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# Atoms and Molecules on Riemannian Symmetric Spaces

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In this announcement we shall describe a relation between atoms and molecules on a non-compact Riemannian symmetric space G/K, and consider a multiplier operator on the atomic Hardy space. This is continuous a line of study in [7]. The details will appear elsewhere.

§1. Introduction. Before to state the aim, we shall recall some results on the theory of Hardy space $H^p(\mathbf{R})$ $(0<p\leq\infty)$ on one dimensional Euclidean space $\mathbf{R}$. The classical Hardy space is the space of analytic functions f on the upper half plane $\{(x,t);x\in\mathbf{R},\ t>0\}$ with finite $H^p$-norm:

$$\|f\|_{H^p}=\sup_{t>0}\left(\int_{-\infty}^{+\infty}|f(x,t)|^p dx\right)^{1/p}<\infty. \tag{1.1}$$

Moreover, taking the limit as $t\to+0$, this space is identified with the subspace of $\mathcal{S}'(\mathbf{R})$ consisting of boundary distributions f(x,0). In this definition the concept of "analytic functions" is necessary. However, new characterizations of $H^p(\mathbf{R})$ are recently obtained without using the concept of analytic functions. That is, D.L.Burkholder-R.F.Gundy-M.L.Silverstein and C.Fefferman-E.M.Stein showed that $H^p(\mathbf{R})$ is chracterized by <u>the tangential maximal functions</u>:

$$f^*(x)=\sup_{(y,t)\in\Gamma(x)}|f(y,t)|, \tag{1.2}$$

where $\Gamma(x)=\{(y,t);\ y\in\mathbf{R},\ t>0,\ |x-y|<t\}$. They obtained the following

<u>Theorem A</u> ([1],[5]). $c_p\|f\|_{H^p}\le\|f^*\|_{L^p}\le C_p\|f\|_{H^p}$.

Moreover, R.Coifman showed that $H^p(\mathbf{R})$ $(0<p\le 1)$ can be characterized in terms of "atoms". Let $(p,q,s)$ be a triplet such that $0<p\le 1$, $1<q\le\infty$ and $s\in\mathbf{N}$, $s\ge[1/p-1]$. Then a $(p,q,s)$-<u>atom</u> is a mesurable function on $\mathbf{R}$ such that the support is contained in an interval $I$ and satisfies the following two conditions:

$$\begin{aligned}&\text{(i)}\quad \|f\|_q\le|I|^{1/q-1/p}\\ &\text{(ii)}\quad \int_{\mathbf{R}} f(t)t^k dt=0\ \ (0\le k\le s).\end{aligned}\tag{1.3}$$

Then <u>the atomic Hardy space</u> $H^p_{q,s}(\mathbf{R})$ is the space consisting of distributions of the form

$$f=\sum_{i=1}^{\infty}\lambda_i f_i, \tag{1.4}$$

where $f_i$'s are $(p,q,s)$-atoms and $\lambda_i\ge 0$, $\Sigma\lambda_i^p<\infty$. He obtained

<u>Theorem B</u> ([2]). $H^p(\mathbf{R})=H^p_{q,s}(\mathbf{R})$ and $c_p\|f\|^p_{H^p}\le\rho^p_{q,s}(f)\le C_p\|f\|^p_{H^p}$, where $\rho^p_{q,s}(f)$ is defined by the infimum of $\Sigma\lambda_i^p$ being taken over all decompositions (1.4).

Here let us define molecules corresponding to atoms. For a quartet $(p,q,s,\varepsilon)$ such that $(p,q,s)$ is as above and $\varepsilon>\max(s,1/p-1)$, we put $a=1-1/p+\varepsilon$ and $b=1-1/q+\varepsilon$. Then a $(p,q,s,\varepsilon)$-

molecule centerd at $x_0$ is a function f on $\mathbf{R}$ such that f, $f|x|^b$ belong to $L^q(\mathbf{R})$ and satisfies the following two conditions:

$$\begin{aligned} &\text{(i)} \quad \|f\|_q^{a/b}\,\|f|x-x_0|^b\|_q^{1-a/b}=M(f) < \infty, \\ &\text{(ii)} \quad \int_{\mathbf{R}} f(x)x^k dx=0 \quad (0\le k\le s). \end{aligned} \tag{1.5}$$

Then M.H.Taibleson-G.Weiss showed the following

Theorem C ([10]).

(i) If f is a (p,q,s)-atom, then f is a (p,q,s,$\varepsilon$)-molecule for all $\varepsilon>0$ and $M(f)\le C$, where C is independent of the atom.

