

TWO TOPICS ON FLEMING-VIOT PROCESSES

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1. INTRODUCTION

For a Fleming-Viot process Y_t (which is a probability measure-valued process) on a compact metric space S , it is well-known that if its mutation operator A is bounded, then Y_t is pure atomic for every $t > 0$ (Ethier and Kurtz in [2], [4]). We shall extend this result to some jump-type measure-valued processes which are called “jump-type Fleming-Viot processes” introduced by the author in [6].

It is also well-known that the normalized binary branching process is a time inhomogeneous Fleming-Viot process. We shall introduce another new class of probability measure-valued diffusion, which are called “space-time inhomogeneous Fleming-Viot processes” and show that the normalized space inhomogeneous binary branching process is a space-time inhomogeneous Fleming-Viot process.

Let S be a compact metric space, fix $r \geq 0$ and set $D_r = \mathbf{D}([r, \infty) \rightarrow S)$ be a path space of right continuous functions with left-hand limit. Let $(w(t), P_x)_{t \geq r, x \in S}$ be a S -valued Markov process starting from x at $t = r$ with sample paths in D_r . We denote the transition semi-group by (P_t) and the generator by $(A, D(A))$, where $D(A)$ is a domain of A . We suppose that (P_t) is a Feller semi-group on $(C(S), \|\cdot\|)$, where $C(S)$ is a family of continuous functions on S and $\|\cdot\| = \|\cdot\|_\infty$ denotes the supremum norm.

Let $\mathcal{M}_F = \mathcal{M}_F(S)$ be a family of finite Radon measures on S with the weak topology, that is, $\mu_n \rightarrow \mu$ in $\mathcal{M}_F \iff \langle \mu_n, f \rangle \rightarrow \langle \mu, f \rangle$ for every $f \in C(S)$, where $\langle \mu, f \rangle = \int f d\mu$. Then, $\mathcal{M}_F$ is a Polish space, i.e., complete separable metrizable space. The family of probability measures on S , $\mathcal{M}_1 = \mathcal{M}_1(S) \subset \mathcal{M}_F$, is a compact metric space (cf. Chap. 3 of [3]). For $\mu \in \mathcal{M}_F \setminus \{0\}$, we always denote the normalized measure as $\bar{\mu} = \mu / \langle \mu, 1 \rangle$.

Let $(Y_t, \mathbf{P}_\mu^{FV})$ be a Fleming-Viot process on S , with a mutation operator A , that is, $(Y_t, \mathbf{P}_\mu^{FV})$ is an $\mathcal{M}_1$ -valued process on S such that $\mathbf{P}_\mu^{FV}(Y_0 = \mu) = 1$ and

$$\langle Y_t, f \rangle = \langle Y_0, f \rangle + \int_0^t \langle Y_s, Af \rangle + M_t(f),$$

where $\{M_t(f)\}$ is a continuous martingale with quadratic variation

$$\langle\langle M(f) \rangle\rangle_t = c \int_0^t (\langle Y_s, f^2 \rangle - \langle Y_s, f \rangle^2) ds \quad (c > 0),$$

A is a generator (with a domain $D(A) \subset (C(S), \|\cdot\|)$) of a conservative Feller process $(w(t), P_x)_{t \geq 0, x \in S}$; a S -valued Markov process starting from x with $w(\cdot) \in \mathbf{D} = \mathbf{D}([0, \infty) \rightarrow S)$ and the transition semi-group (P_t) .

The generator $\mathcal{L}$ of this process is given as, for $\eta \in \mathcal{M}_1, f \in D(A)$,

$$\mathcal{L}e^{-\langle \cdot, f \rangle}(\eta) = -\langle \eta, Af \rangle e^{-\langle \eta, f \rangle} + \frac{c}{2} [\langle \eta, f^2 \rangle - \langle \eta, f \rangle^2] e^{-\langle \eta, f \rangle}.$$

It is well-known that if its mutation operator A is bounded, then Y_t is pure atomic for every $t > 0$ (Ethier and Kurtz in [2], [4], see also Th. 8.2.1 in [1]). In particular, if $A = 0$ and if we denote $\eta = \sum_i m_i \delta_{x_i}$, then the generator $\mathcal{L}$ can be expressed as, for a function $\phi(\mathbf{m})$ of $\mathbf{m} = (m_1, m_2, \dots)$,

$$G\phi(\mathbf{m}) = \frac{c}{2} \sum_{i,j} m_i (\delta_{ij} - m_j) \partial_{ij}^2 \phi(\mathbf{m}).$$

The corresponding weight process $\{m_i(t)\}$ is given as

$$dm_i(t) = \sum_j (\delta_{ij} - m_i(t)) \sqrt{cm_j(t)} dB_j(t) \quad (i \in S),$$

where $\{B_j(t)\}$ is a family of independent one-dimensional Brownian motions.

There is another well-known measure-valued process which is a branching process $(Z_t, \mathbf{P}_\mu)$, that is, $\mathcal{M}_F$ -valued process such that $\mathbf{P}_\mu(Z_0 = \mu) = 1$ and

$$\mathbf{P}_\mu \left[e^{-\langle Z_t, f \rangle} \right] = e^{-\langle \mu, V_t f \rangle},$$

where $V_t f$ is a unique solution to the following equation

$$V_t f(x) = P_t f(x) - \int_0^t ds P_s \Psi(V_{t-s} f)(x),$$

or

$$\partial_t V_t f(x) = A V_t f(x) - \Psi(V_t f)(x), \quad V_0 f(x) = f(x)$$

with branching mechanism $\Psi(v)(x)$;

$$\Psi(v)(x) = \frac{1}{2} c(x) v^2 + \int_0^\infty [e^{-vu} - 1 + vu] \nu(x, du) \quad (\geq 0),$$

where $c(x) \geq 0$ is a bounded function and $\nu(x, du)$ is a kernel on $S \times (0, \infty)$ satisfying that

$$\sup_{x \in S} \int_0^\infty (u \wedge u^2) \nu(x, du) < \infty.$$

In particular, if $\Psi(v)(x) = cv^2/2$ ($c > 0$), then $(Z_t, \mathbf{P}_\mu)$ is called a *binary branching process* or a *binary branching measure-valued process*, or a *binary branching superprocess*, or a *Dawson-Watanabe process*, etc.

Let $\tau_0 = \inf\{t > 0; \langle Z_t, 1 \rangle = 0\}$. For $t < \tau_0$ and $f \in D(A)$, $\langle Z_t, f \rangle$ has the following semi-martingale representation:

$$\langle Z_t, f \rangle = \langle Z_0, f \rangle + \int_0^t \langle Z_s, A f \rangle ds + M_t^c(f) + M_t^d(f),$$

where $\{M_t^c(f)\}$ is a continuous L^2 -martingale with quadratic variation $\langle\langle M^c(f) \rangle\rangle_t$ such that

$$\langle\langle M^c(f) \rangle\rangle_t = \int_0^t \langle Z_s, c f^2 \rangle ds,$$

and

$$M_t^d(f) = \int_0^t \int_{\mathcal{M}_F} \langle \eta, f \rangle \widetilde{N}(ds, d\eta),$$

where $\widetilde{N}(ds, d\eta)$ is a martingale measure with compensator

$$\widehat{N}(ds, d\eta) = ds \int_S Z_s(dx) \int_0^\infty \nu(du) \delta_{u\delta_x}(d\eta).$$

The generator of this process $\mathcal{L}^Z$ is given as

$$\mathcal{L}^Z e^{-\langle \cdot, f \rangle}(\eta) = [-\langle \eta, Af \rangle + \langle \eta, \Psi(f) \rangle] e^{-\langle \eta, f \rangle}.$$

It is also well-known that the normalized binary branching process is a time inhomogeneous Fleming-Viot process. More exactly, in 1991, Perkins [8] established that the conditional law of the binary branching process given the total mass process is a time inhomogeneous Fleming-Viot process.

Fix $r \geq 0$ and let

$$C_{r,+} \equiv \left\{ g : [r, \infty) \rightarrow [0, \infty); g \text{ is continuous, and} \right. \\ \left. \text{there is } \tau_g \in (r, \infty] \text{ such that } g > 0 \text{ on } [r, \tau_g), g = 0 \text{ on } [\tau_g, \infty) \right\}.$$

Let $\mu \in \mathcal{M}_1$, $g \in C_{r,+}$ and $c > 0$.

The time inhomogeneous Fleming-Viot process $(Y_t, \mathbf{P}_\mu^{FV})$ satisfies the following:

- (i) $Y_r = \mu$, $Y_t = Y_{\tau_g-}$ ($t \geq \tau_g$), $\mathbf{P}_{\tau_g, \mu}^{FV}$ -a.s.,
- (ii) For $f \in D(A)$, $\langle Y_t, f \rangle$ has the following semi-martingale representation:

$$\langle Y_t, f \rangle = \langle Y_r, f \rangle + \int_r^t \langle Y_s, Af \rangle ds + M_{r,t}(f)$$

such that $\{M_{r,t}(f)\}$ is a continuous L^2 -martingale with quadratic variation

$$\langle\langle M(f) \rangle\rangle_{r,t} = c \int_r^t g(s)^{-1} [\langle Y_s, f^2 \rangle - \langle Y_s, f \rangle^2] I(s < \tau_g) ds.$$

We denote normalized measure $\bar{\mu} = \mu / \langle \mu, 1 \rangle$ for $\mu \in \mathcal{M}_F \setminus \{0\}$.

