

Some Alternative Theorems of Set-Valued Maps and their Applications

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Abstract. We establish some theorems for a certain minimization problem whose constraints are presented by set-valued maps. For this, we prove two alternative theorems for set-valued maps. By using those theorems, we show some theorems for this minimization problem.

Key Words. Mathematical programming, set-valued analysis, convex analysis, convexity of set-valued maps, continuity of set-valued maps, alternative theorem.

1. Introduction and Preliminaries

我々は、集合値写像を用いて表される次の問題 (P) を考える：

$$(P) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & F(x) \cap (-P) \neq \emptyset \end{array}$$

ただし、 X ：実ベクトル空間、 C ： X の空でない凸集合、 Y ：実線形位相空間、 P ： Y の凸錐、 $f : C \rightarrow \mathbf{R}$ 、 $F : C \rightarrow Y$ 。

この問題 (P) は、従来の不等式制約型の問題：

$$(P') \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, 2, \dots, n \end{array}$$

(ただし、 $g_i : C \rightarrow \mathbf{R}$, $i = 1, 2, \dots, n$) を含み、さらに、問題 (P') に定式化されないような問題も (P) においては扱うことができる。

本論文における目的は、問題 (P) について考察することである。具体的には、

- (1) (P) の双対問題 (D) を考える。
- (2) (P) と (D) の値が等しくなるような条件を求める。

などを考察するが、このときに非常に重要な役割を果たすのが、二者択一の定理 (alternative theorem) である。二者択一の定理の古典的な例としては、Gordan の定理、Farkas の定理などがあり、いずれも応用する上で、非常に有用な定理である。

そこで、二者択一の条件を定式化し、どのような条件の下で、二者択一の定理が成立するのかを観察していく。

まず、(P) の双対問題 (D) を次のように定義する。

$$(D) \quad \begin{array}{ll} \text{maximize} & \phi(y^*) \\ \text{subject to} & y^* \in P^+ \end{array}$$

ただし, $\phi(y^*) \equiv \inf_{(x,y) \in \text{Graph}(F)} \{f(x) + \langle y^*, y \rangle\}$, $P^+ \equiv \{y^* \in Y^* | \langle y^*, y \rangle \geq 0, \forall y \in P\}$.

このとき, 次が成立する.

Proposition 1.1. (Weak Duality)

$$\text{val}(D) \leq \text{val}(P).$$

この等号を成立させる条件の一つが, 関数の凸性である. 従って, 次の章においては, 集合値関数の凸性を定義する.

2. Convexity of Set-Valued Maps and their Relations

この章では, 集合値写像の凸性をいくつか定義し, それらの間にある関係について述べていく. 集合値写像の凸性は, ベクトル値関数の凸性を基にして定義する. その拡張の方法は, いくつかの方法がある. [4]

Definition 2.1. A set-valued map $F : C \rightsquigarrow Y$ is said to be

- (i) *convex* if for every $x_1, x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0, 1)$, there exists $y \in F(\lambda x_1 + (1 - \lambda)x_2)$ such that $y \leq_P \lambda y_1 + (1 - \lambda)y_2$;
- (ii) *convexlike* if for every $x_1, x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0, 1)$, there exists $(x, y) \in \text{Graph}(F)$ such that $y \leq_P \lambda y_1 + (1 - \lambda)y_2$;
- (iii) *properly quasiconvex* if for every $x_1, x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0, 1)$, there exists $y \in F(\lambda x_1 + (1 - \lambda)x_2)$ such that either $y \leq_P y_1$ or $y \leq_P y_2$;
- (iv) *quasiconvex* if for every $x_1, x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0, 1)$, if $y \in Y$ satisfies $y_1 \leq_P y$ and $y_2 \leq_P y$, then there exists $y' \in F(\lambda x_1 + (1 - \lambda)x_2)$ such that $y' \leq_P y$;
- (v) *naturally quasiconvex* (c.f. [7]) if for every $x_1, x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0, 1)$, there exists $y \in F(\lambda x_1 + (1 - \lambda)x_2)$ and $\eta \in [0, 1]$ such that $y \leq_P \eta y_1 + (1 - \eta)y_2$;
- (vi) **-quasiconvex* (c.f. [3]) if for each $y^* \in P^+$, function $x \mapsto \inf_{y \in F(x)} \langle y^*, y \rangle$ is quasiconvex on C .

ただし, $y_1 \leq_P y_2 \iff y_2 - y_1 \in P$.