(ii) If f is a (p,q,s,$\varepsilon$)-molecule, then $f\in H^p_{q,s}(\mathbf{R})$ and $\rho^p_{q,s}(f)\le C'M(f)$, where C' is independent of the molecule.

By many people, these concepts: maximal functions, atoms and molecules on $\mathbf{R}$ were extended to $\mathbf{R}^n$ and moreover, to the general setting of spaces of homogeneous type (cf. [3],[6],[8]). But, our aim in this note is to extend these concepts to non-compact symmetric spaces G/K, which are not of homogeneous type. In §2, we shall give some notations about G, and in §3, define "radial maximal functions" and "atoms" on G/K and obtain a relation between them. In §4, we shall introduce "molecules" on G/K and obtain a theorem corresponding to Theorem C in $\mathbf{R}$. Next we shall constract an atomic Hardy space by using the K-biinvariant, (p,q,s)-atoms on G centered at the unit element of G, and in §5, give a slightly simple characterization of this spce. In last §6, we shall consider convolution (or multiplier) operators on it.

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§2. Notations. Let $G$ be a connected, real rank one semisimple Lie group with finite center, $G=KAN$ an Iwasawa decomposition of $G$ and $\underline{g}=\underline{k}+\underline{a}+\underline{n}$ the corresponding decomposition of the Lie algebra $\underline{g}$ of $G$. For any real vector space $V$ let $V_c$ and $V^*$ denote the complexification and the dual space of $V$ respectively. Let $\alpha$ be a reduced simple root of $(\underline{g}_c, \underline{a}_c)$ and $H_0$ the element of $\underline{a}$ such that $\alpha(H_0)=1$. In the following we identify $A$ (resp. $\underline{a}_c^*$) with $\mathbf{R}$ by $a_t=\exp(tH_0) \leftrightarrow \alpha(\log(a_t))=t$ (resp. $\lambda \leftrightarrow \lambda(H_0)$) and moreover, by using the Cartan decomposition $G=KCL(A^+)K$ of $G$, we identify each $K$-biinvariant function $f$ on $G$ with the even function on $\mathbf{R}$ defined by, which we denote by the same letter, $f(t(x))=f(a_{t(x)})=f(x)$ for $x=k_1 a_{t(x)} k_2 \in KCL(A^+)K$. Let $m_1$ and $m_2$ denote the multiplicities of the root $\alpha$ and $2\alpha$ respectively and put $\rho=(m_1+2m_2)/2$. Then for any $K$-biinvariant functions $f$ on $G$ with compact support its integral on $G$ is given by the integral on $\mathbf{R}^+$ with weight $\Delta(t)=(\mathrm{sh}t)^{m_1}(\mathrm{sh}2t)^{m_2}$:

$$\int_G f(x)\,dx = \int_0^\infty f(t)\Delta(t)\,dt. \tag{2.1}$$

Let $B(r,x)$ denote the open ball with radius $r$ and centerd at $x$ and $|B(r,x)|$ the volume of it, i.e. $B(r,x)=\{y\in G;\ \sigma(x^{-1}y)<r\}$, where $\sigma(x)$ is the Riemannian distance between $x$ and the unit element $e$ of $G$, and $|B(r,x)|=\int_{B(r,x)} 1dg = \int_0^r \Delta(t)\,dt$. For simplicity we put $B(r)=B(r,e)$. Then the order of $|B(r)|$ with respect to $r$ is given by

$$B(r) = \begin{cases} O(e^{2\rho r}) & (r\to\infty) \\ O(r^{m_1+m_2+1}) & (r\to 0). \end{cases} \tag{2.2}$$

This property means that $G$ is not of homogeneous type in the sence of [3].

§3. Maximal functions and atoms on G/K. First we shall define maximal functions on G/K (see [7,§3]). Let $\phi$ be a K-biinvariant function on G with finite $L^1$-norm. Then for a positive number $\varepsilon>0$, we put

$$\phi_\varepsilon(x)=\frac{1}{\varepsilon}\,\frac{\Delta(t(x)/\varepsilon)}{\Delta(t(x))}\;\phi(t(x)/\varepsilon). \tag{3.1}$$

Now for any locally integrable functions f on G/K, we define the radial maximal function $M_\phi f$ of f as follows.

$$M_\phi f(x)=\sup_{\varepsilon>0}|f*\phi_\varepsilon(x)|, \tag{3.2}$$

where $*$ is the convolution on G. Then the following theorem is valid (see [7, Theroem 3.3]).

