$ and $\eta(t) \in C_0^{\infty}([0, T])$.

On the other hand, from (2.6), we get the following estimates.

$$(3.2) \quad \text{ess sup}_{-1 < t < T} \int_{\Omega} \left\| \frac{\partial u_h}{\partial t} \right\|_a^2 dx \leq 2K,$$

$$(3.3) \quad \int_{-1}^T \int_{\Omega} \left\| \frac{\partial u_h}{\partial t} \right\|_a^2 dx dt \leq 2K(T+1),$$

$$(3.4) \quad \int_{-1}^T \mathcal{E}(u_h) dt \leq 2K(T+1),$$

$$(3.5) \quad \int_0^T \mathcal{E}(\bar{u}_h) dt \leq 2KT.$$

Using Banach-Alaoglu theorem, from (3.2), (3.3) and (3.4) we can deduce that

$$(3.6) \quad \frac{\partial}{\partial t} u_h \rightharpoonup \frac{\partial}{\partial t} u, \quad D_{\alpha} u_h \rightharpoonup D_{\alpha} u \quad \text{weakly in } L^2(\Omega \times (-1, T)),$$

$$(3.7) \quad u_h \rightharpoonup u \quad \text{weakly in } L^{m'}(\Omega \times (-1, T)),$$

$$(3.8) \quad \frac{\partial}{\partial t} u_h \rightharpoonup u' \quad \text{weakly star in } L^{\infty}(-1, T; L^2(\Omega))$$

for some $u \in L^m \cap H^{1,2}(\Omega \times (-1, T))$ and $u' \in L^\infty(-1, T; L^2(\Omega))$ taking a subsequence if necessary, where $m' = \max\{2, m\}$. Since (3.6) and (3.8) imply that $u_t = u'$ a.e. on $\Omega \times (-1, T)$, we can see that $u_t \in L^\infty(-1, T; L^2(\Omega))$. Moreover, using Rellich's compactness theorem, from (3.6) and (3.7), we get

$$(3.9) \quad u_h \rightarrow u \text{ strongly in } L^2(\Omega \times (-1, T)).$$

Using Banach-Alaoglu theorem again, by (3.5) we obtain that

$$(3.10) \quad \begin{aligned} D_\alpha \bar{u}_h &\rightharpoonup D_\alpha \tilde{u} \text{ weakly in } L^2(\Omega \times (0, T)), \\ \bar{u}_h &\rightharpoonup \tilde{u} \text{ weakly in } L^{m'}(\Omega \times (0, T)), \end{aligned}$$

for some $\tilde{u} \in L^{m'}(\Omega \times (0, T))$ with $D_\alpha \tilde{u} \in L^2(\Omega \times (0, T))$ taking a subsequence if necessary.

Moreover, by the definition of u_h and $\bar{u}_h$ and (3.2), we have

$$(3.11) \quad \int_0^T \int_\Omega \|\bar{u}_h - u_h\|_a^2 dx dt \leq ch^2 KT \rightarrow 0 \quad \text{as } h \rightarrow 0$$

for some constant c depending only on the matrix (a_{ij}) . Hence, using (3.9) and (3.11), we see that $\bar{u}_h \rightarrow u$ in $L^2(\Omega \times (0, T))$. This implies that $\tilde{u} = u$ a.e. and therefore $D_\alpha \tilde{u} = D_\alpha u$ a.e. on $\Omega \times (0, T)$.

Now, letting $h \rightarrow 0$ in (3.1), we obtain

$$(3.12) \quad \begin{aligned} &\int_0^T \int_\Omega \left\{ -a_{ij}(x) \frac{\partial u^i}{\partial t} \frac{\partial \eta(t)}{\partial t} \varphi^j(x) + b_{ij}^{\alpha\beta} D_\alpha u^i D_\beta \varphi^j(x) \eta(t) \right. \\ &\quad \left. + c_{ij}(x) \|u\|_c^{m-2} u^i \varphi^j(x) \eta(t) \right\} dx dt \\ &= \int_\Omega a_{ij}(x) v_0^i(x) \eta(0) \varphi^j(x) dx, \end{aligned}$$

$$\forall \varphi(x) \in C_0^\infty(\Omega), \quad \forall \eta(t) \in C_0^\infty([0, T]).$$

Since functions of the form $\varphi(x)\eta(t)$ are total in the space $C^1([0, T]; C_0(\Omega)) \cap C([0, T]; C^1(\Omega))$, (3.12) means that u satisfies (1.6).

On the other hand, since $u_h(x, 0) = u_0(x)$, $u_h|_{\partial\Omega} = w$ and $u_h \rightarrow u$ in $H^{1,2}(\Omega \times (-1, T))$, we can see that u satisfies (ii) also. Thus Theorem 1.1 is proved.

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