

# Dynamical systems of Certain non-holomorphic maps

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## 1 Introduction

Complex dynamical systems given by quadratic polynomials

$$f_c(z) = z^2 + c$$

have many well-known important analytic and geometric properties. When  $c = -2$ ,  $f_{-2}(z)$  has relation to a Chebyshev polynomial. The Julia set of  $f_{-2}(z)$  is the interval  $[-2, 2]$ . For any  $z$  in  $[-2, 2]$ , we set

$$z = \psi(\theta) = e^{\pi\theta i} + e^{-\pi\theta i},$$

where  $\psi$  is a function from  $\mathbf{R}$  to  $[-2, 2]$ . Clearly

$$\psi(\theta) = \psi(\theta + 2) = \psi(2 - \theta).$$

Then we introduce an equivalence relation  $\sim$  generated by  $\theta \sim \theta + 2$ ,  $\theta \sim 2 - \theta$ . Hence we have an induced map

$$\psi : \mathbf{R}/\sim \rightarrow [-2, 2].$$

Let  $t$  be a transformation on  $\mathbf{R}/\sim$  defined by  $t(\theta) = 2\theta$ . Then two dynamical systems  $\{[-2, 2], f_{-2}\}$  and  $\{\mathbf{R}/\sim, t\}$  are topological conjugate and the induced map  $\psi$  is a topological conjugacy.  $\{\mathbf{R}/\sim, t\}$  is equivalent to the dynamical system given by a tent map on  $[0, 1]$ .

Two-dimensional extension of these dynamical systems is considered. Let

$$z = \Psi(\sigma, \tau) = e^{2\pi\sigma i} + e^{-2\pi\tau i} + e^{2\pi(\tau-\sigma)i}.$$

Then  $\Psi$  maps  $\mathbf{R}^2$  to a subset  $S$  in the complex plane  $\mathbf{C}$ . Clearly

$$\Psi(\sigma, \tau) = \Psi(\sigma + 1, \tau) = \Psi(\sigma, \tau + 1).$$

Then  $\Psi$  may be considered as a map from two-dimensional torus  $T^2$  to  $S$ . Further

$$\Psi(\sigma, \tau) = \Psi(\sigma, \sigma - \tau) = \Psi(-\sigma + \tau, \tau) = \Psi(1 - \tau, 1 - \sigma).$$

Then we introduce an equivalence relation  $\sim$  on  $T^2$  defined by

$$(\sigma, \tau) \sim (\sigma, \sigma - \tau) \sim (-\sigma + \tau, \tau) \sim (1 - \tau, 1 - \sigma). \quad (1.1)$$

Hence we have an induced map

$$\Psi : T^2 / \sim \rightarrow S.$$

An extension of  $f_{-2}(z)$  is a map :

$$F_2(z) = z^2 - 2\bar{z}.$$

Then

$$F_2(\Psi(\sigma, \tau)) = \Psi(2\sigma, 2\tau).$$

Let  $d$  be a double angle map of  $T^2 / \sim$  onto itself. That is,

$$d(\sigma, \tau) = (2\sigma, 2\tau).$$

Then  $\{S, F_2\}$  and  $\{T^2 / \sim, d\}$  are topological conjugate. Hence  $F_2(z)$  is a two-dimension extension of  $f_{-2}(z)$ . Uchimura [1996] shows that  $\{S, F_2\}$  is chaotic.

## 2 The dynamics of $F_2$

We study the dynamical system given by

$$F_2(z) = z^2 - 2\bar{z}. \quad (2.1)$$

We shall show that  $F_2$  partition the complex plane  $\mathbb{C}$  into two sets. One is a closed domain  $S$  and the other is its complement. The dynamics of  $F_2$  on  $S$  is chaotic and the complement of  $S$  is the basin of  $\infty$  of  $F_2$ .

Since  $F_2(z)$  is not holomorphic, we regard the function  $F_2$  as a function of two real variables. That is,

$$F_2((x, y)) = (x^2 - y^2 - 2x, 2xy + 2y). \quad (2.2)$$

Here  $z = x + iy$ . The Jacobian determinant of  $F_2$  is  $4x^2 + 4y^2 - 4$ . So the set of critical values is an algebraic curve of the fourth degree which is known as Steiner's hypocycloid. It takes the form

$$(x^2 + y^2 + 9)^2 + 8(-x^3 + 3xy^2) - 108 = 0. \quad (2.3)$$

Let  $S$  be a closed region bounded by Steiner's hypocycloid (2.3). See [Koornwinder, 1974].