Theorem 3.1. If there exist a constant C and a positive number $\delta>0$ such that $|\phi(x)|\le Ce^{-2\rho\sigma(x)/\delta}$ $(x\in G)$, the operator $M_\phi$ is of type $(L^p,L^p)$ $(1<p\le\infty)$ and of weak type $(L^1,L^1)$.

Next we shall define atoms on G/K (see [7,§4]). Let (p,q,s) be a triplet such that $0<p\le 1$, $2(\alpha+1)/3<q\le\infty$ and $s\in\mathbf{N}$, $s\ge[2(\alpha+1)/(1/p-1)]$, where $\alpha$ and $\beta$ are defined by $m_1=2(\alpha-\beta)$ and $m_2=2\beta$. Then we say that a function f on G/K is a (p,q,s)-atom centered at x if the support is contained in an open ball B(r,x) and satisfies the following conditions:

$$\begin{aligned}
&\text{(i)} && \|f\|_q \le |B(r)|^{1/q-1/p},\\
&\text{(ii)} && \text{if } r<r_p=(\alpha+1)p/\rho(1-p), \text{ then}\\
& && \int_0^\infty f_{x,K}(t)t^k\Delta(t)\,dt=0 \quad (0\le k\le s),
\end{aligned} \tag{3.3}$$

where $f_{x,K}$ is the K-biinvariant function on G defined by $f_{x,K}(g)=$

$\int_K f(xkg)dk$. Of course, if we put $\alpha=\beta=-1/2$, i.e., $\Delta=1$ and $\rho=0$, this definition of atoms on G/K coincides with one on $\mathbf{R}$. If f satisfies the condition (ii) of (3.3), we say that f has vanishing moments. Here we define the modified radial maximal function $M'_\phi f$ for $f\in L^q(G/K)$ $(1\leq q\leq\infty)$ as follows.

$$M'_\phi f(x)=\sup_{0<\varepsilon<\varepsilon_p} |f*\phi_\varepsilon(x)|, \tag{3.4}$$

where $\varepsilon_p=(1-1/\delta)/(1-1/p)$ if $|\phi(t)|\leq Ce^{-2\rho|t|/\delta}$. Then the following theorem was obtained in [7,Theroem 4.1].

Theroem 3.2. Let $G\neq SL(2,\mathbf{R})$ and (p,q,s) be as above. If there exist a constant C, $0<\delta<1$ and $\lambda>1/p$ $(0<p<1)$ such that $|((\frac{d}{dt})^\ell\phi(t))(1+|t|)^\ell|\leq Ce^{-2\rho|t|/\delta}(1+|t|)^{-\lambda}$ for all $0\leq\ell\leq s+1$, then there exists a constant $c=c(C,p,q,s,\delta,\lambda)$ such that $\|M'_\phi f_{x,K}\|_p\leq c$ for all (p,q,s)-atoms f on G/K.

§4. Molecules on G/K. In the following we shall restrict our attension to K-biinvariant functions on G. Then the natural extension to G/K of the definition of molecules centered at 0 in $\mathbf{R}$ is given as follows. Let $(p,q,s,\varepsilon)$ be a quartet such that (p,q,s) is as above , $\varepsilon>1/p-1$ and put $a=1-1/p+\varepsilon$, $b=1-1/q+\varepsilon$. Let B(x) denote the K-biinvariant function on G defined by $B(x)=|B(\sigma(x))|$. Then we say that a function f is a K-biinvariant, $(p,q,s,\varepsilon)$-molecule centered at e if it satisfies the following two conditions:

$$\text{(i)}\quad \|f\|_q^{a/b}\|fB^b\|_q^{1-a/b}=M(f)<\infty,$$

$$\text{(ii)}\quad \int_0^\infty f(t)t^k\Delta(t)dt=0\ (0\leq k\leq s) \tag{4.1}$$

$$\text{or}\ \|f\|_q\leq|B(r_p)|^{a-b}.$$

Of course, if we put $\alpha=\beta=-1/2$, this definition coincides with one of $(p,q,s,\varepsilon)$-molecules centered at 0 in **R**. Then we can obtain the following

Theorem 4.1.

(i) If $f$ is a K-biinvariant, $(p,q,s)$-atom centered at e, then $f$ is a K-biinvariant, molecule centered at e for all $\varepsilon>0$ and $M(f)\leq C$, where C is independent of the atom.

