

4.

The affine algebraic curve of Newton maps in Przytycki's cubic rational maps

Kiyoko NISHIZAWA

(Dept. Math. Josai Univ.)

Masayo FUJIMURA

(College of Sci. and Tech.

Nihon Univ.)

4.1 Introduction

Przytycki studies a family of cubic rational maps:

$$\mathcal{PC}(a, b, c) = \{f(z) = z^2 + c + \frac{b}{z-a} : a, b, c \in \mathbb{C}\},$$

and defines the exotic map to be a map of $\mathcal{PC}(a, b, c)$ having two super-attracting fixed points and a critical point of period 2. An example of exotic map is given in Przytycki [Prz94], obtained by computer experiment.

Example: $f(z) = z^2 + c + b/(z-a) : c = -3.121092, a = 1.719727, b = 0.3142117$.

He considers 1-parameter families joining exotic examples with Newton maps for degree 3 polynomials. The subfamily with a super-attracting fixed point except of ∞ can be parametrized by two parameters (k, w) as follows:

let w be super-attracting fixed point, and $a = kw$. Then

$$a = kw, \quad b = 2w^3(1-k)^2, \quad c = w^2(2k-3) + w,$$

and

$$\mathcal{PC}(w, k) = \left\{ f(z) = z^2 + w^2(2k - 3) + w + \frac{2w^3(1 - k)^2}{z - kw} \right\}.$$

Four critical points are ∞ , w and

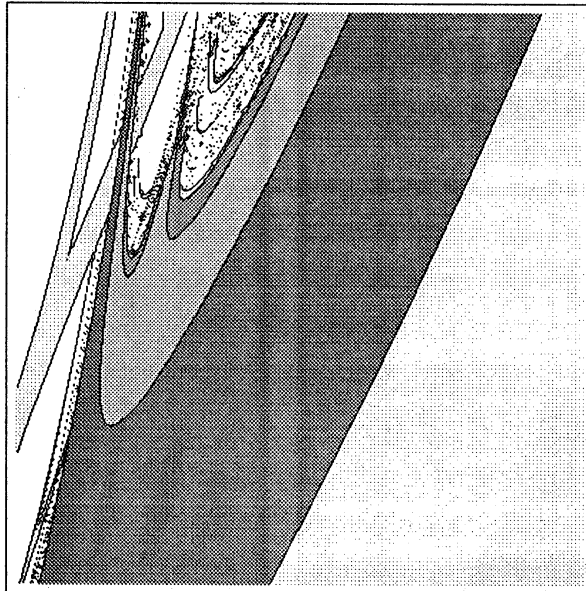
$$u = \frac{w(-1 + 2k - \sqrt{4k - 3})}{2}, \quad v = \frac{w(-1 + 2k + \sqrt{4k - 3})}{2}$$

Przytycki gives two questions in [Prz94]:

Question 1 In the set of Newton maps for the polynomials $P_\lambda = z^3 + (\lambda - 1)z - \lambda$, there exist Mandelbrot-like sets where the free critical point converges to a periodic attracting orbit. For a critical point $v = g(k, w)$, what happens to $\mathcal{M}(v)$ -sections of the set of exotic maps when we change parameters from Newton maps to the exotic ones? : namely, do these sets move to $\mathcal{M}(v)$ (or $\mathcal{M}(u)$)-sections of the set of exotic maps when we change parameters from Newton maps to the exotic ones?

Question 2 Describe precisely how does the dynamics bifurcate for real parameters k, w .

Concerning about these questions, we obtain following results: to the first question, there is an affine algebraic curve consisting of Newton maps for degree 3 polynomials, and to the second one, we partly give an answer. Namely, for any fixed parameter k , we consider the bifurcation of this family as the parameter w varies monotonely. We observe complex bifurcations for $3/4 < k < 1$.



⊠ 1 Bifurcations in real (k, w) -plane

4.2 Cubic Newton Curve

We show that there is an affine algebraic curve consisted by all Newton maps in $\mathcal{PC}(a, b, c)$. We call hereafter this curve **cubic Newton curve**, denoted by $\mathcal{N}(w, k)$.