Theorem 1 (Perkins '91). *Let $\mu \in \mathcal{M}_F \setminus \{0\}$ and set $y = \langle \mu, 1 \rangle$. For a binary branching process $(Z_t, \mathbf{P}_\mu)$, set $x_t = \langle Z_t, 1 \rangle$ and $Q_y = \mathbf{P}_\mu \circ x^{-1}$. Then*

$$\mathbf{P}_\mu (\bar{Z} \in B \mid \langle Z, 1 \rangle = g(\cdot)) = \mathbf{P}_{0, \bar{\mu}}^{FV}(Y \in B), \quad Q_y\text{-a.a. } g \in C_{0,+},$$

where $(Y_t, \mathbf{P}_{0, \bar{\mu}}^{FV})$ is a time inhomogeneous Fleming-Viot process associated with (A, g, c) starting from $Y_0 = \bar{\mu}$.

We would like to extend these results to some wide class which include Fleming-Viot processes. The first one to “jump-type Fleming-Viot processes” introduced by the author in [6], and second one to “space-dependent Fleming-Viot processes” introduced by this paper.

2. PURE ATOMIC JUMP-TYPE FLEMING-VIOT PROCESSES

According to [6], we give characterizations of jump-type Fleming-Viot processes.

For each $x \in S$, we define an operator T_x from the space of Dirac measures $\delta_x(dy)$ on S to $\mathcal{M}_1$ by

$$\langle T_x \delta, f \rangle = \langle \delta_x, T f \rangle = T f(x) = \int_S f(y) T(x, dy),$$

where $T(x, dy)$ is a non-negative kernel on S such that $T(x, S) = 1$.

We fix $\gamma > 0$ and let $\nu(dv)$ be a measure on $(0, \infty)$ such that

$$\int_0^\infty (v \wedge v^2) \nu(dv) < \infty.$$

Let $(Y_t, \mathbf{P}_\mu)$ be a jump-type Fleming-Viot process associated with (A, γ, ν, T_x) starting from $\mu \in \mathcal{M}_1$. That is, $(Y_t, \mathbf{P}_\mu)$ is an $\mathcal{M}_1$ -valued process such that $\langle Y_t, f \rangle$ ($f \in D(A)$) has the following semi-martingale representation:

$$\langle Y_t, f \rangle = \langle Y_0, f \rangle + \int_0^t \langle Y_s, Af \rangle + M_t^c(f) + M_t^d(f),$$

where $\{M_t^c(f)\}$ is a continuous martingale with quadratic variation

$$\langle\langle M^c(f) \rangle\rangle_t = \gamma \int_0^t (\langle Y_s, f^2 \rangle - \langle Y_s, f \rangle^2) ds \quad (\gamma > 0),$$

and $\{M_t^d(f)\}$ is a pure discontinuous martingale such that

$$M_t^d(f) = \int_0^t \int_{\mathcal{M}_F} \frac{\langle \eta, 1 \rangle}{1 + \langle \eta, 1 \rangle} \langle \bar{\eta} - Y_{s-}, f \rangle \widetilde{N}(ds, d\eta),$$

where $\widetilde{N}(ds, d\eta)$ is the martingale measure with compensator

$$\widehat{N}(ds, d\eta) = ds \int_S Y_s(dx) \int_0^\infty \nu(dv) \delta_{vT_x\delta}(d\eta)$$

Remark 1. In [6] the process is defined only the case of $T_x = I$, i.e., $T_x\delta = \delta_x$. However the extension is possible and easy.

The generator $\mathcal{L}$ of this process for Laplace functionals $e^{-\langle \mu, f \rangle}$ ($\mu \in \mathcal{M}_1, f \in D(A)$) is given as

$$\begin{aligned} \mathcal{L}e^{-\langle \cdot, f \rangle}(\mu) &= -\langle \mu, Af \rangle e^{-\langle \mu, f \rangle} + \frac{\gamma}{2} [\langle \mu, f^2 \rangle - \langle \mu, f \rangle^2] e^{-\langle \mu, f \rangle} \\ &\quad + \int \mu(dx) \int_0^\infty \nu(dv) \\ &\quad \left\{ \exp \left[-\frac{v}{1+v} \langle T_x\delta - \mu, f \rangle \right] - 1 + \frac{v}{1+v} \langle T_x\delta - \mu, f \rangle \right\} e^{-\langle \mu, f \rangle}. \end{aligned}$$

For a functional $F(\mu)$, a derivative at $x \in S$ is defined by

$$\frac{\delta F(\mu)}{\delta \mu(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [F(\mu + \epsilon \delta_x) - F(\mu)] \quad (\text{if exists}),$$

and higher order derivatives $\delta^2 F(\mu)/(\delta \mu(x) \delta \mu(y)), \dots$ are defined similarly.

Note that the generator can be expressed as

$$\begin{aligned} (2.1) \quad \mathcal{L}F(\mu) &= \langle \mu, A \frac{\delta F(\mu)}{\delta \mu(\cdot)} \rangle + \frac{\gamma}{2} \iint \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} Q(\mu; dx, dy) \\ &\quad + \int_{\mathcal{M}_F} \left[F(\mu + g(\mu, \eta)) - F(\mu) - \langle g(\mu, \eta), \frac{\delta F(\mu)}{\delta \mu(\cdot)} \rangle \right] n(\mu; d\eta), \end{aligned}$$

where

$$\begin{aligned} Q(\mu; dx, dy) &= \mu(dx) \delta_x(dy) - \mu(dx) \mu(dy), \\ g(\mu, \eta) &= \frac{\mu + \eta}{1 + \langle \eta, 1 \rangle} - \mu = \frac{\langle \eta, 1 \rangle}{1 + \langle \eta, 1 \rangle} (\bar{\eta} - \mu) \in \mathcal{M}_F \end{aligned}$$

and

$$n(\mu; d\eta) = \int \mu(dx) \int_0^\infty \nu(dv) \delta_{vT_x\delta}(d\eta).$$

Here we give a formal calculation for general measure-valued processes, i.e., Ito's formula for measure-valued processes.

For a set B , let $\mathcal{M}_F^\pm(B)$ be the class of finite signed measures on B , i.e., $\eta \in \mathcal{M}_F^\pm(B) \iff \eta = \eta^+ - \eta^-$; $\eta^\pm \in \mathcal{M}_F(B)$. We denote $\|\eta\| = (\eta^+ + \eta^-)(B)$. For simplicity, if $B = S$, then $\mathcal{M}_F^\pm = \mathcal{M}_F^\pm(S)$. Let $Q(\mu; dx, dy) : \mathcal{M}_F \rightarrow \mathcal{M}_F^\pm(S \times S)$ be measurable such that $Q(\mu; dx, dx) \leq C\mu(dx)$ for some $C > 0$. Let $g(\mu, \eta) : \mathcal{M}_F \times \mathcal{M}_F^\pm \rightarrow \mathcal{M}_F^\pm$ and $n(\mu; d\eta) : \mathcal{M}_F \rightarrow \mathcal{M}_F(\mathcal{M}_F^\pm)$ be measurable such that

$$\sup_{\mu \in \mathcal{M}_F; \mu(S) \leq K} \int_{\mathcal{M}_F^\pm} \|g(\mu, \eta)\| \wedge \|g(\mu, \eta)\|^2 n(\mu; d\eta) < \infty \quad \text{for every } K > 1.$$

In general, let X_t be an $\mathcal{M}_F$ -valued Markov process such that

$$X_t = X_0 + \int_0^t A^* X_s ds + M_t^c + M_t^d,$$

where $A^* X_s$ is defined by $\langle A^* X_s, f \rangle = \langle X_s, Af \rangle$, M_t^c is a continuous martingale and M_t^d is a pure discontinuous martingale;

$$M_t^c(dx) = \int_0^t M(ds, dx)$$

$$M_t^d(dx) = \int_0^t \int_{\mathcal{M}_F^\pm} g(X_{s-}, \eta)(dx) \widetilde{N}(ds, d\eta)$$

with a continuous martingale measure $M(ds, dx)$ and a pure discontinuous martingale measure $\widetilde{N}(ds, d\eta)$. Suppose that the covariance of $M^c(dx)M^c(dy)$ is given as

$$\langle\langle M^c(dx), M^c(dy) \rangle\rangle_t = \gamma \int_0^t Q(X_s; dx, dy) ds$$

and the compensator of $\widetilde{N}$ is $\widehat{N}(ds, d\eta) = ds n(X_s, d\eta)$ (cf. for martingale measures, see Walsh [10]).

Let $F(\mu)$ be a suitable functional of $\mu \in \mathcal{M}_F$ and let $\mathcal{L}F(\mu)$ be given as in (2.1). Then, by a formal calculation we have the following Ito's formula:

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t \mathcal{L}F(X_s) ds + M_t^c(F) \\ &\quad + \int_0^t \int_{\mathcal{M}_F^\pm} [F(X_{s-} + g(X_{s-}, \eta)) - F(X_{s-})] \widetilde{N}(ds, d\eta), \end{aligned}$$

where

$$M_t^c(F) = \int_0^t \int \frac{\delta F(X_s)}{\delta X_s(x)} M(ds, dx)$$

is a martingale with quadratic variation

$$\langle\langle M^c(F) \rangle\rangle_t = \gamma \int_0^t ds \iint \frac{\delta F(X_s)}{\delta X_s(x)} \frac{\delta F(X_s)}{\delta X_s(y)} Q(X_s; dx, dy)$$

(for the stochastic integrals corresponding to the martingale measures, see Walsh [10] and also Dawson [1]).

We denote the space of pure atomic probability measures on S by $\mathcal{M}_{1,a} = \mathcal{M}_{1,a}(S)$. We also denote the pure atomic part of a measure η by η_a , and for a process Y_t by $Y_{t,a}$ or $(Y_t)_a$.