(ii) If $f$ is a K-biinvariant, $(p,q,s,\varepsilon)$-molecule centered at e with vanishing moments, then $f$ has an atomic decomposition $f=\Sigma\lambda_i f_i$ such that $f_i$'s are K-biinvariant, $(p,q,s)$-atoms centered at e with vanishing moments and $\lambda_i\geq 0$, $(\Sigma\lambda_i^p)^{1/p}\leq C'M(f)(1+N(f))^s$, where C' is independent of the molecule and $N(f)$ is defined by $\|f\|_q=M(f)|B(N(f))|^{a-b}$.

Sketch of the proof: As in **R**, (i) is obvious from the definition. To prove (ii), without loss of generality, we may assume that $M(f)=1$. We define the number $N=N(f)$ by $\|f\|_q=|B(N)|^{a-b}$ and $k_0$ by the smallest integer such that $2^{k_0}N\geq 1$. Then we put

$$(4.2)\qquad G=\bigcup_{k=0}^{\infty} B_k,\quad B_k=\begin{cases} B(0,N) & (k=0)\\ B(2^{k-1}N,2^kN) & (0<k\leq k_0)\\ B(N_0+k-k_0-1,N_0+k-k_0) & (k_0<k),\end{cases}$$

where $B(r,r')=B(r)_c\cap B(r')$ and $N_0=2^{k_0}N$. Let $f_k$ denote the restriction of $f$ to $B_k$. Obviously, $f=\Sigma f_k$ and $f_k$'s are K-biinvariant functions on G. To obtain the desired decomposition, we modify this to the desired one as in **R** (see [10,Theorem 2.9]). In this step we use the following lemma.

Lemma 4.2. For each k, there exist K-biinvariant functions $h_k^i$ $(0\leq i\leq s)$ satisfying the following conditions:

(i) $\mathrm{supp}(h_k^i) \subset B_k$ $(0\leq i\leq s)$,

(ii) $\int_0^\infty h_k^i(t)t^j\Delta(t)dt=\delta_{ij}$ $(0\leq i,j\leq s)$,

(iii)

$$\|h_k^i\|_\infty \leq C \begin{cases} N^{-(i+2\alpha+2)} & (k=0,\ N<1) \\ N^{s-i}|B(N)|^{-1} & (k=0,\ N\geq 1) \\ (2^{k-1}N)^{-(i+2\alpha+2)} & (0<k\leq k_0) \\ N_0^{s-i}(k-k_0)^{s-i}|B(N_0+k-k_0+1)|^{-1} & (k_0<k). \end{cases}$$

Remark 1. If we use the decomposition of G such that $G=\bigcup_{k=0}^\infty B_k'$, $B_k'=B(2^{k-1}N,2^kN)$ instead of (4.2) (this corresponds to the case of **R**), we have an atomic decomposition $f=\Sigma\lambda_i f_i$ such that $(\Sigma\lambda_i^p)^{1/p} \leq C'M(f)e^{2\rho N(f)}$.

Remark 2. When f is a K-biinvariant $(p,q,s,\varepsilon)$-molecule $(0<p<1)$ which satisfies the latter condition of (ii) in (4.1), the similar result is valid. In this case f has an atomic decomposition consisting of atoms which satisfy (i) in (3.3) only.

§5. Atomic Hardy space on G/K. Let (p,q,s) be as above. Now let $L_+^p=L_+^p(G//K)$ denote the space of all K-biinvariant functions f on G having a non-increasing, K-biinvariant function $f^+\varepsilon L^p(G)$ such that $|f|\leq f^+$. We call such a $f^+$ the $L^p$ non-increasing dominator ($L^p$ n.i.d.). In this section we shall consider the following three spaces:

$$^\circ L_+^p=\{f\varepsilon L_+^p;\ f \text{ has a } L^p \text{ n.i.d. } f^+ \text{ such that } |B(r)|^{-1}\int_{B(r)_c} f(x)dx \leq f^+(r)\}. \qquad (5.1)$$

$H^p=\{f\varepsilon L^1_{loc}(G//K)\,;\ M'_\phi f\varepsilon L^p_+$ for all $\phi$ satisfying the condition in Theorem 3.2$\}$.

$H^p_{q,s}=\{f=\Sigma\lambda_i f_i$; all $f_i$'s are K-biinvariant, $(p,q,s)$-atoms centered at e, $\lambda_i\geq 0$ and $\Sigma\lambda_i^p<\infty\}$.