Proposition 1 *The defining equation of the cubic Newton curve is the following:*

$$\mathcal{N}(w, k) : k^2 w^2 - 6kw^2 + 9w^2 - 2kw - 3w + 1 = 0.$$

The cubic Newton curve is irreducible of genus one with two singular points

$$(0, 0, 1), (0, 1, 0)$$

on $L_\infty : Z = 0$ of $P^2\mathbf{C} : (Z, k, w)$.

Outline of proof:

$$f(k, w, x) = x^2 + w^2(2k - 3) + w + \frac{2w^3(1 - k)^2}{(x - kw)^2},$$

$$f'(k, w, x) = 2x - \frac{2(1 - k)^2 w^3}{(x - kw)^2}.$$

The critical points are ∞ , w , and

$$u = \frac{-w\sqrt{4k - 3} - (1 - 2k)w}{2}, \quad v = \frac{w\sqrt{4k - 3} - (1 - 2k)w}{2}.$$

For f to be a Newton map, we claim $f(k, w, u) = u$, or $f(k, w, v) = v$.

Therefore the equation of the Newton curve is

$$(k^2 w^2 - 6kw^2 + 9w^2 - 2kw - 3w + 1) = 0.$$

Let

$$F(Z, k, w) = Z^4 - 3wZ^3 - 2z^2kw + 9Z^2w^2 - 6Zkw^2 + k^2w^2 = 0,$$

and $P_k : (0, 0, 1)$, $P_w : (0, 1, 0)$. The singular points are P_k , P_w . The principal part of F is $(3Z - k)^2$. Therefore by Plücker's formula, we can calculate the genus one.

We define dynamical curves in the parameter space:

Definition Let $\text{Per}_p(\mu)$ consist of all parameter pairs (k, w) for which the associated cubic rational map $f(k, w, x)$ has a periodic orbit of period p with multiplier $(f^p)'$ equal to μ .

In particular, $\text{Per}_1(0)$ consists of all parameter pairs with a super-attracting fixed point. The real part of the curve $\text{Per}_1(1)$ consists of all parameter pairs for which the graph of f is tangent to the diagonal. Such points of tangency are called **saddle nodes** of period 1. On $\text{Per}_1(-1)$ attracting period one orbits bifurcate into attracting period two orbits.

It is clear that (k, w) of $Per_1(-1)$ belongs to $Per_2(1)$, but it is not known whether inverse inclusion holds. In the quadratic rational maps, we have $Per_2(1) = Per_1(-1)$.

Proposition 2 We obtain defining equations of two dynamical curves:

$$Per_1(1): k^2 w^2 - 6kw^2 + 9w^2 - 2kw - 2w + 1 = 0,$$

$$Per_1(-1): 3k^2 w^2 - 18kw^2 + 27w^2 - 6kw - 14w + 3 = 0.$$

See Figures for plots of these curves in the real case.

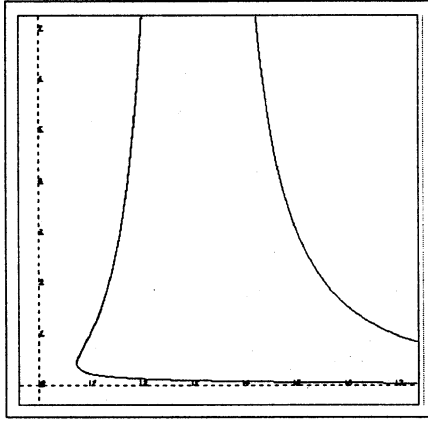


Figure 2 Newton

curve : $(k^2 - 6k + 9)w^2 - (2k + 3)w + 1 = 0$.