これらの集合値写像の凸性に関して, 次が成立する. [4]

Proposition 2.1. The following statements hold:

- (i) F is convex if and only if $\text{Graph}(F) + \{\theta_X\} \times P$ is a convex set;
 - (ii) F is convexlike if and only if $F(C) + P$ is a convex set;
 - (iii) F is quasiconvex if and only if for all $y \in Y$, the set $F^{-1}(y - P)$ is a convex set.
- ただし, $F^{-1}(M) \equiv \{x \in C | F(x) \cap M \neq \emptyset\}$; $F^{+1}(M) \equiv \{x \in C | F(x) \subset M\}$.

Proposition 2.2. *The following statements hold:*

- (i) every convex map is also convexlike;
- (ii) every convex map is also naturally quasiconvex;
- (iii) properly quasiconvex map is also naturally quasiconvex;
- (iv) naturally quasiconvex map is also quasiconvex;
- (v) naturally quasiconvex map is also $*$ -quasiconvex.

Theorem 2.1. *Assume that Y is a locally convex space and $F(x) + P$ is closed convex for all $x \in C$. If F is $*$ -quasiconvex, then F is also naturally quasiconvex.*

Theorem 2.2. *If We assume that P is closed and F is upper semicontinuous and convex valued. If F is naturally quasiconvex then it is convexlike.*

3. Alternative Theorems for Some Set-Valued Maps

この章では, 2つの二者択一の定理を示す. これらの定理は, 最適化問題を解く上で, 非常に重要であり, この論文の主定理の主な道具である. まず, 最初の二者択一の定理で使われる条件を述べる.

- (A1) $Q \neq \emptyset$;
- (A2) Q is open;
- (A3) F is convexlike,

where $Q \equiv \{y \in Y | \langle y^*, y \rangle > 0, \forall y^* \in P^+ \setminus \{\theta_{Y^*}\}\}$.

Remark 3.1. *It is easy to show that $\text{int}P \subset Q$, and if $\text{int}P \neq \emptyset$, $\text{int}P = Q$. Also, assumption (A2) is fulfilled when the function $(y^*, y) \mapsto \langle y^*, y \rangle$ is continuous in $\sigma(Y^*, Y) \times \mathcal{O}_Y$, where $\mathcal{O}_Y$ is the topology of Y . We recall that this continuity is satisfied if Y is a normed space.*

このとき, 次の定理を得る.

Theorem 3.1. Under the assumptions (A1), (A2), and (A3), exactly one of the following statements (i) and (ii) is true :

- (i) there exists $x_0 \in C$ such that $F(x_0) \cap (-Q) \neq \emptyset$;
- (ii) there exists $y_0^* \in P^+ \setminus \{\theta_{Y^*}\}$ such that for any $(x, y) \in \text{Graph}(F)$, $\langle y^*, y \rangle \geq 0$.

Remark 3.2. If F is a vector-valued map and $\text{int}P \neq \emptyset$, then Theorem 3.1 becomes Lemma 2.1 of [2].

次に、2つめの二者択一の定理を述べる。そこで使われる条件を述べるために、まず、集合値写像のある連続性を定義する。

Definition 3.1. A set-valued map $F : C \rightsquigarrow Y$ is said to be **-lower semicontinuous* (*-l.s.c.) at $x \in C$ if for any $y^* \in P^+$, the function $z \mapsto \inf_{y \in F(z)} \langle y^*, y \rangle$ is lower semicontinuous at x . F is said to be **-lower semicontinuous* if and only if it is *-lower semicontinuous at every point of C .

Remark 3.3. Every upper-semicontinuous set-valued map is also *-lower semicontinuous.

- (B1) X is a topological vector space;
- (B2) Y is a locally convex space;
- (B3) P^+ has a w^* -compact convex base D ;
- (B4) F is *-quasiconvex on C ;
- (B5) F is *-lower semicontinuous on C .

Remark 3.4. In (B3), P^+ has a w^* -compact convex base D , means that there exists a w^* -compact convex subset D of Y^* such that $\theta_{Y^*} \notin D$ and $P^+ = \bigcup_{\lambda \geq 0} \lambda D$. Assumption (B3) is satisfied when $\text{int}P \neq \emptyset$, see [3].

