

A description of a space of holomorphic discrete series
by means of the Fourier transform on the $\check{S}$ ilov boundary

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1. Let G be a connected non-compact real simple Lie group of matrices and K a maximal compact subgroup of G . Assume that G/K is a hermitian symmetric space. Then, G/K can be realized as a Siegel domain of type II, D , in $W \times V$, where W and V are finite dimensional complex vector spaces. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in the Lie algebra $\mathfrak{k}$ of K , Δ the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. We introduce an order in Δ compatible with the complex structure of G/K . For each K -dominant K -integral linear form Λ on $\mathfrak{h}_{\mathbb{C}}$ satisfying the Harish-Chandra's non-vanishing condition [1], the holomorphic discrete series Π_{Λ} of G with Λ as a parameter is realized on a Hilbert space $H(\Lambda)^{(\dagger)}$ of vector valued holomorphic functions on D . Let $S(D)$ be the $\check{S}$ ilov boundary of D . Then, one knows that $S(D)$ is the orbit of the origin in $W \times V$ under a certain nilpotent subgroup $N(D)$ of the affine automorphisms of D , and that $S(D)$ is diffeomorphic to $N(D)$. By identifying $S(D)$ with $N(D)$, the aim of this talk is a description of the space $H(\Lambda)$ by using the Fourier transform on $N(D)$.

If D reduces to a tube domain, then $N(D)$ is abelian and therefore the Fourier transform on $N(D)$ is euclidean. Since this case is treated

($\dagger$) For precise definition see p.7. 1

in Rossi-Vergne [4], we assume from now on that D does not reduce to a tube domain. Then, $N(D)$ is a simply connected 2-step nilpotent Lie group (in fact $N(D)$ is the nilradical of a maximal parabolic subgroup of G) and the Fourier transform on $N(D)$ is non-euclidean.

2. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with the associated Cartan involution θ and $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) be the sum of all the root subspaces corresponding to positive (resp. negative) non-compact roots in Δ . Both $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras in $\mathfrak{g}_\mathbb{C}$ normalized by $\mathfrak{k}_\mathbb{C}$. Let $P_\pm$ and $K_\mathbb{C}$ be the analytic subgroups of $G_\mathbb{C}$ corresponding to $\mathfrak{p}_\pm$ and $\mathfrak{k}_\mathbb{C}$ respectively, where $G_\mathbb{C}$ is the complexified connected matrix group having $\mathfrak{g}_\mathbb{C}$ as Lie algebra. Every $x \in P_+ K_\mathbb{C} P_-$ can be expressed in a unique way as

$$x = \exp \zeta(x) \cdot k(x) \cdot \exp \zeta'(x)$$

with $\zeta(x) \in \mathfrak{p}_+$, $k(x) \in K_\mathbb{C}$ and $\zeta'(x) \in \mathfrak{p}_-$. We know that G is contained in $P_+ K_\mathbb{C} P_-$ and that the image $M = \zeta(G)$ is a bounded domain in $\mathfrak{p}_+$. M is known as the Harish-Chandra's realization of G/K .

3. Let $\{\gamma_1, \dots, \gamma_\ell\}$ be a maximal system of positive non-compact pairwise strongly orthogonal roots constructed as follows: for each j , γ_j is the largest positive non-compact root strongly orthogonal to $\gamma_{j+1}, \dots, \gamma_\ell$. For every $\alpha \in \Delta$, we choose $H_\alpha \in \mathfrak{h}_\mathbb{C}$ and $X_\alpha \in \mathfrak{g}_\alpha \subset \mathfrak{g}_\mathbb{C}$ in such a way that the following are valid:

$$\begin{aligned} B(H_\alpha, H) &= \alpha(H) \quad (H \in \mathfrak{h}_\mathbb{C}), & X_\alpha - X_{-\alpha} &\in \mathfrak{k} + i\mathfrak{p}, \\ i(X_\alpha + X_{-\alpha}) &\in \mathfrak{k} + i\mathfrak{p}, & [X_\alpha, X_{-\alpha}] &= 2H_\alpha / \alpha(H_\alpha), \end{aligned}$$