We set

$$\begin{aligned}\mathbf{M} &= \{\mathbf{m} = (m_1, m_2, \dots); m_1 \geq m_2 \geq \dots \geq 0, \sum_i m_i = 1\} \\ \overline{\mathbf{M}} &= \{\mathbf{m} = (m_1, m_2, \dots); m_1 \geq m_2 \geq \dots \geq 0, \sum_i m_i \leq 1\}\end{aligned}$$

and let $M : \mathcal{M}_1 \rightarrow \overline{\mathbf{M}}$; $M(\eta) = M(\eta_a)$ = (the vector of descending order statistics of the masses of the atoms of η_a).

Theorem 2. *Let $(Y_t, \mathbf{P}_\mu)$ be a jump-type Fleming-Viot process associated with (A, γ, ν, T_x) starting from $\mu \in \mathcal{M}_1$. Suppose that A is a bounded operator having the following form:*

$$Af(x) = a(x) \int_S [f(y) - f(x)] B(x, dy),$$

where $a(x)$ is a nonnegative bounded function on S and $B(x, dy)$ is a nonnegative kernel such that $B(x, S) = 1$ for all $x \in S$. Also assume that $\gamma > 0$ and $T_x \delta \in \mathcal{M}_{1,a}$ for every $x \in S$. Then Y_t is pure atomic for all $t > 0$ a.s., i.e.,

$$\mathbf{P}_\mu(Y_t \in \mathcal{M}_{1,a} \text{ for all } t > 0) = 1.$$

Moreover if $a(\cdot) \equiv a(\geq 0)$, $T_x \delta = \delta_x$ for every $x \in S$ and $B(x, \cdot)$ has no atoms for every $x \in S$, then $\{\mathbf{m}_t = M(Y_t)\}$ is a solution of the $\mathbf{D}([0, \infty), \mathbf{M})$ -martingale problem for the infinite dimensional operator $(G, \mathcal{D}(G))$, where

$$\begin{aligned}G\phi(\mathbf{m}) &= \frac{\gamma}{2} \sum_{i,j} m_i (\delta_{ij} - m_j) \partial_{ij}^2 \phi(\mathbf{m}) - a \sum_i m_i \partial_i \phi(\mathbf{m}) \\ &+ \sum_i m_i \int_0^\infty \left[\phi \left(\left(\frac{m_j + v \delta_{ij}}{1+v} \right)_j \right) - \phi(\mathbf{m}) - \sum_j \frac{v}{1+v} (\delta_{ij} - m_j) \partial_j \phi(\mathbf{m}) \right] \nu(dv)\end{aligned}$$

with $\partial_i = \partial / \partial m_i$, $\partial_{ij}^2 = \partial^2 / (\partial m_i \partial m_j)$ and $\mathcal{D}(G)$ is the algebra generated by $\{1, \phi^2, \phi^3, \dots\}$ with $\phi^\beta(\mathbf{m}) = \sum_i m_i^\beta$ ($\beta > 1$). That is, Y_t is the size ordered atom process for the infinitely many neutral alleles (jump-type) model.

Proof. It is enough to consider the case that $a(\cdot) \equiv a > 0$ and $B(x, \cdot)$ has no atoms. Because if we set $\overline{S} = S \times [0, 1]$, $\overline{a} = \sup a(x) > 0$ (note that the case of $\overline{a} = 0$ is trivial),

$$\overline{B}(x, u, dydv) = \frac{a(x)}{\overline{a}} B(x, dy) dv + \frac{\overline{a} - a(x)}{\overline{a}} \delta_x(dy) dv$$

and

$$\overline{A}f(x, u) = \overline{a} \int_0^1 \int_S [f(y, v) - f(x, u)] \overline{B}(x, u, dydv),$$

then $Y_t(\cdot) := \overline{Y}_t(\cdot \times [0, 1])$, with the solution $\overline{Y}_t$ of the martingale problem for $\overline{A}$, is the solution of the martingale problem for A .

For $\mu \in \mathcal{M}_1$, if $\mu_a = \sum_i m_i \delta_{x_i}$, then set $F_\beta(\mu) = \phi_\beta(M(\mu)) = \sum_i m_i^\beta$ ($\beta > 1$), and

$$F_{1+}(\mu) = \lim_{\beta \downarrow 1} F_\beta(\mu) = \sum_j m_j (= \langle \mu_a, 1 \rangle).$$

If $\beta > 2$, then

$$\frac{\delta F_\beta(\mu)}{\delta \mu(x)} = \sum_i \beta m_i^{\beta-1} 1_{x_i}(x) \quad \text{and} \quad \frac{\delta^2 F_\beta(\mu)}{\delta \mu(x) \delta \mu(y)} = \sum_i \beta(\beta-1) m_i^{\beta-2} 1_{x_i}(x) 1_{x_i}(y),$$

We first give some formal calculations. For each fixed x we denote all atoms of $\mu + T_x$ by $\{x_j\}$. Since $B(x, \cdot)$ has no atoms, by (2.1) we have

$$\begin{aligned} \mathcal{L}F_\beta(\mu) &= \sum_i \left[-a\beta m_i^\beta + \frac{\gamma}{2}\beta(\beta-1)(m_i - m_i^2) m_i^{\beta-2} \right] \\ &\quad + \int_S \mu(dx) \int_0^\infty \left[F_\beta\left(\frac{\mu + vT_x\delta}{1+v}\right) - F_\beta(\mu) \right. \\ &\quad \left. - \frac{v}{1+v} \sum_j (T_x\delta - \mu)(\{x_j\})\beta m_j^{\beta-1} \right] \nu(dv) \\ &= -a\beta F_\beta(\mu) + \frac{\gamma}{2}\beta(\beta-1)(F_{\beta-1}(\mu) - F_\beta(\mu)) \\ &\quad + \int_S \mu(dx) \int_0^\infty \left[F_\beta\left(\frac{\mu + vT_x\delta}{1+v}\right) - F_\beta(\mu) \right. \\ &\quad \left. - \frac{\beta v}{1+v} \sum_j (T_x\delta(\{x_j\}) - \mu(\{x_j\}))\mu(\{x_j\})^{\beta-1} \right] \nu(dv) \end{aligned}$$

Moreover if we set $F_1(\mu) \equiv 1$, then the above formula is still valid for $\beta = 2$. Hence by formal Ito's formula

$$(2.2) \quad M_t(\beta) := M_t(F_\beta(Y_t)) = F_\beta(Y_t) - F_\beta(Y_0) - \int_0^t \mathcal{L}F_\beta(Y_s) ds$$

is an L^2 -martingale such that $M_t(\beta) = M_t^{c,\beta} + M_t^{d,\beta}$, where $M_t^{c,\beta}$ is a continuous martingale with quadratic variation

$$\langle\langle M^{c,\beta} \rangle\rangle_t = \beta^2 \gamma \int_0^t [F_{2\beta-1}(Y_s) - F_\beta(Y_s)^2] ds$$

and

$$M_t^{d,\beta} = \int_0^t \int \left[F_\beta\left(\frac{Y_{s-} + \eta}{1 + \langle \eta, 1 \rangle}\right) - F_\beta(Y_{s-}) \right] \widetilde{N}(ds, d\eta)$$

is a pure discontinuous martingale with compensator

$$\widehat{N}(ds, d\eta) = ds \int_S Y_s(dx) \int_0^\infty \nu(dv) \delta_{vT_x\delta}(d\eta).$$

To verify the above result we use an approximation method. Let $\{S_j^n, x_j^n, \rho_n; n, j \in \mathbb{N}\}$ be a partition family for S , i.e.,

- (a) $S_j^n \subset S, x_j^n \in S_j^n, S_j^n \cap S_k^n = \emptyset$ if $j \neq k$,
- (b) $\bigcup_j S_j^n = S$ for all $n \in \mathbb{N}$,
- (c) for each j , there is k such that $S_j^{n+1} \subset S_k^n$,
- (d) $\rho_n := \sup_j \text{diam}(S_j^n) \rightarrow 0$ ($n \rightarrow \infty$).

For $\mu \in \mathcal{M}_1$, set $\xi^n(\mu) = \sum_j \mu(S_j^n) \delta_{x_j^n}$ and $\xi_t^n = \xi^n(Y_t)$. For $\beta \geq 1$, let $F_{\beta,n}(\mu) = \sum_j \mu(S_j^n) = \phi^\beta(\xi^n(\mu))$. Note that for $\beta \geq 1$, $F_{\beta,n}(\mu) \rightarrow F_\beta(\mu)$ as $n \rightarrow \infty$ by the definition of $F_\beta(\mu)$.