このとき、次の定理を得る。

Theorem 3.2. Under the assumptions (B1), (B2), (B3), (B4), and (B5), exactly one of the following statements (i) and (ii) is true:

- (i) there exists $x_0 \in C$ such that for any $y^* \in P^+ \setminus \{\theta_{Y^*}\}$, $\inf_{y \in F(x_0)} \langle y^*, y \rangle < 0$;
- (ii) there exists $y_0^* \in P^+ \setminus \{\theta_{Y^*}\}$ such that for any $x \in C$, $\inf_{y \in F(x)} \langle y_0^*, y \rangle \geq 0$.

Remark 3.5. If F is a vector-valued map, then Theorem 3.2 becomes Theorem 2.1 of [3].

4. Applications to Optimization Problem

この章では、最初に与えた問題 (P) に対して、前章における Theorem 3.1, Theorem 3.2 を適用して、その双対問題 (D) との関連を調べていく。まず、1 章で述べた Weak Duality を証明する。

Proof of Proposition 1.1. For each $y^* \in P^+$,

$$\begin{aligned} \text{val}(P) &= \inf_{x \in F^{-1}(-P)} f(x) \\ &\geq \inf_{x \in F^{-1}(-P)} \{f(x) + \langle y^*, y \rangle\} \quad (\forall y \in F(x) \cap (-P)) \\ &\geq \inf_{(x,y) \in \text{Graph}(F)} \{f(x) + \langle y^*, y \rangle\} \\ &= \phi(y^*). \end{aligned}$$

Hence,

$$\text{val}(P) \geq \sup_{y^* \in P^+} \{\phi(y^*)\} = \text{val}(D).$$

This completes the proof. □

次に、主問題 (P) の値が、その双対問題 (D) の値に一致するための条件について考察していく。まず、問題 (P) に対して、拡張された Slater condition を定義する。

(AS) $F^{-1}(-Q) \neq \emptyset$;

(BS) there exists $x_0 \in C$ such that for any $y^* \in P^+ \setminus \{\theta_{Y^*}\}$, $\inf_{y \in F(x_0)} \langle y^*, y \rangle < 0$.

Remark 4.1. If F is a vector-valued map, then condition (BS) becomes the generalized Slater condition in [3]. Moreover $\text{int}P \neq \emptyset$, then condition (AS) becomes the Slater condition in [2].

条件 (AS) と (BS) の間には、次のような関係がある。

Proposition 4.1. For each problem (P),

- (i) if (AS) is satisfied, then (BS) is also satisfied;
- (ii) if (BS) is satisfied and for each $x \in C$, $F(x) + P$ is closed convex, then (AS) is also satisfied;
- (iii) if conditions (BS), (A1), (A2), and (A3) are satisfied, then (AS) is also satisfied;

また、次のように条件 (A3'), (B4'), (B5'), を置き直す。

(A3') (f, F) is convexlike;

(B4') (f, F) is $*$ -quasiconvex on C ;

(B5') (f, F) is $*$ -lower semicontinuous on C ,

where (f, F) is the set-valued map from C to $\mathbf{R} \times Y$ defined by $(f, F)(x) \equiv (\{f(x)\}, F(x))$ for each $x \in C$. In this case, we consider $\mathbf{R}_+ \times P$ as the convex cone in **(A3')**, and $(\mathbf{R}_+ \times P)^+ = \mathbf{R}_+ \times P^+$ as the positive polar cone in **(B4')** and **(B5')**.

さらに、次の条件 **(B6)** を定義する。

(B6) $F(x) + P$ is closed convex for any $x \in C$.

Remark 4.2. From Theorem 2.1, we have the following: under assumption **(B6)**, condition **(B4')** holds if and only if (f, F) is naturally quasiconvex on C .

このとき、Theorem 3.1, Theorem 3.2 より、次の主定理を得る。

Theorem 4.1. For problem (P), assume that one of the following assumptions:

- (i) **(AS)**, **(A1)**, **(A2)**, and **(A3')** are satisfied;
- (ii) **(BS)**, **(B1)**, **(B2)**, **(B3)**, **(B4')**, **(B5')**, and **(B6)** are satisfied.

Then $\text{val}(P) = \text{val}(D)$, and there exists $y_0^* \in P^+$ such that $\phi(y_0^*) = \text{val}(D)$. Moreover, if there exists $x_0 \in C$ such that $\text{val}(P) = f(x_0)$ and $x_0 \in F^{-1}(-P)$, then $\langle y_0^*, y \rangle = 0$ for all $y \in F(x_0) \cap (-P)$.

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