where B is the Killing form of $\mathfrak{g}_C$. Note that $H_\alpha \in i\mathfrak{h}$. Put $\mathfrak{a} = \sum_{1 \leq i \leq \ell} R(X_{\gamma_i} + X_{-\gamma_i})$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, so that ℓ equals the real rank of G . Let

$$(1) \quad c = \exp \frac{\pi}{4} \sum_{j=1}^{\ell} (X_{\gamma_j} - X_{-\gamma_j}) \in G_C$$

and $\nu = \text{Ad } c$. It should be noted that c belongs to $P_+ K P_-$. Put $\mathfrak{h}^- = \sum_{1 \leq j \leq \ell} R\mathfrak{h}_{\gamma_j}$ and let $\mathfrak{h}^+$ be the orthogonal complement of $\mathfrak{h}^-$ in $i\mathfrak{h}$ relative to the inner product $-B_\tau|_{i\mathfrak{h} \times i\mathfrak{h}}$, where $B_\tau(X, Y) = B(X, \tau Y)$ and τ is the conjugation in $\mathfrak{g}_C$ with respect to the compact real form $\mathfrak{k} + i\mathfrak{p}$. Since $\mathfrak{h}_C^+ + \mathfrak{a}_C = \nu^{-1}(\mathfrak{h}_C)$, $\mathfrak{h}_C^+ + \mathfrak{a}_C$ is also a Cartan subalgebra of $\mathfrak{g}_C$. As we have assumed that G/K does not reduce to a tube domain, there is only one possibility of the positive $\mathfrak{a}$ -root system $\Phi(\mathfrak{a})^+$ compatible with the original order in Δ through ν^* (see Moore [2]): put $\lambda_j = \nu^*(\gamma_j)$, then

$$\Phi(\mathfrak{a})^+ = \left\{ \frac{\lambda_i + \lambda_j}{2} ; 1 \leq j < i \leq \ell \right\} \cup \left\{ \frac{\lambda_i - \lambda_j}{2} ; 1 \leq j < i \leq \ell \right\} \cup \left\{ \frac{\lambda_i}{2} ; 1 \leq i \leq \ell \right\}.$$

We denote by $\mathfrak{n}$ the sum of all the positive $\mathfrak{a}$ -root subspaces and put $\mathfrak{s} = \mathfrak{a} + \mathfrak{n}$. It is easy to see that the map $r(X) = (X - \theta(X))/2$ is a linear isomorphism of $\mathfrak{s}$ onto $\mathfrak{p}$. Let j be the complex structure on the underlying vector space $\mathfrak{s}$ obtained by transforming the complex structure on $\mathfrak{p}$ by means of r . We set

$$\mathfrak{s}(0) = \mathfrak{a} + \sum_{k < m} \mathfrak{n}_{(\lambda_m - \lambda_k)/2}, \quad \mathfrak{s}(1/2) = \sum_{k=1}^{\ell} \mathfrak{n}_{\lambda_k/2}, \quad \mathfrak{s}(1) = \sum_{k \leq m} \mathfrak{n}_{(\lambda_m + \lambda_k)/2}.$$

Then, clearly we have $\mathfrak{s} = \mathfrak{s}(0) + \mathfrak{s}(1/2) + \mathfrak{s}(1)$ and $\mathfrak{s}(0)$ is a subalgebra of $\mathfrak{g}$. Let $S(0)$ be the analytic subgroup of G corresponding to $\mathfrak{s}(0)$.