For each j, n , if we take $f_k^{j,n} \in D(A)$; $0 \leq f_k^{j,n} \uparrow 1_{S_j^n}$ ($k \uparrow \infty$), and use Ito's formula for $\langle Y_t, f_k^{j,n} \rangle$, then by letting $k \rightarrow \infty$ and summing up on j we can get the following: If

$\beta \geq 2$, then

$$M_{n,t}(\beta) := M_t(F_{\beta,n}(Y_t)) = F_{\beta,n}(Y_t) - F_{\beta,n}(Y_0) - \int_0^t \mathcal{L}F_{\beta,n}(Y_s) ds$$

is a bounded (uniformly in n) L^2 -martingale, where

$$\begin{aligned} \mathcal{L}F_{\beta,n}(\mu) = & \sum_i \left[\frac{\gamma}{2} \beta(\beta-1) (\mu(S_i^n) - \mu(S_i^n)^2) \mu(S_i^n)^{\beta-2} \right. \\ & \left. + a\beta [\langle \mu, B(\cdot, S_i^n) \rangle - \mu(S_i^n)] \mu(S_i^n)^{\beta-1} \right] \\ & + \int_S \mu(dx) \int_0^\infty \sum_j \left[\left(\frac{(\mu + vT_x\delta)(S_j^n)}{1+v} \right)^\beta - \mu(S_j^n)^\beta \right. \\ & \left. - \frac{v}{1+v} (T_x\delta - \mu)(S_j^n) \beta \mu(S_j^n)^{\beta-1} \right] \nu(dv). \end{aligned}$$

Note that if we set $p = 1/(1+v)$, $q = v/(1+v)$ and

$$(2.3) \quad h(a, b) := (pa + qb)^\beta - a^\beta - \beta q(b-a)a^{\beta-1} \geq 0$$

for $0 \leq a, b \leq 1$, then by $\beta \geq 2$

$$\begin{aligned} h(a, b) &= \int_0^1 ds \int_0^s \beta(\beta-1)(tq(b-a) + a)^{\beta-2} q^2(b-a)^2 dt \\ &\leq \beta(\beta-1)q^2(a^2 + b^2). \end{aligned}$$

Since $B(x, \cdot)$ has no atoms, we can see that as $n \rightarrow \infty$ $\mathcal{L}F_{\beta,n}(\mu) \rightarrow \mathcal{L}F_\beta(\mu)$ (bounded-pointwisely), and hence $M_{n,t}(\beta) \rightarrow M_t(\beta)$ (a.s., in L^2). Moreover the limit process $\{M_t(\beta)\}$ is an L^2 -martingale with $M_t(\beta) = M_t^{c,\beta} + M_t^{d,\beta}$ as mentioned above (note that this decomposition can be shown by using the uniqueness of special martingales; cf. Theorem 6.1.3 in [1]).

By mean zero in (2.2) and by taking a limit of $\mathcal{L}F_\beta(\mu)$ as $\beta \downarrow 2$ carefully, we have

$$0 = \lim_{\beta \downarrow 2} \mathbf{E}_\mu [M_t(\beta) - M_t(2)] = \mathbf{E}_\mu \left[\gamma \int_0^t (1 - F_{1+}(Y_s)) ds \right] (\geq 0).$$

By $\gamma > 0$ this implies

$$\langle (Y_t)_a, 1 \rangle = F_{1+}(Y_t) = 1, \quad \text{i.e., } Y_t \in \mathcal{M}_{1,a} \text{ for a.a. } t > 0, \text{ a.s.}$$

In order to show that $Y_t \in \mathcal{M}_{1,a}$ for all $t > 0$, a.s., we mention that $M_t^{c,\beta}$, $M_t^{d,\beta}$ are still L^2 -martingales for $\beta \in (1, 2)$. In fact, to verify this, let $F_{\beta,\epsilon}(\mu) = \phi_{\beta,\epsilon}(M(\mu))$, where

$$\phi_{\beta,\epsilon}(\mathbf{m}) = \sum_i \psi_\epsilon(m_i) \quad \text{with} \quad \psi_\epsilon(m) = (m + \epsilon)^\beta - \epsilon^\beta - \beta \epsilon^{\beta-1} m.$$

Apply Ito's formula to $F_{\beta,\epsilon}(Y_t)$ and take the limit $\epsilon \downarrow 0$, then the corresponding martingale parts $M_t^{c,\beta,\epsilon}$, $M_t^{d,\beta,\epsilon}$ converge to $M_t^{c,\beta}$, $M_t^{d,\beta}$ respectively in L^2 . Moreover for $M_t(\beta) = M_t^{c,\beta} + M_t^{d,\beta}$, the formula (2.2) also holds. These can be checked as follows: First note that for $1 < \beta < 2$,

$$\psi_\epsilon(m) = \int_0^1 ds \int_0^s \beta(\beta-1)(tm + \epsilon)^{\beta-2} m^2 dt \uparrow m^\beta \quad (\beta \downarrow 1),$$

$\psi_\epsilon(m)$ is convex in $0 \leq m \leq 1$ and

$$\begin{aligned}\frac{\delta F_{\beta,\epsilon}(\mu)}{\delta \mu(x)} &= \sum_i \beta \left\{ (m_i + \epsilon)^{\beta-1} - \epsilon^{\beta-1} \right\} 1_{x_i}(x), \\ \frac{\delta^2 F_{\beta,\epsilon}(\mu)}{\delta \mu(x) \delta \mu(y)} &= \sum_i \beta(\beta-1) (m_i + \epsilon)^{\beta-2} 1_{x_i}(x) 1_{x_i}(y).\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{L}F_{\beta,\epsilon}(\mu) &= \sum_i \left[-a\beta \left\{ (m_i + \epsilon)^{\beta-1} - \epsilon^{\beta-1} \right\} m_i + \frac{\gamma}{2} \beta(\beta-1) (m_i - m_i^2) (m_i + \epsilon)^{\beta-2} \right] \\ &\quad + \int_S \mu(dx) \int_0^\infty \left[F_{\beta,\epsilon} \left(\frac{\mu + vT_x \delta}{1+v} \right) - F_{\beta,\epsilon}(\mu) \right. \\ &\quad \left. - \frac{\beta v}{1+v} \sum_j (T_x \delta - \mu)(\{x_j\}) \left\{ (m_i + \epsilon)^{\beta-1} - \epsilon^{\beta-1} \right\} \right] \nu(dv)\end{aligned}$$

converges to $\mathcal{L}F_\beta(\mu)$ as $\epsilon \downarrow 0$ by using

$$(m + \epsilon)^{\beta-1} - \epsilon^{\beta-1} = (\beta-1)m \int_0^1 (tm + \epsilon)^{\beta-2} dt \uparrow m^{\beta-1} \quad (\epsilon \downarrow 0)$$

and monotone convergence theorem. By the same way we also have

$$\int_0^t \mathcal{L}F_{\beta,\epsilon}(Y_s) ds \rightarrow \int_0^t \mathcal{L}F_\beta(Y_s) ds.$$

If we set $(Y_t)_a = \sum_j m_j(t) \delta_{x(t)}$, then as $\epsilon \downarrow 0$,

$$\begin{aligned}\langle\langle M^{c,\beta,\epsilon} \rangle\rangle_t &= \gamma\beta^2 \int_0^t \left[\sum_j \left\{ (m_j(s) + \epsilon)^{\beta-1} - \epsilon^{\beta-1} \right\}^2 m_j(s) \right. \\ &\quad \left. - \left\{ \sum_j \left((m_j(s) + \epsilon)^{\beta-1} - \epsilon^{\beta-1} \right) m_j(s) \right\}^2 \right] ds \\ &\rightarrow \gamma\beta^2 \int_0^t \sum_j \left\{ m_j(s)^{2\beta-1} - \left(\sum_j m_j(s)^\beta \right)^2 \right\} ds \\ &= \langle\langle M^{c,\beta} \rangle\rangle_t.\end{aligned}$$

Hence

$$M_t^{c,\beta,\epsilon} = B_{\langle\langle M^{c,\beta,\epsilon} \rangle\rangle_t} \rightarrow M_t^{c,\beta} = B_{\langle\langle M^{c,\beta} \rangle\rangle_t},$$

where $\{B_t\}$ is a one-dimensional Brownian motion. The L^2 convergence can be shown by the following:

$$\begin{aligned}\langle\langle M^{c,\beta,\epsilon} - M^{c,\beta} \rangle\rangle_t &= \gamma\beta^2 \int_0^t \left[\sum_j \left\{ (m_j(s) + \epsilon)^{\beta-1} - \epsilon^{\beta-1} - m_j(s)^{\beta-1} \right\}^2 m_j(s) \right. \\ &\quad \left. - \left\{ \sum_j \left((m_j(s) + \epsilon)^{\beta-1} - \epsilon^{\beta-1} - m_j(s)^{\beta-1} \right) m_j(s) \right\}^2 \right] ds \\ &\rightarrow 0 \quad \text{in } L^1.\end{aligned}$$

For the discontinuous parts, we can see that

$$\int_0^t ds \int Y_s(dx) \int_0^\infty \nu(dv) \left| F_{\beta,\epsilon} \left(\frac{Y_s + vT_x\delta}{1+v} \right) - F_{\beta,\epsilon}(Y_s) \right. \\ \left. - \left\{ F_\beta \left(\frac{Y_s + vT_x\delta}{1+v} \right) - F_\beta(Y_s) \right\} \right|^2 \rightarrow 0 \quad \text{in } L^1$$

by Lebesgue's convergence theorem. Hence $M_t^{d,\beta,\epsilon} \rightarrow M_t^{d,\beta}$ in L^2 . Moreover by taking a suitable subsequence we get the a.s. convergence. Therefore (2.2) is also valid.

