

On the identification of noncausal functions from the SFCs *

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1 Introduction.

Let $\{W_t, t \in [0, 1]\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, starting at the origin. Let $f(t, \omega)$ be a random function on $[0, 1] \times \Omega$ and $\{\varphi_n(t)\}$ be a CONS in $L^2([0, 1]; \mathbb{C})$. The system $\{\tilde{f}_n(\omega) = \int_0^1 f(t, \omega) \overline{\varphi_n(t)} dW_t\}$ is called the stochastic Fourier coefficients (SFCs in abbr.) of $f(t, \omega)$. Here $\bar{z}$ ($z \in \mathbb{C}$) denotes the complex conjugate of z . It is of course understood that the stochastic integral $\int dW$ in the definition of SFCs should be chosen adequately. We are concerned with the problem whether $f(t, \omega)$ is identified from the SFCs of $f(t, \omega)$ or not.

We here roughly present some preceding studies of this problem without rigorous assumptions on $f(t, \omega)$. Suppose $f(t, \omega)$ be a square integrable Wiener functional. Then $f(t, \omega)$ admits the Wiener-Itô expansion

$$f(t, \omega) = \sum_{n=0}^{\infty} I_n(k_n^f(t; \cdot)),$$

where $I_n(k_n^f(t; \cdot))$ denotes the multiple Wiener-Itô integral of n th degree of the kernel function $k_n^f(t; \cdot)$. In S.Ogawa[4] and S.Ogawa and H.Uemura[6] the author(s) investigated this problem in this framework of the theory of the Homogeneous Chaos, and obtained some affirmative answers by determining the kernel functions $k_n^f(t; \cdot)$, $n \in \mathbb{N} \cup \{0\}$. In [4], $f(t, \omega)$ was assumed to be causal and SFCs were defined by any uniformly bounded basis through the Itô integral, whereas in [6], $f(t, \omega)$ was assumed to be noncausal and SFCs were defined by the system of trigonometric functions through the Skorokhod integral.

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In these procedures some simple multiple Wiener integrals constructed by $\{\varphi_n(t)\}$ were used as test functions. Moreover, we needed the multiple Wiener integrals of these functions to reconstruct $f(t, \omega)$ from $\{k_n^f(t, \cdot)\}$. In the same framework of [6] we applied the Bohr convolution method to identify the Fourier coefficients of $f(t, \omega)$ in S.Ogawa and H.Uemura[7]. In this procedure we utilized $\int_0^1 \varphi_n(t) dW_t, n \in \mathbb{Z}$. In any studies mentioned above we needed the Brownian motion as 'a catalyst'.

Recently, however, S. Ogawa [5] obtained an affirmative answer without the aid of a Brownian motion from the stochastic Fourier transform (SFT in abbr.) if $f(t, \omega)$ is a non-negative or nonpositive causal function. In [5] Itô type stochastic process $\int_0^t f(s, \omega) dW_s$ was gotten from the stochastic Fourier transform and the quadratic variation of this process gave $f(t, \omega)$ since $f(t, \omega)$ being of definite sign.

In this note we develop this SFT method to the case where $f(t, \omega)$ is noncausal. In Section 2, we state our setting. We next introduce the SFT in Section 3, and Section 4 is devoted to our main theorem and some remarks.

2 Our Setting.

Our settings are as follows:

- [H1] $f(t, \omega)$ is a nonnegative function which is differentiable for almost all ω and satisfies $\int_0^1 f(t, \omega) dt \in L^2(\Omega, dP)$ and $f'(t, \omega) \in L^2([0, 1] \times \Omega, dt dP)$, where $f'(t, \omega) = \partial f(t, \omega) / \partial t$. We do not assume $f(t, \omega)$ is causal.
- [H2] We employ the system of trigonometric functions $\{e_n(t) = e^{2\pi i n t}, n \in \mathbb{Z}\}$ to define the SFCs of $f(t, \omega)$.
- [H3] We choose the Ogawa integral for the stochastic integral to define the SFCs of $f(t, \omega)$.

Here, $f(t, \omega)$ is called Ogawa integrable if

$$\sum_n \int_0^1 f(t, \omega) \overline{\varphi_n(t)} dt \int_0^1 \varphi_n(t) dW_t$$

converges in probability for any CONS $\{\varphi_n(t)\}$ and moreover if this limit is identical. We call this limit by the Ogawa integral of $f(t, \omega)$ and denote by $\int_0^1 f(t, \omega) d_* W_t$. We note that the equation

$$\int_0^1 f(t, \omega) d_* W_t = f(1, \omega) W_1 - \int_0^1 W_t f'(t, \omega) dt \quad (1)$$

holds under our hypothesis [H1]. Refer S.Ogawa [1, 2] for details. We employ the notation $\tilde{f}_n(\omega) = \int_0^1 f(t, \omega) \overline{e_n(t)} d_* W_t$ for SFC.

3 Stochastic Fourier Transform.

In this section, we first introduce the stochastic Fourier transform of $f(t, \omega)$. Let $\{\varphi_n(t)\}$ be a CONS. Let $\{\varepsilon_n\}$ be a sequence such that $\varepsilon_n \neq 0$ for all n and that

$$\mathcal{T}_{\varphi, \varepsilon}(f) = \sum_n \varepsilon_n \tilde{f}_n(\omega) \varphi_n(t)$$

converges. Then $\mathcal{T}_{\varphi, \varepsilon}(f)$ is called the $\{\varepsilon_n\}$ -stochastic Fourier transform ($\{\varepsilon_n\}$ -SFT in abbr.) of $f(t, \omega)$. Refer S.Ogawa[3] for details.