Put

$$s = \frac{i}{2} \sum_{k=1}^{\ell} (2H_{\gamma_k} / \langle \gamma_k, \gamma_k \rangle - X_{\gamma_k} + X_{-\gamma_k}).$$

Then, s belongs to $\mathfrak{s}(1)$. Let Ω be the $S(0)$ -orbit of s in $\mathfrak{s}(1)$ under the adjoint representation. By Rossi-Vergne [4, Theorem 4.15], Ω is a regular open convex cone in $\mathfrak{s}(1)$ and diffeomorphic to $S(0)$. For every $t \in \Omega$, we denote by $\eta_0(t)$ the unique element in $S(0)$ for which $(\text{Ad } \eta_0(t))s = t$. On the other hand, it is known that j leaves $\mathfrak{s}(1/2)$ invariant, and so $\mathfrak{s}(1/2)$ can be considered as a complex vector space V by $j|_{\mathfrak{s}(1/2)}$. Let W be the complexification of $\mathfrak{s}(1)$. The $\mathbb{R}$ -bilinear map $Q : V \times V \rightarrow W$ defined by $Q(x, y) = ([jx, y] + i[x, y])/4$ turns out to be an Ω -positive sesqui-linear hermitian map. By using this pair of Ω and Q , we now define a Siegel domain of type II, $D = D(\Omega, Q) :$

$$D = \{(w, v) \in W \times V ; \text{Im } w - Q(v, v) \in \Omega\}.$$

Then, $S(D) = \{(x + iQ(\zeta, \zeta), \zeta) ; x \in \mathfrak{s}(1), \zeta \in V\}$ and $N(D) = \{(x, \zeta) ; x \in \mathfrak{s}(1), \zeta \in V\}$ with the multiplication

$$(x, \zeta) \cdot (x', \zeta') = (x + x' + 2\text{Im } Q(\zeta, \zeta'), \zeta + \zeta').$$

Let $\alpha : G \rightarrow D$ be the map which induces a G -equivariant biholomorphism of G/K onto D . Then it holds that

$$D \ni (x + iy, \zeta) = \alpha(\exp(x, \zeta) \cdot \eta_0(y - Q(\zeta, \zeta))).$$

4. Let Ξ (resp. Ξ^+) be the set of all $\lambda \in \mathfrak{s}(1)^*$ such that the hermitian form $\langle \lambda, Q(\cdot, \cdot) \rangle$ on $V \times V$ is non-degenerate (resp. positive definite). It can be proved that Ξ^+ is equal to the dual cone

$$\Omega^* = \{\lambda \in \mathfrak{s}(1)^* : \langle \lambda, x \rangle > 0 \text{ for all } x \in \overline{\Omega} - \{0\}\}.$$

In particular, Ξ is non-empty. As in Ogden-Vági [3], we have a family $(\pi_\lambda)_{\lambda \in \Xi}$ of concrete irreducible unitary representations of $N(D)$ enough to decompose $L^2(N(D))$. The space of π_λ is the L^2 -space on a real subspace E_λ of V with $\dim_{\mathbb{R}} E_\lambda = \dim_{\mathbb{C}} V (= n, \text{ say})$. For $\lambda \in \Xi$, let $\rho(\lambda)$ be the Pfaffian of the alternating bilinear form $\text{Im}\langle \lambda, Q(\cdot, \cdot) \rangle$ on the real vector space $\mathfrak{s}(1/2) \times \mathfrak{s}(1/2)$. The Fourier transform $\hat{f}$ of $f \in L^1(N(D))$ is by definition

$$\hat{f}(\lambda) = \int_{N(D)} f(n) \pi_\lambda(n^{-1}) \, dn,$$

where dn is the Haar measure on $N(D)$. Then, the Plancherel formula for $N(D)$ is as follows:

$$\|f\|^2 = c \int_{\Xi} \|\hat{f}(\lambda)\|_{\text{HS}}^2 \rho(\lambda) \, d\lambda.$$

The positive constant c depends only on the normalization of dn . One can define the Fourier transform of $f \in L^2(N(D))$ in the standard way.