Form the above results as $\beta \downarrow 1$,

$$\mathbf{E}_\mu \left[(M_t^{c,\beta})^2 \right] = \mathbf{E}_\mu \left[\langle M^{c,\beta} \rangle_t \right] \rightarrow \gamma \mathbf{E}_\mu \left[\int_0^t F_{1+}(Y_s) (1 - F_{1+}(Y_s)) ds \right] = 0, \\ \mathbf{E}_\mu \left[(M_t^{d,\beta})^2 \right] = \mathbf{E}_\mu \left[\int_0^t ds \int Y_s(dx) \int_0^\infty \nu(dv) \left| F_\beta \left(\frac{Y_s + vT_x\delta}{1+v} \right) - F_\beta(Y_s) \right|^2 \right] \\ \rightarrow \mathbf{E}_\mu \left[\int_0^t ds \int Y_s(dx) \int_0^\infty \nu(dv) \left(\frac{v}{1+v} \right)^2 \langle (T_x\delta - Y_s)_a, 1 \rangle^2 \right] = 0$$

by $T_x \in \mathcal{M}_{1,a}$ for all $x \in S$ and $Y_t \in \mathcal{M}_{1,a}$ for a.a. $t > 0$, a.s. Furthermore

$$\lim_{\beta \downarrow 1} \mathbf{E}_\mu \left[M_t(\beta)^2 \right] \leq 2 \lim_{\beta \downarrow 1} \left(\mathbf{E}_\mu \left[(M_t^{c,\beta})^2 \right] + \mathbf{E}_\mu \left[(M_t^{d,\beta})^2 \right] \right) \\ = 0.$$

Hence by Doob's maximal inequality and by taking a sequence $\{\beta_n\}; \beta_n \downarrow 1$,

$$\sup_{t \leq T} |M_t(\beta_n)| \rightarrow 0 \quad \text{a.s. for each } T > 0.$$

Note that

$$H^\beta(\mu) := \int_S \mu(dx) \int_0^\infty \left[F_\beta \left(\frac{\mu + vT_x\delta}{1+v} \right) - F_\beta(\mu) \right. \\ \left. - \frac{\beta v}{1+v} \left(\sum_j T_x\delta(\{x_j\}) \mu(\{x_j\})^{\beta-1} - F_\beta(\mu) \right) \right] \nu(dv) \geq 0$$

and that for $\mu \in \mathcal{M}_1$,

$$\limsup_{\beta \downarrow 1} \mathcal{L} F_\beta(\mu) = -a F_{1+}(\mu) + \limsup_{\beta \downarrow 1} \left[\frac{\gamma}{2} \beta(\beta-1) F_{\beta-1}(\mu) + H^\beta(\mu) \right].$$

Thus if we set

$$R_t = \limsup_{n \rightarrow \infty} \int_0^t \left[\frac{\gamma}{2} \beta_n(\beta_n-1) F_{\beta_n-1}(Y_s) + H^{\beta_n}(Y_s) \right] ds$$

(which is nondecreasing in t), then by (2.2) and the above result we have

$$F_{1+}(Y_t) - F_{1+}(Y_0) + a \int_0^t F_{1+}(Y_s) ds - R_t = 0 \quad \text{for all } 0 < t \leq T, \text{ a.s.}$$

By $F_{1+}(Y_t) = 1$ for a.a. $t > 0$, a.s.,

$$F_{1+}(Y_t) - F_{1+}(Y_0) = R_t - at \quad \text{for all } t \geq 0, \text{ a.s.}$$

and

$$0 = F_{1+}(Y_t) - F_{1+}(Y_s) = R_t - R_s - a(t - s) \quad \text{for a.a. } t > s > 0, \text{ a.s.},$$

that is,

$$R_t - R_s = a(t - s) \quad \text{for a.a. } t > s > 0, \text{ a.s.}$$

The left hand side is nondecreasing in $t > s$. Hence it is easy to see that

$$R_t = at \quad \text{for all } t \geq 0, \text{ a.s.}$$

This implies that $Y_t \in \mathcal{M}_{1,a}$ for all $t > 0$, a.s. Finally in case of $T_x \delta = \delta_x$ ($x \in S$), it is easy to check that $\{\mathbf{m}_t\}$ is a solution of the martingale problem for $(G, \mathcal{D}(G))$. $\square$

Remark 2 (Pure jump case). In case of $\gamma = 0$, even if A is bounded, it is not ensure that Y_t is pure atomic for all $t > 0$ $\mathbf{P}_\mu$ -a.s. For instance, suppose that $T_x \delta \in \mathcal{M}_{1,a}$ for every $x \in S$ and $Y_0 = \mu \in \mathcal{M}_{1,a}$ $\mathbf{P}_\mu$ -a.s. Also assume that $a(\cdot) \equiv a > 0$ and $B(x, \cdot)$ has no atoms for each $x \in S$. Let $H^\beta(\mu)$ be defined as in the previous proof. If $\int_0^\infty v \nu(dv) < \infty$, then it is easy to see that

$$\lim_{\beta \downarrow 1} H^\beta(\mu) = 0.$$

In fact, for $\beta > 1$, let $h(a, b) \geq 0$ be in (2.3) with $p = 1/(1 + v)$, $q = v/(1 + v)$, then it holds that for $0 \leq a, b \leq 1$,

$$\begin{aligned} h(a, b) &\leq pa^\beta + qb^\beta - a^\beta - \beta q(b - a)a^{\beta-1} \\ &= q(-a^\beta + b^\beta - \beta(b - a)a^{\beta-1}) \\ &\begin{cases} \leq q(1 + \beta)(a + b), \\ \rightarrow 0 \quad (\beta \downarrow 1). \end{cases} \end{aligned}$$

Therefore we can apply Lebesgue's convergence theorem for $H(\eta)$. Now by mean zero

$$\begin{aligned} 0 &= \lim_{\beta \downarrow 1} \mathbf{E}_\mu[M_t^{a, \beta_n}] \\ &= \mathbf{E}_\mu \left[F_{1+}(Y_t) - F_{1+}(Y_0) + a \int_0^t F_{1+}(Y_s) ds \right]. \end{aligned}$$

This implies that $\mathbf{E}_\mu[F_{1+}(Y_t)] = F_{1+}(\mu)e^{-at} = e^{-at} < 1$, i.e., Y_t is not pure atomic.

(Note that if $a = 0$, then $\mathbf{E}_\mu[F_{1+}(Y_t)] = 1$, i.e., $\mathbf{E}_\mu[\int_0^T F_{1+}(Y_t) dt] = T$ for all $T > 0$. thus, $Y_t \in \mathcal{M}_{1,a}$ for a.a. $t > 0$, a.s. Moreover by the same way as in the previous proof we have

$$F_{1+}(Y_t) - 1 = R_t - at = H_t - at = 0 \quad \text{for all } t > 0, \text{ a.s.}$$

That is,

$$\mathbf{P}_\mu(Y_t \in \mathcal{M}_{1,a} \quad \text{for all } 0 \leq t \leq T) = 1.)$$

3. SPACE-TIME INHOMOGENEOUS FLEMING-VIOT PROCESSES

Next we would like to extend the Perkins result to the space(-time) inhomogeneous case.

Let $c(x) \geq 0$ be a bounded function.

According to Dawson [1], we first give a characterization of the space inhomogeneous binary branching process $(Z_t, \mathbf{P}_\mu)_{t \geq 0}$ ($\mu \in \mathcal{M}_F$).

$(Z_t, \mathbf{P}_\mu)_{t \geq 0}$ is an $\mathcal{M}_F$ -valued process such that $\mathbf{P}_\mu(Z_0 = \mu) = 1$, $Z_t = Z_{t \wedge \tau_0}$ ($\tau_0 = \inf\{t \geq 0; \langle Z_t, 1 \rangle = 0\}$) and

$$\mathbf{P}_\mu \left[e^{-\langle Z_t, f \rangle} \right] = e^{-\langle \mu, V_t f \rangle},$$

where $V_t f$ is a unique solution to the following equation

$$V_t f(x) = P_t f(x) - \frac{1}{2} \int_0^t ds P_s \left(c(\cdot) (V_{t-s} f)(\cdot)^2 \right) (x),$$

or

$$\partial_t V_t f(x) = A V_t f(x) - \frac{1}{2} c(x) (V_t f)^2(x), \quad V_0 f(x) = f(x).$$

Moreover $\langle Z_t, f \rangle$ ($f \in D(A)$) has the following semi-martingale representation:

$$\langle Z_t, f \rangle = \langle Z_0, f \rangle + \int_0^t \langle Z_s, A f \rangle ds + M_t(f),$$

where $\{M_t(f)\}$ is a continuous L^2 -martingale with quadratic variation $\langle\langle M(f) \rangle\rangle_t$ such that

$$\langle\langle M(f) \rangle\rangle_t = \int_0^t \langle Z_s, c f^2 \rangle ds \quad (t < \tau_0).$$

The generator of this process $\mathcal{L}^Z$ is given as

$$\mathcal{L}^Z e^{-\langle \cdot, f \rangle}(\mu) = \left[-\langle \mu, A f \rangle + \frac{1}{2} \langle \mu, c f^2 \rangle \right] e^{-\langle \mu, f \rangle}.$$

For a functional $F(\eta)$ of $\eta \in \mathcal{M}_F$, a derivative at $x \in S$ is defined by

$$\frac{\delta F(\eta)}{\delta \eta(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [F(\eta + \epsilon \delta_x) - F(\eta)] \quad (\text{if exists}),$$

and higher order derivatives $\delta^2 F(\eta) / (\delta \eta(x) \delta \eta(y)), \dots$ are defined similarly. Note that the generator can be also expressed as

$$(3.1) \quad \mathcal{L}^Z F(\eta) = \langle \eta, A \frac{\delta F(\eta)}{\delta \eta(\cdot)} \rangle + \frac{1}{2} \iint \frac{\delta^2 F(\eta)}{\delta \eta(x) \delta \eta(y)} Q(\eta; dx, dy)$$

where $Q(\eta; dx, dy) = c(x) \eta(dx) \delta_x(dy)$.