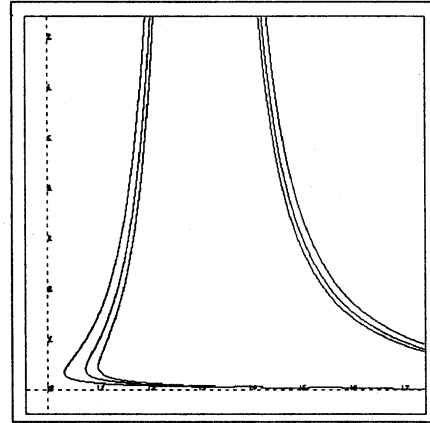


Figure 3 Dynamical curves : $Per_1(1), Per_1(-1)$.

Outline of proof:

The fixed points of $f(k, w, x)$ are ∞, w and

$$x_1 = \frac{(1-k)w - 1 - \sqrt{(k^2 - 6k + 9)w^2 + (-2k - 2)w + 1}}{2},$$

$$x_2 = \frac{-(1-k)w + 1 + \sqrt{(k^2 - 6k + 9)w^2 + (-2k - 2)w + 1}}{2}.$$

From $f'(k, w, x_i) = 1$, (resp. $= -1$) for $i = 1$, or 2 , we obtain the equation of $Per_1(1)$ (resp. $Per_1(-1)$).

4.3 bifurcations

Proposition 3 For the real parameter k , we can roughly divide (k, w) -plane into four distinct classes as follows: (1) $k < 3/4$, (2) $3/4 \leq k < 1$, (3) $1 < k < 3/2$, (4) $3/2 \leq k$.

Outline of Proof:

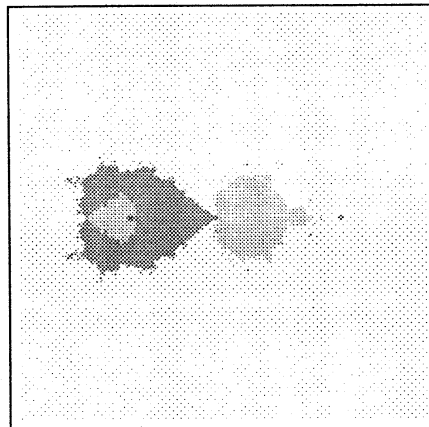
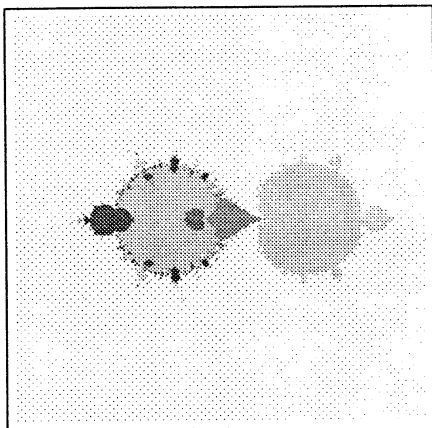


图 4 Bifurcation: $k = 1.15$, $w : (-1.5, -2) - (2.5, 2)$ 图 5 Bifurcation: $k = 1.5$, $w : (-2, -4) - (6, 4)$

- For $k : 3/4 \leq k \leq 1$, the critical points u, v, w except ∞ are ordered as $u < v < a < w$.
- For $k : 1 \leq k < 3/2$, $u < w < a < v$.
- For $k : 3/2 \leq k$, $w < u < a < v$. in case (3) (resp.(4)), the dynamics of u (resp. v) is analyzed as the dynamics of a quadratic map by Douady-Hubbard-Sullivan's theory([D-H85]). In case (2), the dynamical behavior is very complex. In case (1), two critical points u, v are complex numbers, mutually complex conjugate. The dynamical behavior is very complex: namely we can observe "swallow" and "tri-corn" configurations ([Mil90]).

参考文献

- [D-H85] A. Douady & J. Hubbard, Etude dynamique des polynomes complexes I(1984) & II(1985), Publ. Math. Orsay.
- [Mil90] J. Milnor, Remarks on iterated cubic maps. Preprint # 1990/6, SUNY Stony Brook.
- [Prz94] F. Przytycki. Iterations of Rational Function: Which Hyperbolic Components Contain Polynomials ?. Preprint # 1994/6, SUNY Stony Brook, 1994.