We shall show that the function  $F_2(z)$  restricted to the set  $S$  is a finite-to-one factor of the algebraic endomorphism of the torus given by the matrix  $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ . We set

$$z = \exp(2\pi\sigma i) + \exp(-2\pi\tau i) + \exp(2\pi(\tau - \sigma)i). \quad (2.4)$$

Then the point  $(\sigma, \tau)$  is regarded as a point in the torus  $T \simeq \mathbb{R}^2/\mathbb{Z}^2$ . From Then, we have a torus endomorphism  $l$  on  $T$  given by  $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ . From (2.4), we know that the point  $(\sigma, \tau)$  is equivalent to the points  $(-\sigma + \tau, \tau)$ ,  $(\sigma, \sigma - \tau)$  and  $(1 - \tau, 1 - \sigma)$ . That is to say, we have to consider an equivalence relation given by

$$(\sigma, \tau) \sim (-\sigma + \tau, \tau), (\sigma, \tau) \sim (\sigma, \sigma - \tau), (\sigma, \tau) \sim (1 - \tau, 1 - \sigma). \quad (2.5)$$

Let  $R = T/\sim$ .

The set  $R$  is regarded as a closed region bounded by a regular triangle  $OAB$  with  $O = (0, 0)$ ,  $A = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ ,  $B = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$ , in the  $(s, t)$ -plane, where

$$s = (\sigma + \tau)/2, t = \sqrt{3}(\sigma - \tau)/2.$$

Now we consider the symbolic dynamics. For any point  $x \in R$ , we define the itinerary  $h(x)$  by the rule

$$h(x) = (s_0 s_1 s_2 \dots)$$

where  $s_j = k$  if and only if  $d^j(x) \in \Delta(k)$ , ( $k = 0, 1, 2, 3$ ). Here  $\Delta(0) = \triangle DEF$ ,  $\Delta(1) = \triangle OEF$ ,  $\Delta(2) = \triangle ADF$ ,  $\Delta(3) = \triangle BED$ , where  $D, E$  and  $F$  are the midpoints of  $AB, OB$  and  $OA$ , respectively. Set

$$\Sigma_4 = \{(s_0 s_1 s_2 \dots) \mid s_j = 0, 1, 2 \text{ or } 3\}.$$

We introduce an equivalence relation  $\sim$  on  $\Sigma_4$  which is defined by

$$\begin{aligned} (w_1 0 w_2) &\sim (w_1 2 w_2), \text{ for } w_1 \in \{0, 1, 2, 3\}^*, (w_2) \in h(OA), \\ (w_1 0 w_2) &\sim (w_1 3 w_2), \text{ for } w_1 \in \{0, 1, 2, 3\}^*, (w_2) \in h(OB), \\ (w_1 0 w_2) &\sim (w_1 1 w_2), \text{ for } w_1 \in \{0, 1, 2, 3\}^*, (w_2) \in h(AB). \end{aligned}$$

Let  $\Sigma'$  denote the quotient  $\Sigma_4/\sim$ . Note that  $\sigma$  is naturally defined on  $\Sigma'$ . That is,

$$\sigma[(s_0 s_1 \dots s_n \dots)] = [(s_1 \dots s_n \dots)].$$

Clearly  $\sigma$  is well-defined.

**Theorem 2.1.** The mapping  $\tilde{h} : R \rightarrow \Sigma'$  is a bijection and it makes the following diagram commutative:

$$\begin{array}{ccc} R & \xrightarrow{d} & R \\ \tilde{h} \downarrow & & \downarrow \tilde{h} \\ \Sigma' & \xrightarrow{\sigma} & \Sigma' \end{array}$$

Let  $\Gamma$  be a Sierpinsky gasket whose outermost triangle is  $OAB$ . Then it can be easily seen that

$$d(\Gamma) = \Gamma.$$

Let

$$\Sigma'' = \{[(s_0 s_1 \dots s_n \dots)] \mid s_i \in \{1, 2, 3\}, \text{ for all } i\}.$$

The shift map  $\sigma$  is naturally defined on  $\Sigma''$ . Hence we have the following corollary.