Next in order to introduce space-time inhomogeneous Fleming-Viot processes we define a family of operators $\mathcal{L}^g = (\mathcal{L}_t^g)_{r \leq t < \tau_g}$ for a fixed $g \in C_{r,+}$ as follows: For functionals $\exp[-\langle \eta, f \rangle]$ ($\eta \in \mathcal{M}_1, f \in D(A)$) and $r \leq t < \tau_g$,

$$\begin{aligned} \mathcal{L}_t^g e^{-\langle \cdot, f \rangle}(\eta) &= \left\{ -\langle \eta, A f \rangle - g(t)^{-1} [\langle \eta, c \rangle \langle \eta, f \rangle - \langle \eta, c f \rangle] \right\} e^{-\langle \eta, f \rangle} \\ &\quad + \frac{1}{2g(t)} \left[\langle \eta, c f^2 \rangle + \langle \eta, c \rangle \langle \eta, f \rangle^2 - 2 \langle \eta, c f \rangle \langle \eta, f \rangle \right] e^{-\langle \eta, f \rangle}. \end{aligned}$$

with a domain

$$\mathcal{D}_0(\mathcal{L}^g) := \text{lin span} \left\{ e^{-\langle \cdot, f \rangle}; f \in D(A), f \geq 0 \right\}.$$

This operator can be also expressed as in (3.1) by

$$\begin{aligned}\mathcal{L}_t^g F(\eta) &= \langle \eta, A \frac{\delta F(\eta)}{\delta \eta(\cdot)} \rangle + g(t)^{-1} \left[\langle \eta, c \rangle \langle \eta, \frac{\delta F(\eta)}{\delta \eta(\cdot)} \rangle - \langle \eta, c \frac{\delta F(\eta)}{\delta \eta(\cdot)} \rangle \right] \\ &\quad + \frac{1}{2} \iint \frac{\delta^2 F(\eta)}{\delta \eta(x) \delta \eta(y)} Q_t(\eta; dx, dy),\end{aligned}$$

where

$$Q_t(\eta; dx, dy) = \frac{1}{g(t)} [c(x)\eta(dx)\delta_x(dy) + (\langle \eta, c \rangle - c(x) - c(y))\eta(dx)\eta(dy)].$$

We need the following condition.

Condition 3.1. $c \in C(S)$ satisfies $0 \leq \sup c(x) \leq 2 \inf c(x)$.

This condition is equivalent to $c(x) + c(y) \geq c(z) \geq 0$ for every $x, y, z \in S$.

The following result gives the definition of the space-time inhomogeneous Fleming-Viot process $(Y_t, \mathbf{P}_{r,\mu}^{FV})$ associated with (A, g, c) starting from $\mu \in \mathcal{M}_1$ at $t = r$.

Theorem 3. Let $\mu \in \mathcal{M}_1$, $g \in C_{r,+}$ and $c \in C(S); c(x) \geq 0$. For $\omega \in \mathbf{C}_{r,\tau_g-} := \mathbf{C}([r, \tau_g) \rightarrow \mathcal{M}_1)$, set $Y_t(\omega) = \omega(t)$. Then, under Condition 3.1 on $c(x)$, there is a solution $\mathbf{P}_{r,\mu}^{FV}$ on $\mathbf{C}_{r,\tau_g-}$ to the martingale problem for $(\mathcal{L}_t^g, \mathcal{D}_0(\mathcal{L}^g))_{t \in [r, \tau_g)}$ satisfying the following:

- (i) $Y_r = \mu$, $\mathbf{P}_{r,\mu}^{FV}$ -a.s.,
- (ii) For $f \in D(A)$ and $r \leq t < \tau_g$, $\langle Y_t, f \rangle$ has the following semi-martingale representation:

$$\begin{aligned}\langle Y_t, f \rangle &= \langle Y_r, f \rangle + \int_r^t \left\{ \langle Y_s, Af \rangle + g(s)^{-1} [\langle Y_s, c \rangle \langle Y_s, f \rangle - \langle Y_s, cf \rangle] \right\} ds \\ &\quad + M_{r,t}(f)\end{aligned}$$

such that $\{M_{r,t}(f)\}$ is a continuous L^2 -martingale with quadratic variation

$$\langle\langle M(f) \rangle\rangle_{r,t} = \int_r^t g(s)^{-1} [\langle Y_s, cf^2 \rangle + \langle Y_s, c \rangle \langle Y_s, f \rangle^2 - 2 \langle Y_s, cf \rangle \langle Y_s, f \rangle] ds.$$

Moreover if $A = 0$, then the solution $(Y_t, \mathbf{P}_\mu^{FV})$ is unique.

By using the same argument as in Perkins [8] and the author [6] it is possible to obtain the following result. Recall $\bar{\mu} = \mu / \langle \mu, 1 \rangle$.

Corollary 1. Let $\mu \in \mathcal{M}_F \setminus \{0\}$ and set $y = \langle \mu, 1 \rangle$. For a fixed $r \geq 0$, let $(Z_t, \mathbf{P}_\mu)$ be a space-inhomogeneous binary branching process with $A = 0$ on $\mathbf{C}([r, \infty), \mathcal{M}_F)$ such that $Z_r = \mu$. Set $x_t = \langle Z_t, 1 \rangle$ and $\tau_0 = \inf\{t \geq r; x_t = 0\}$. If $Q_y = \mathbf{P}_\mu \circ (\{x_t\}_{t \in [r, \tau_0)})^{-1}$, then

$$\mathbf{P}_\mu \left(\bar{Z} \cdot |_{[r, \tau_0)} \in B \mid \langle Z_\cdot, 1 \rangle = g(\cdot) |_{[r, \tau_g)} \right) = \mathbf{P}_{r,\bar{\mu}}^{FV}(Y_\cdot |_{[r, \tau_g)} \in B), \quad Q_y\text{-a.a. } g \in C_{r,+},$$

where $(Y_t, \mathbf{P}_{r,\bar{\mu}}^{FV})$ is a space-time inhomogeneous Fleming-Viot process associated with $(0, g, c)$ starting from $Y_r = \bar{\mu}$.

Note that $x_t = \langle Z_t, 1 \rangle$ has a decomposition $x_t = \langle \mu, f \rangle + m_t$, where $\{m_t\}$ is a continuous martingale starting from 0 with quadratic variation,

$$\langle\langle m \rangle\rangle_t = \int_r^t \langle Z_s, c \rangle ds \quad (r \leq t < \tau_0).$$

Proof. For simplicity of the notations, as in [6] we set $Z_t(f) = \langle Z_t, f \rangle$, $|Z_t| = \langle Z_t, 1 \rangle = x_t$. Recall that

$$dZ_t(f) = Z_t(Af)dt + dM_t(f), \quad Z_0(f) = \langle \mu, f \rangle,$$

where $\{M_t(f)\}$ is a continuous L^2 -martingale with quadratic variation $d\langle\langle M(f) \rangle\rangle_t = \langle Z_t, cf^2 \rangle dt$. Thus by using Ito's formula we have

$$d(1/|Z_t|) = -d|Z_t|/|Z_t|^2 + d\langle\langle M(1) \rangle\rangle_t/|Z_t|^3$$

and, noting that

$$d\langle\langle Z(f), (1/|Z|) \rangle\rangle_t = -d\langle\langle M(f), M(1) \rangle\rangle_t/|Z_t|^2 = -[\langle Z_t, cf \rangle/|Z_t|^2]dt,$$

we also have

$$d\bar{Z}_t(f) = \bar{Z}_t(Af)dt + dU_t(f) + [\bar{Z}_t(c)\bar{Z}_t(f) - \bar{Z}_t(cf)]/|Z_t|dt,$$

where

$$dU_t(f) = dM_t(f)/|Z_t| - [\bar{Z}_t(f)/|Z_t|]dM_t(1)$$

is a continuous L^2 -martingale with quadratic variation

$$\begin{aligned} d\langle\langle U(f) \rangle\rangle_t &= \frac{d\langle\langle M(f) \rangle\rangle_t}{|Z_t|^2} + \left[\frac{\bar{Z}_t(f)}{|Z_t|} \right]^2 d\langle\langle M(1) \rangle\rangle_t - 2 \left[\frac{\bar{Z}_t(f)}{|Z_t|^2} \right] d\langle\langle M(f), M(1) \rangle\rangle_t \\ &= \left[\bar{Z}_t(cf^2) + \bar{Z}_t(f)^2 \bar{Z}_t(c) - 2\bar{Z}_t(f)\bar{Z}_t(cf) \right] |Z_t|^{-1} dt. \end{aligned}$$

Hence by the same way as in [8] we can show that $\{\bar{Z}_t\}$ under the condition $|Z_t| = g(\cdot)$ is the space inhomogeneous Fleming-Viot process. $\square$

Proof of Theorem 3.

Let $c(x) \geq 0$ be in $C(S)$ and satisfy Condition 3.1. Fix $g \in C_{r,+}$ and set $c_t^g(x) = c^g(t; x) := c(x)/g(t)$ ($t \leq \tau_g$),

$$c_t^g(x, y, z) = c^g(t; x, y, z) := \frac{1}{2} [c^g(t, x) + c^g(t, y) - c^g(t, z)] \quad (\geq 0).$$

It is enough to consider the uniqueness and the existence in $\mathbf{C}_{r,T} := \mathbf{C}([r, T] \rightarrow \mathcal{M}_1)$ for each fixed $r < T < \tau_g$.