5. Let ψ be a continuous everywhere positive function on Ω such that $\psi(at) = a^\delta \psi(t)$ ($a > 0, t \in \Omega$) for some $\delta \in \mathbb{R}$. We consider first the Hilbert space $H^2(D, \psi)$ of $\mathbb{C}$ -valued holomorphic functions on D satisfying

$$\int_D |F(x + iy, \zeta)|^2 \psi(y - Q(\zeta, \zeta)) \, dx dy d\zeta < \infty \quad (x + iy \in W, \zeta \in V).$$

When the irreducible unitary representation of K with highest weight Λ is one dimensional, $H(\Lambda)$ is of this type for a certain ψ . For $F \in H^2(D, \psi)$, we put

$$(2) \quad f_t(x, \zeta) = F(x + i(t + Q(\zeta, \zeta)), \zeta)$$

for every $t \in \Omega$. Then, f_t belongs to $L^2(N(D))$, so that one can consider the Fourier transform $\hat{(f_t)}$ of f_t . Identify $L^2(E_\lambda)$ with $L^2(\mathbb{R}^n)$ and let $\{\phi_m^\lambda; m \in (\mathbb{Z}_+)^n\}$ be the complete orthonormal system consisting of the Hermite functions and V_λ the one dimensional subspace of $L^2(E_\lambda)$ spanned by ϕ_0^λ . We now define a Hilbert space $H^2(\Omega^*, \psi)$: every $\Phi \in H^2(\Omega^*, \psi)$ is a measurable function on Ξ taking its value at $\lambda \in \Xi$ in the Hilbert space of Hilbert-Schmidt operators on $L^2(E_\lambda)$ such that

- (i) $\Phi(\lambda) = 0$ if $\lambda \notin \Omega^*$,
- (ii) Range $\Phi(\lambda)$ is contained in V_λ if $\lambda \in \Omega^*$,
- (iii) $\|\Phi\|^2 = c \int_{\Omega^*} \|\Phi(\lambda)\|_{HS}^2 I_\psi(\lambda) \rho(\lambda) \, d\lambda < \infty$,

where $I_\psi(\lambda) = \int_\Omega e^{-2\langle \lambda, x \rangle} \psi(x) \, dx$.

Theorem 1. Let $F \in H^2(D, \psi)$ be given and define f_t by (2). Then, $\Phi(\lambda) = e^{\langle \lambda, t \rangle} \hat{(f_t)}(\lambda)$ is independent of $t \in \Omega$ and belongs to $H^2(\Omega^*, \psi)$. Conversely, let $\Phi \in H^2(\Omega^*, \psi)$ be given. Then,

$$F(x + i(t + Q(\zeta, \zeta)), \zeta) = c \int_{\Omega^*} e^{-\langle \lambda, t \rangle} \text{Tr}[\pi_\lambda(x, \zeta) \Phi(\lambda)] \rho(\lambda) d\lambda$$

is absolutely convergent and gives an element $F \in H^2(D, \psi)$ such that $\Phi(\lambda) = e^{\langle \lambda, t \rangle} (f_t)^\wedge(\lambda)$. Moreover, the map $F \rightarrow \Phi$ is unitary.

6. Now we treat $H(\Lambda)$. Let Λ be as in 1 and τ_Λ the irreducible unitary representation of K on a finite dimensional Hilbert space E with highest weight Λ . Since P_+K_C is a semidirect product, τ_Λ can be naturally extended to a representation of P_+K_C . Let $c \in G_C$ be the element defined by (1) and put

$$\Phi_\Lambda(g) = \tau_\Lambda(k(c)^{-1}) \tau_\Lambda(k(cg)).$$

We note that it makes sense to write $k(cg)$ for $g \in G$, for one can show that $cg \in P_+K_C P_-$. Let

$$\theta_0(t) = |\det_{\mathfrak{s}(1/2)} \text{Ad } \eta_0(t)|^{-1} |\det_{\mathfrak{s}(1)} \text{Ad } \eta_0(t)|^{-2} \quad (t \in \Omega)$$

and $\theta_\Lambda(\alpha(h)) = \Phi_\Lambda(h)$ ($h \in S = \exp \mathfrak{s}$), where α is the map $G \rightarrow D$ which induces a G -equivariant biholomorphism of G/K onto D .