**Corollary 2.1.** The mapping  $\tilde{h} : \Gamma \rightarrow \Sigma''$  is a bijection and it makes the following diagram commutative :

$$\begin{array}{ccc} \Gamma & \xrightarrow{d} & \Gamma \\ \tilde{h} \downarrow & & \downarrow \tilde{h} \\ \Sigma'' & \xrightarrow{\sigma} & \Sigma'' \end{array}$$

**Theorem 2.2.** The dynamical system  $\{S, F_2\}$  is chaotic. That is,

- (1) it is transitive;
- (2) it is sensitive to initial conditions;
- (3) the set of periodic orbits of  $F_2$  is dense in  $S$ .

For any complex number  $z \notin S$ , we have the following theorem.

**Theorem 2.3** For any complex number  $z \notin S$ ,

$$(F_2)^n(z) \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

### 3 Attracting basins of $F_{\frac{4}{3}}$

In this section we consider perturbations of the mapping  $F_2(z)$ . We study the dynamical systems given by

$$F_c(z) = z^2 - c\bar{z}.$$

Such a map is one of the simplest q.c. mappings except for affine mappings. The mapping also satisfies a simple property

$$\frac{\partial^2}{\partial z \partial \bar{z}} F_c^2 = -4cz,$$

for Laplace operator.

First we note that there are four fixed points and 12 periodic points of prime period 2 which are listed in Uchimura[1977].

We study the dynamics of  $F_c(z)$  when  $c = \frac{4}{3}$ . The value  $c = \frac{4}{3}$  is a special value in the sense that all attractive fixed points and attractive periodic points of period 2 lie on a critical set. We will study attractive basins of  $F_{\frac{4}{3}}$ .

When  $c = \frac{4}{3}$ , two attracting fixed points  $P_{(1)}^2$  and  $P_{(1)}^3$  are written as

$$P_{(1)}^2 = -\frac{1}{6} + i\sqrt{\frac{5}{12}}$$

$$P_{(1)}^3 = \overline{P_{(1)}^2}.$$

There are four attracting periodic points of period 2 which are written as

$$P_{(2)}^2 = \omega P_{(1)}^2, P_{(2)}^3 = \omega^2 P_{(1)}^2$$

$$P_{(2)}^4 = \omega P_{(1)}^3, P_{(2)}^5 = \omega^2 P_{(1)}^3.$$

Critical points of  $F_{\frac{4}{3}}(z)$  are the zeros of Jacobian determinant of  $F_{\frac{4}{3}}$ . Then critical set of  $F_{\frac{4}{3}}(z)$  is equal to a circle  $\{z; 2|z| = \frac{4}{3}\}$ . Note that the attracting periodic points  $P_{(1)}^2, P_{(1)}^3, P_{(2)}^2, P_{(2)}^3, P_{(2)}^4, P_{(2)}^5$  lie on the circle. Set

$$A = (\frac{4}{3}, 0), \quad B = (-\frac{2}{3}, \frac{2\sqrt{3}}{3}) \text{ and } C = (-\frac{2}{3}, -\frac{2\sqrt{3}}{3}).$$

**Theorem 3.1.** The interiors of the triangular regions  $\triangle OBD$  and  $\triangle OCF$  are the immediate basins of fixed points  $P_{(1)}^2$  and  $P_{(1)}^3$ . The interiors of  $\triangle OBE$  and  $\triangle OAD$  are immediate basins of periodic points  $P_{(2)}^4$  and  $P_{(2)}^5$ . The interiors of  $\triangle OCE$  and  $\triangle OAF$  are immediate basin of periodic points  $P_{(2)}^2$  and  $P_{(2)}^3$ .

By the symmetry we know that to prove Theorem 3.1 it suffices to prove the following proposition.

**Proposition 3.1.** Let  $z$  be any interior point of  $\triangle OBD$ . Then

$$(F_{\frac{4}{3}})^n(z) \rightarrow P_{(1)}^2 \quad (n \rightarrow \infty).$$

This proposition is proved in Uchimura[1997].

## References

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