We first show the existence of the solution. Fix $n \geq 1$. Let $\mu^{(n)} = \sum_{k=1}^n \delta_{x_k}$. Let $(X_t^0, \mathbf{Q}_{r, \mu^{(n)}}^0)$ be the independent particle system associated with the motion process $(w(t), P_x)$ starting from $X_r^0 = \mu^{(n)}$. Let $(X_t, \mathbf{Q}_{r, \mu^{(n)}})$ be the Markov particle system starting from $X_r = \mu^{(n)}$ such that the Laplace functional $L_{r,t}(\mu^{(n)}) = \mathbf{Q}_{r, \mu^{(n)}}[\exp - \langle X_t, f \rangle]$ is

the unique solution to the following equation:

$$\begin{aligned}
L_{r,t}(\mu^{(n)}) = & \frac{1}{n} \sum_{k=1}^n \mathbf{Q}_{r,\mu^{(n)}}^0 \left[\exp \left(- \sum_m c_m^g(r,t) - \sum_{i \neq j} c_{i,j,k}^g(r,t) - \langle X_t^0, f \rangle \right) \right. \\
& + \sum_m \int_r^t ds c_m^g(s) \exp \left(- c_m^g(r,s) - \sum_{m' \neq m} c_{m'}^g(r,t) - \sum_{i \neq j} c_{i,j,k}^g(r,t) \right) \\
& \quad L_{s,t} \left(X_s^0 - \delta_{w_m(s)} + \delta_{w_k(s)} \right) \\
& + \sum_{i \neq j} \int_r^t ds c_{i,j,k}^g(s) \exp \left(c_{i,j,k}^g(r,s) - \sum_m c_m^g(r,t) - \sum_{i' \neq j'; (i',j') \neq (i,j)} c_{i',j',k}^g(r,t) \right) \\
& \quad \left. L_{s,t} \left(X_s^0 - \delta_{w_i(s)} + \delta_{w_j(s)} \right) \right],
\end{aligned}$$

where $c_m^g(t) := c^g(t; w_m(t))$, $c_m^g(r,t) := \int_r^t c_m^g(s) ds$ and $c_{i,j,k}^g(t) := c^g(t; w_i(t), w_j(t), w_k(t))$, $c_{i,j,k}^g(r,t) := \int_r^t c_{i,j,k}^g(s) ds$ ($i \neq j$).

This particle system can be constructed directly as follows: first n -particles $\{w_i(t); i = 1, 2, \dots, n\}$ ($\subset D_r$) move independently and one particle (e.g. k_1 -th particle w_{k_1}) is selected with probability $1/n$ at the starting time $t = r$. Let $\sigma_m^{(p)} = \sigma_m^{(p)}$, $\tau_{i,j,k}^{(p)}$ ($p = 1, 2, \dots$, and $m, i, j, k \in \{1, \dots, n\}; i \neq j$) be independent random variables such that $P_{\mathbf{w}}(\sigma_m^{(p)} > t) = \exp[-c_m^g(r,t)]$ and $P_{\mathbf{w}}(\tau_{i,j,k}^{(p)} > t) = \exp[-c_{i,j,k}^g(r,t)]$, where $\mathbf{w} = \{w_i(t); i = 1, 2, \dots, n\}$. The index m_1 and the pair (i_1, j_1) is uniquely defined by $\min_m \sigma_m^{(1)} = \sigma_{m_1}^{(1)}$ and $\min_{i \neq j} \tau_{i,j,k_1}^{(1)} = \tau_{i_1,j_1,k_1}^{(1)}$. If $\sigma_{m_1}^{(1)} < \tau_{i_1,j_1,k_1}^{(1)}$, then after the random time $\sigma_{m_1}^{(1)}$, m_1 -th particle jumps to the location of the k_1 -th particle and at the same time k_2 -th particle is selected with probability $1/n$. If $\sigma_{m_1}^{(1)} > \tau_{i_1,j_1,k_1}^{(1)}$, then after the random time $\tau_{i_1,j_1,k_1}^{(1)}$, i_1 -th particle jumps to the location of the another j_1 -th particle and at the same time k_2 -th particle is selected with probability $1/n$. Again these particles move independently according to the same law. For these particles, we use the same notations $\{w_i(t)\}$. Next the random times $\sigma_{m_2}^{(2)}, \tau_{i_2,j_2,k_2}^{(2)}$ are defined as above by using $\{\sigma_m^{(2)}\}, \{\tau_{i,j,k}^{(2)}\}$. Then according to $\sigma_{m_2}^{(2)} > \tau_{i_2,j_2,k_2}^{(2)}$ or $\sigma_{m_2}^{(2)} < \tau_{i_2,j_2,k_2}^{(2)}$ particles moves similarly to above. These operations are continued.

This particle system $(X_t, \mathbf{Q}_{r,\mu^{(n)}})$ is called the *space-time dependent Moran particle system starting from $\mu^{(n)}$ at $t = r$ associated with the motion process $(w(t), P_{r,x})$, sampling rate function $c(t; x)$.*

We denote the generator of independent particle system $\{w_k(t)\}$ by $\mathcal{G}^0$, which is defined as

$$\mathcal{G}^0 e^{-\langle \cdot, f \rangle}(\eta) = -\langle \eta, e^f A(1 - e^{-f}) \rangle e^{-\langle \eta, f \rangle}.$$

Let $\mu^{(n)} = \sum_{i=1}^n \delta_{x_i}$. The generator $\mathcal{G}_t$ of $(X_t, \mathbf{Q}_{r, \mu^{(n)}})$ is given as

$$\begin{aligned} \mathcal{G}_t e^{-\langle \cdot, f \rangle}(\mu^{(n)}) &= \frac{1}{n} \sum_{k=1}^n \left[\mathcal{G}^0 e^{-\langle \cdot, f \rangle}(\mu^{(n)}) + \sum_m c_t^g(x_m) \left(e^{-|f(x_k) - f(x_m)|} - 1 \right) e^{-\langle \mu^{(n)}, f \rangle} \right. \\ &\quad \left. + \sum_{i \neq j} c_t^g(x_i, x_j, x_k) \left(e^{-|f(x_j) - f(x_i)|} - 1 \right) e^{-\langle \mu^{(n)}, f \rangle} \right] \\ &= \left\{ -\langle \eta, e^f A(1 - e^{-f}) \rangle + \frac{1}{n} \sum_{k=1}^n \left[\sum_m c_t^g(x_m) \left(e^{-|f(x_k) - f(x_m)|} - 1 \right) \right. \right. \\ &\quad \left. \left. + \sum_{i \neq j} c_t^g(x_i, x_j, x_k) \left(e^{-|f(x_j) - f(x_i)|} - 1 \right) \right] \right\} e^{-\langle \mu^{(n)}, f \rangle}. \end{aligned}$$

Set the domain of $\mathcal{G} = (\mathcal{G}_t)$ by

$$\mathcal{D}_0(\mathcal{G}) := \text{lin span} \left\{ e^{-\langle \cdot, f \rangle}; f = -\log(1 - h), 0 \leq h < 1, h \in D(A) \right\}.$$

Then it is easy to see that $(X_t, \mathbf{Q}_{r, \mu^{(n)}})$ is a Markov process with sample paths in $\mathbf{D}([r, \infty) \rightarrow \mathcal{M}_F(S))$ and the unique solution to the martingale problem for $(\mathcal{G}_t, \mathcal{D}_0(\mathcal{G}))$ on $\mathbf{D}([r, \infty) \rightarrow \mathcal{M}_F(S))$.

Now we consider the scaled Moran particle system $(Y_{n,t}, \mathbf{P}_{r, \mu_n}^{(n)})$, where $Y_{n,t} = X_t/n$ with $\mu_n = \mu^{(n)}/n$ and $\mathbf{P}_{r, \mu_n}^{(n)}$ is its probability law. We also denote the generator by $\mathcal{L}_{n,t}^g$. We shall show that if $\mu_n \rightarrow \mu$ in $\mathcal{M}_1$, then the scaling limit $(Y_t, \mathbf{P}_{r, \mu}^{FV})$ exists as a space-time inhomogeneous Fleming-Viot process associated with $(A, 1, c^g) = (A, g, c)$ and has the following generator $\mathcal{L}_t^g$; for $r \leq t < \tau_g$,

$$\begin{aligned} \mathcal{L}_t^g e^{-\langle \cdot, f \rangle}(\eta) &= -\langle \eta, Af \rangle e^{-\langle \eta, f \rangle} \\ &\quad - \frac{1}{g(t)} [\langle \eta, c \rangle \langle \eta, f \rangle - \langle \eta, cf \rangle] e^{-\langle \eta, f \rangle} \\ &\quad + \frac{1}{2g(t)} [\langle \eta, cf^2 \rangle + \langle \eta, c \rangle \langle \eta, f \rangle^2 - 2\langle \eta, cf \rangle \langle \eta, f \rangle] e^{-\langle \eta, f \rangle}. \end{aligned}$$

For $f \in D(A)$,

$$\langle Y_{n,t}, f \rangle - \langle Y_{n,r}, f \rangle - \int_r^t \langle Y_{n,s}, Af \rangle ds$$

is a $\mathbf{P}_{r, \mu_n}^{(n)}$ -martingale and

$$\sup_n \mathbf{P}_{r, \mu_n}^{(n)} \left[\text{ess sup}_{r \leq t \leq T} |\langle Y_{n,t}, Af \rangle| \right] \leq \|Af\|.$$

Hence by Th. 9.4 in Chap. 3 (p 145) of [3] $\{\langle Y_{n,t}, f \rangle\}$ is relatively compact, i.e., tight in $\mathbf{D}([r, T], \mathbf{R})$ (because $\mathbf{D}([r, T], \mathbf{R})$ is Polish). Moreover since S is compact and $D(A)$ is dense in $C(S)$ and closed under addition, by Th. 3.7.1 in [1] $\{Y_{n,t}\}$ is tight, i.e., relatively compact in $\mathbf{D}([r, T], \mathcal{M}_1)$. Therefore there exist a subsequence $\{(Y_{n_k, t}, \mathbf{P}_{r, \mu_{n_k}}^{(n)})\}$ and a limit point $(Y_t, \mathbf{P}_{r, \mu})$ such that $\{Y_{n_k, t}\}$ converges weakly to $\{Y_t\}$ in $\mathbf{D}([r, T], \mathcal{M}_1)$.