Then we put $\rho^p_+(f)=\inf_{f^+}\|f^+\|^p_p$ for $f\varepsilon{}^\circ L^p_+$, where the infimum being taken over all $L^p$ n.i.d. $f^+$ of f satisfying (5.1), $\rho^p(f)=\sup_\phi\inf\|(M'f)^+\|^p_p$ for $f\varepsilon H^p$, where the supremum (resp. the infimum) being taken over all $\phi$ satisfying the condition in Theorem 3.2. with C=1 (resp. all $L^p$ n.i.d. of M'f) and $\rho^p_{q,s}(f)=\inf\Sigma\lambda^p_i$ for f $\varepsilon H^p_{q,s}$, where the infimum being taken over all K-biinvariant (p,q,s) -atomic decompositions of f. Obviously, $H^p_{q,s}\subset H^p_{q',s}$ $(q\geq q')$. The following proposition was obtained in [7,Proposition 5.1].

Proposition 5.1. $H^p_{q,s}\subset H^p$.

Moreover we can prove

Theorem 5.2. $H^p_{\infty,0}={}^\circ L^p_+$ and $\rho^p_{\infty,0}\sim\rho^p_+$.

Sketch of the proof: Let f be in $H^p_{\infty,0}$. Then f has an atomic decomposition $f=\Sigma\lambda_i f_i$ such that all $f_i$'s are K-biinvariant, $(p,\infty,0)$ -atoms centered at e. That is, $\mathrm{supp}(f_i)\subset B(r_i)$ and $\|f_i\|_\infty\leq |B(r_i)|^{-1/p}$. Therefore, $|f|\leq\Sigma\lambda_i|f_i|\leq\sum_{\sigma(x)\leq r_i}\lambda_i|B(r_i)|^{-1/p}$. Here we define $f^+$ by the right hand side. Then we can show that $f^+$ is a $L^p$ n.i.d. of f satisfying the condition (5.1). To prove the converse, we use Theorem 4.1 (ii) and the similar argument in the proof of the theorem.

Corollary 5.3. $H^p_{\infty,0}$ is complete.

Conjecture. $H^p_{\infty,s} = {}^\circ L^p_+ = H^p$.

Remark. As in $\mathbf{R}$, if the integral: $|B(r)|^{-1} \int_{B(r)_c} f(x)dx$ can be expressed suitably in terms of the convolutions on G and be bounded by the maximal functions of f, this conjecture is valid.

§6. Multiplier operators on $H^p_{q,s}$. In this section we shall consider convolution (or multiplier) operators on $H^p_{q,s}$. First, as in $\mathbf{R}$, we see that

Proposition 6.1. If a linear operator T maps each K-biinvariant, (p,q,s)-atoms centered at e into a K-biinvariant, (p,q,s)-molecule T(f) centered at e and $M(T(f)) \le C$, where C is independent of the atom f, then T is a bounded operator on $H^p_{q,s}$.

By using this proposition we can obtain the following results. For a K-biinvariant function f on G (resp. an even function $\mu$ on $\underline{a}^*$) with a suitable condition, the Spherical Fourier transform $\hat{f}$ of f (resp. the inverse Fourier transform $\check{\mu}$ of $\mu$) is defined as follows (cf. [11,Chap.9.2]).

$$\hat{f}(\nu) = \int_G f(x)\phi_\nu(x)dx$$
$$(\text{resp. } \check{\mu}(x) = \int_{\underline{a}^*} \mu(\nu)\phi_\nu(x)|C(\nu)|^{-2}d\nu).$$

Now we put $F(\xi) = \{\nu \in \underline{a}^*_c;\ |Im(\xi)| < \xi\rho\}$. Then we have the following

Theorem 6.1. Suppose that $\mu$ is an even function on $\underline{a}^*$ such that $\mu$ is bounded and holomorphic on $F(\xi)$ $(\xi > 2/p-1)$ and $\mu(\nu)(1+|\nu|)^{1-[p]}C(-\nu)^{-1} \in L^1(\mathbf{R}+\sqrt{-1}\xi\rho)$. Then if the multiplier operator $T_\mu$, i.e., $T_\mu(f) = (\mu\hat{f})^\vee$, is of type $(L^\infty, L^\infty)$, $T_\mu$ is also of type $(H^p_{\infty,0}, H^p_{\infty,0})$ for $2\alpha+2/2\alpha+3 \le p \le 1$.

Moreover, using this theorem and Corollary 5.3, we can obtain

*Corollary 6.3.* *Suppose that* $m$ *is a K-biinvariant function on* $G$ *with finite* $L^1$*-norm and* $\hat{m}(\nu)\ C(-\nu)^{-1} \in L^1(\mathbf{R}+\sqrt{-1}\rho)$. *Then the convolution operator* $T_m$, i.e., $T_m(f)=m*f$, *is of type* $(H^1_{\infty,0}, H^1_{\infty,0})$.

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