The Hilbert space $H(\Lambda)$ consists of E -valued holomorphic functions on D satisfying

$$\|F\|^2 = \int_D \|\theta_\Lambda(iy, \zeta)^{-1} F(x + iy, \zeta)\|^2 \theta_0(y - Q(\zeta, \zeta)) dx dy d\zeta < \infty.$$

Let v_Λ be a highest weight vector for τ_Λ normalized so that $\|v_\Lambda\|$

$= 1$. We take an orthonormal basis $v_1 = v_\Lambda, v_2, \dots, v_d$ ($d = \deg \tau_\Lambda$) in E consisting of weight vectors arranged in order so that any vector in the root subspaces corresponding to positive compact roots in Δ is represented, under τ_Λ , by an upper triangular matrix. We denote by Λ_j the weight for the weight vector v_j . Let E_k be the one dimensional subspace of E spanned by v_k and

$$H_j(\Lambda) = \{F \in H(\Lambda) ; F(w, \zeta) \in E_1 \oplus \dots \oplus E_j\}.$$

Then, it can be proved that $H_j(\Lambda)$ is a closed subspace of $H(\Lambda)$ invariant under $\Pi_\Lambda|_S$. Let $H^1(\Lambda) = H_1(\Lambda)$ and $H^j(\Lambda)$ = the orthogonal complement of $H_{j-1}(\Lambda)$ in $H_j(\Lambda)$ ($j = 2, \dots, d$). Define a positive character χ_j ($j = 1, \dots, d$) of $A = \exp \mathfrak{a}$ by

$$\chi_j(\exp \sum_{i=1}^{\ell} a_i (X_{\gamma_i} + X_{-\gamma_i})) = \prod_{i=1}^{\ell} \exp a_i \Lambda_j(v(X_{\gamma_i} + X_{-\gamma_i})),$$

where $v = \text{Ad } c$ as in 3. Extending χ_j canonically to a character of S , we put

$$\psi_j(t) = \chi_j(\eta_0(t))^{-2} \theta_0(t) \quad (t \in \Omega, j = 1, 2, \dots, d).$$

Then, $\psi_j(at) = a^{\delta_j} \psi_j(t)$ ($a > 0, t \in \Omega$) for some $\delta_j \in \mathbb{R}$. Consider the Hilbert space $H^2(D, \psi_j)$ of the type in 5 and define an operator T_j by

$$T_j F(w, \zeta) = (F(w, \zeta), v_j) \quad (F \in H_j(\Lambda)).$$

T_j is a bounded operator $H_j(\Lambda) \rightarrow H^2(D, \psi_j)$ and its range is dense.

Therefore $H^j(\Lambda)$ is unitarily isomorphic to $H^2(D, \psi_j)$ by U_j , where U_j is the partial isometry appearing in the polar decomposition of the operator T_j . Thus, we have an irreducible decomposition $H(\Lambda) = \bigoplus_{j=1}^d H^j(\Lambda)$ for $\Pi_\Lambda|_S$.

7. Put for $\lambda \in \Omega^*$

$$(3) \quad I_\Lambda(\lambda) = \int_{\Omega} e^{-2\langle \lambda, t \rangle} \Phi_\Lambda(\eta_0(t)^{-1})^2 \theta_0(t) dt.$$

Lemma. The integral in (3) is absolutely convergent.