For each integer n , let $\mathcal{M}_1^{(n)} = \mathcal{M}_1^{(n)}(S)$ be a family of counting measures on S of the form $\eta_n = \sum_{k=1}^n \delta_{x_k}/n$. Moreover let $f_n = -n \log(1 - f/n)$ for $f \in D(A)$ such that

$\|f\| < 1$ and $\inf f > 0$. It is possible to show that for each $r < T < \tau_g$,

$$(3.2) \quad \lim_{n \rightarrow \infty} \sup_{r \leq t \leq T} \sup_{\eta \in \mathcal{M}_1^{(n)}} |\mathcal{L}_{n,t}^g e^{-\langle \cdot, f_n \rangle}(\eta) - \mathcal{L}_t^g e^{-\langle \cdot, f \rangle}(\eta)| = 0$$

(we shall show this at the end). Hence it is easy to see that the limit point $(Y_t, \mathbf{P}_{r,\mu})$ is a solution to the martingale problem for $(\mathcal{L}_t^g, \mathcal{D}_0(\mathcal{L}))$ in $\mathbf{D}([r, T], \mathcal{M}_1)$. We need to show the continuity and the semi-martingale representation of $\{Y_t\}$. However this can be shown by the same way as in the proof of Th. 6.1.3 of [1].

Next in case of $A = 0$ the uniqueness can be shown by the same way as in [9]. In fact, for $r \leq t \leq T$ and $\eta \in \mathcal{M}_1$ let

$$a_t(\eta, f) = g(t)^{-1} [\langle \eta, c \rangle \langle \eta, f \rangle - \langle \eta, cf \rangle]$$

and

$$b_t(\eta, f) = g(t)^{-1} [\langle \eta, cf^2 \rangle + \langle \eta, c \rangle \langle \eta, f \rangle^2 - 2\langle \eta, cf \rangle \langle \eta, f \rangle].$$

We consider the following stochastic differential equation: for $r \leq t \leq T$,

$$(3.3) \quad d\langle Y_t, f \rangle = a_t(Y_t, f)dt + \sqrt{b_t(Y_t, f)}dB_t, \quad Y_r = \mu,$$

where (B_t) is a one-dimensional standard Brownian motion. To show the (law) uniqueness of (Y_t) it is enough to show the pathwise uniqueness of the solution to the above equation in $\mathbf{C}_{r,T}$ (see [7]). However the pathwise uniqueness can be easily checked. Let $(Y_t), (\tilde{Y}_t)$ be solutions for the equation (3.3) defined on the same probability space (we denote the probability measure as $\mathbf{P}_\mu$). By using the following inequality (let $g_* = \inf_{r \leq t \leq T} g(t)$)

$$\begin{aligned} |a_t(\eta, f) - a_t(\tilde{\eta}, f)| &\leq Ng_*^{-1}(\|c\| \vee \|f\|) (|\langle \eta - \tilde{\eta}, c \rangle| + |\langle \eta - \tilde{\eta}, f \rangle|), \\ |b_t(\eta, f) - b_t(\tilde{\eta}, f)| &\leq Ng_*^{-1}(\|cf\| \vee \|f\|^2) (|\langle \eta - \tilde{\eta}, c \rangle| + |\langle \eta - \tilde{\eta}, f \rangle|), \end{aligned}$$

where N is an appropriate number, and we have

$$\mathbf{E}_\mu [|\langle Y_t - \tilde{Y}_t, f \rangle|] \leq C \int_r^t \mathbf{E}_\mu [|\langle Y_s - \tilde{Y}_s, c \rangle| + |\langle Y_s - \tilde{Y}_s, f \rangle|] ds,$$

where $C > 0$ is a constant depending only on $(g_*, \|c\|, \|f\|)$. Thus we get the pathwise uniqueness of $\{\langle Y_t, c \rangle\}$ and of $\{\langle Y_t, f \rangle\}$ ($f \in C(S)$). Hence the law of uniqueness holds.

Therefore if $A = 0$, then the limit process $(Y_t, \mathbf{P}_{r,\mu}^{FV})_{r \leq t \leq T}$ uniquely exists in $\mathbf{C}_{r,T}$.

Finally we show (3.2). Note that for $\eta_n = \sum_j \delta_{x_j}/n \in \mathcal{M}_1^{(n)}$,

$$\begin{aligned} \mathcal{L}_{n,t}^g e^{-\langle \cdot, f_n \rangle}(\eta_n) &= \mathcal{G}_t e^{-\langle \cdot, f_n/n \rangle}(n\eta_n) \\ &= -\langle \eta_n, \frac{Af}{1 - f/n} \rangle e^{-\langle \eta_n, f_n \rangle} \\ &\quad + \frac{1}{n} \sum_{k,m=1}^n c_t^g(x_m) \left(e^{(f_n(x_m) - f_n(x_k))/n} - 1 \right) e^{-\langle \eta_n, f_n \rangle} \\ &\quad + \frac{1}{n} \sum_{k=1}^n \sum_{i \neq j} c_t^g(x_i, x_j, x_k) \left(e^{(f_n(x_i) - f_n(x_j))/n} - 1 \right) e^{-\langle \eta_n, f_n \rangle}. \end{aligned}$$

It is easy to see that

$$\frac{1}{n^2} \sum_{k,m} c(x_m)(f(x_m) - f(x_k)) = -(\langle \eta_n, c \rangle \langle \eta_n, f \rangle - \langle \eta_n, cf \rangle)$$

and that by symmetry of $c_t^g(x_i, x_j, x_k)$ in (i, j)

$$\sum_{i \neq j} c_t^g(x_i, x_j, x_k) [f(x_i) - f(x_j)] = 0.$$

Moreover

$$\begin{aligned} \frac{1}{n^2} \sum_{i \neq j} (f(x_i) - f(x_j))^2 &= \frac{1}{n^2} \sum_{i,j} (f(x_i) - f(x_j))^2 \\ &= 2 \left(\langle \eta_n, f^2 \rangle - \langle \eta_n, f \rangle^2 \right) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{n^2} \sum_{i \neq j} c(x_i) (f(x_i) - f(x_j))^2 &= \frac{1}{n^2} \sum_{i,j} c(x_i) (f(x_i) - f(x_j))^2 \\ &= \langle \eta_n, c f^2 \rangle + \langle \eta_n, c \rangle \langle \eta_n, f^2 \rangle - 2 \langle \eta_n, c f \rangle \langle \eta_n, f \rangle. \end{aligned}$$

Hence by $c_t^g(x_i, x_j, x_k) = [c_t^g(x_i) + c_t^g(x_j) - c_t^g(x_k)]/2$ we have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \sum_{i \neq j} c_t^g(x_i, x_j, x_k) \frac{(f(x_i) - f(x_j))^2}{2n^2} \\ = \frac{1}{2g(t)} \left[\langle \eta_n, c f^2 \rangle + \langle \eta_n, c \rangle \langle \eta_n, f^2 \rangle - 2 \langle \eta_n, c f \rangle \langle \eta_n, f \rangle \right]. \end{aligned}$$

Therefore by using Taylor's expansion the equation (3.2) can be easily checked. $\square$

Remark 3. If $A = 0$ and the starting measure $Y_r = \mu$ is pure atomic, i.e., $\mu = \sum m_i^0 \delta_{x_i^0}$, then clearly the process Y_t is also pure atomic and the corresponding generator is given as

$$\begin{aligned} G\phi(\mathbf{m}) &= \frac{1}{2} \sum_{i,j} \left[m_i c(x_i^0) \delta_{ij} + m_i m_j \left\{ \sum_k m_k c(x_k^0) - c(x_i^0) - c(x_j^0) \right\} \right] \partial_{ij}^2 \phi(\mathbf{m}) \\ &\quad + \sum_i b_i(\mathbf{m}) \partial_i \phi(\mathbf{m}), \end{aligned}$$

where $b_i(\mathbf{m}) = \left(\sum_j c(x_j^0) m_j - c(x_i^0) \right) m_i$. However in this case our result is contained to Shiga's result [9] in 1987. He showed the result under more general conditions on $c(x)$ and $b_i(\mathbf{m})$ such that let $\beta_i = c(x_i^0)$, $\beta_i \geq 0$; $\sup_i \beta_i < \infty$ and for some matrix (q_{ij}) ; $q_{ij} \geq 0$, $\sup_j \sum_i q_{ij} < \infty$, $|b_i(\mathbf{m}) - b_i(\mathbf{m}')| \leq \sum_j q_{ij} |m_j - m'_j|$.

By using the same argument as in the proof of Theorem 3 we can see the following results:

Theorem 4. *In Theorem 3, if the quadratic martingale part is changed to the following*

$$\langle\langle M(f) \rangle\rangle_{r,t} = \int_r^t g(s)^{-1} \left[\langle Y_s, c f^2 \rangle + \langle Y_s, c \rangle \langle Y_s, f^2 \rangle - 2 \langle Y_s, c f \rangle \langle Y_s, f \rangle \right] ds,$$

then the same claim holds for all bounded functions $c \in C(S)$; $c(x) \geq 0$ without Condition 3.1. Moreover it is possible to construct the processes without the drift term $g(t)^{-1} [\langle \eta, c \rangle \langle \eta, f \rangle - \langle \eta, c f \rangle]$.

Proof. We have to consider the approximating particle systems $\{X_t\}$. However it is enough to change $c_t^g(x, y, z)$ to $c_t^g(x, y) = c^g(t; x, y) := [c^g(t, x) + c^g(t, y)]/2$, thus $c_{i,j,k}^g(t)$ to $c_{i,j}^g(t) := c_t^g(t; w_i(t), w_j(t))$. Moreover for the processes without drift term, it enough to delete the terms corresponding to $c_m^g(s)$ and $c_t^g(x_m)$. $\square$

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