Now the matrix of $I_\Lambda(\lambda)$ with respect to the basis (v_k) is upper triangular with (k,k) -entry $I_{\psi_k}(\lambda) > 0$. Therefore we can give a meaning to $I_\Lambda(\lambda)^{1/2}$. Put $\mathcal{H}_\lambda = L^2(E_\lambda)$ and let $B_2(\mathcal{H}_\lambda)$ be the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}_\lambda$. Let us put

$$A(\mathcal{H}_\lambda) = \{T \in B_2(\mathcal{H}_\lambda) ; \text{Range } T \subset V_\lambda\}.$$

It is evident that $A(\mathcal{H}_\lambda)$ is a closed subspace of $B_2(\mathcal{H}_\lambda)$ and so $A(\mathcal{H}_\lambda)$ itself is a Hilbert space. Consider the Hilbert space tensor product $A(\mathcal{H}_\lambda) \otimes E$ of two Hilbert spaces $A(\mathcal{H}_\lambda)$ and E . This tensor product space $A(\mathcal{H}_\lambda) \otimes E$ is regarded as the Hilbert space $B_2(E, A(\mathcal{H}_\lambda))$ of anti-linear Hilbert-Schmidt operators mapping E to $A(\mathcal{H}_\lambda)$ via $(T \otimes v)(u) = (v, u)T$. For $\lambda \in \Omega^*$, we define an operator $M_\Lambda(\lambda)$ on $A(\mathcal{H}_\lambda) \otimes E$ by

$$M_\Lambda(\lambda)(T \otimes v) = T \otimes I_\Lambda(\lambda)^{1/2}v.$$

We are now in a position to define a Hilbert space $H(\Lambda)$: it consists of measurable functions Ψ on E taking their value at $\lambda \in E$ in $A(\mathfrak{H}_\lambda) \otimes E$ such that

$$(i) \quad \Psi(\lambda) = 0 \quad \text{if} \quad \lambda \notin \Omega^*,$$

$$(ii) \quad \|\Psi\|^2 = c \int_{\Omega^*} \|M_\Lambda(\lambda)\Psi(\lambda)\|^2 \rho(\lambda) d\lambda < \infty.$$

Put $H_j(\Lambda) = \{\Psi \in H(\Lambda) ; \Psi(\lambda) \in A(\mathfrak{H}_\lambda) \otimes (E_1 \oplus \dots \oplus E_j)\}$ and

$$T_j \Psi(\lambda) = \Psi(\lambda) v_j \in A(\mathfrak{H}_\lambda) \quad (\Psi \in H_j(\Lambda)).$$

T_j is a bounded operator $H_j(\Lambda) \rightarrow H^2(\Omega^*, \psi_j)$ and its range is dense. Let $H^1(\Lambda) = H_1(\Lambda)$ and $H^j(\Lambda)$ is the orthogonal complement of $H_{j-1}(\Lambda)$ in $H_j(\Lambda)$ ($j = 2, \dots, d$). Then, $H^j(\Lambda)$ is unitarily isomorphic to $H^2(\Omega^*, \psi_j)$ by U_j , where U_j is the partial isometry appearing in the polar decomposition of the operator T_j . Therefore we have an orthogonal decomposition $H(\Lambda) = \bigoplus_{j=1}^d H^j(\Lambda)$.

Theorem 2. $H(\Lambda)$ is unitarily isomorphic to $H(\Lambda)$ via the following diagram:

$$\begin{array}{ccc} \bigoplus_{j=1}^d H^j(\Lambda) = H(\Lambda) & \xrightarrow{\quad \approx \quad} & H(\Lambda) = \bigoplus_{j=1}^d H^j(\Lambda) \\ \text{(for each } j) \quad \parallel \{ U_j & & \parallel \{ U_j \\ H^2(D, \psi_j) & \xrightarrow[\text{(Theorem 1)}]{\quad \approx \quad} & H^2(\Omega^*, \psi_j) \end{array}$$

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