We construct a SFT of $f(t, \omega)$ for some suitable sequence. To this end we first note the following proposition:

Proposition 1. *Under the hypotheses [H.1], [H.2] and [H.3], $\{\tilde{f}_n(\omega), n \in \mathbb{Z}\}$ is uniformly bounded in $L^1(dP)$.*

Proof. From (1) we have

$$\begin{aligned} \tilde{f}_n(\omega) &= \int_0^1 f(t, \omega) dt \int_0^1 e_{-n}(t) dW_t \\ &\quad + (f(1, \omega) - f(0, \omega)) \sum_{k \neq -n} \frac{1}{-2\pi i(n+k)} \int_0^1 e_k(t) dW_t \\ &\quad - \sum_{k \neq -n} \int_0^1 f'(t, \omega) \overline{e_{n+k}(t)} dt \frac{1}{-2\pi i(n+k)} \int_0^1 e_k(t) dW_t. \end{aligned} \quad (2)$$

Applying the Schwarz inequality

$$\begin{aligned} E|\tilde{f}_n(\omega)| &\leq \sqrt{E \left| \int_0^1 f(t, \omega) dt \right|^2} \\ &\quad + \sqrt{E \left| \int_0^1 f'(t, \omega) dt \right|^2} \sqrt{\sum_{k \neq -n} \frac{1}{4\pi^2(n+k)^2}} \\ &\quad + \sqrt{E \left[\int_0^1 |f'(t, \omega)|^2 dt \right]} \sqrt{\sum_{k \neq -n} \frac{1}{4\pi^2(n+k)^2}} < \infty, \end{aligned}$$

which complete the proof since the right hand side is independent of n . $\square$

Thus it is enough to employ an ℓ^1 sequence to construct a SFT, and we here choose the following $\{\tau_n\}$:

$$\tau_n = \begin{cases} \frac{1}{-4\pi^2 n^2} & \text{if } n \neq 0 \\ 1 & \text{if } n = 0. \end{cases}$$

Then

$$\left\{ \sum_{|n| \leq N} \tau_n \tilde{f}_n(\omega) e_n(t) \right\}_{N=1,2,\dots}$$

forms a Cauchy sequence in $(L^1(\Omega \rightarrow C([0, 1]; \mathbb{C})), dP, \|\cdot\|)$, where

$$\|X(\cdot)\| = E \left[\sup_{0 \leq t \leq 1} |X(t)| \right].$$

This is because

$$\begin{aligned} & E \left[\sup_{0 \leq t \leq 1} \left| \sum_{n \neq 0, |n| \leq N} \frac{1}{-4\pi^2 n^2} \tilde{f}_n(\omega) e_n(t) - \sum_{n \neq 0, |n| \leq M} \frac{1}{-4\pi^2 n^2} \tilde{f}_n(\omega) e_n(t) \right| \right] \\ & \leq \sum_{N < |n| \leq M} \frac{1}{4\pi^2 n^2} E |\tilde{f}_n(\omega)|, \end{aligned}$$

which goes to 0 as $N, M \rightarrow \infty$. Thus we obtain the $\{\tau_n\}$ -SFT of $f(t, \omega)$, say $S(t, \omega)$:

Proposition 2. *Under the hypotheses [H.1], [H.2] and [H.3], there exists a random function $S(t, \omega) \in C([0, 1])$ a.s. such that*

$$\lim_{N \rightarrow \infty} E \left[\sup_{0 \leq t \leq 1} \left| \sum_{|n| \leq N} \tau_n \tilde{f}_n(\omega) e_n(t) - S(t, \omega) \right| \right] = 0.$$

4 Main Theorem.

From (2) we have

$$\begin{aligned} S(t, \omega) &= \tilde{f}_0(\omega) + \sum_{n \neq 0} \frac{1}{-4\pi^2 n^2} \tilde{f}_n(\omega) e_n(t) \\ &= \tilde{f}_0(\omega) - \frac{1}{2} \left(f(1, \omega) W_1 - \int_0^1 W_t f'(t, \omega) dt \right) \left(\frac{1}{6} - t + t^2 \right) \\ &\quad - \left(\int_0^1 \left(\int_0^t W_s f'(s, \omega) ds \right) dt - \int_0^1 W_t f(t, \omega) dt \right) \left(\frac{1}{2} - t \right) \end{aligned}$$

$$\begin{aligned}
& - \left(\int_0^t \int_0^s W_u f'(u, \omega) du ds - \int_0^1 \int_0^t \int_0^s W_u f'(u, \omega) du ds dt \right) \\
& + \left(\int_0^t W_s f(s, \omega) ds - \int_0^1 \int_0^t W_s f(s, \omega) ds dt \right)
\end{aligned} \tag{3}$$

for all $t \in (0, 1)$ and almost all ω , noting that

$$\lim_{N \rightarrow \infty} \sum_{n \neq 0, |n| \leq N} \frac{1}{2\pi i n} e_n(t) = \frac{1}{2} - t$$

and

$$\lim_{N \rightarrow \infty} \sum_{n \neq 0, |n| \leq N} \frac{1}{-4\pi^2 n^2} e_n(t) = -\frac{1}{2} \left(\frac{1}{6} - t + t^2 \right)$$

for all $t \in (0, 1)$. Since the right hand side of (3) is differentiable with respect to $t \in (0, 1)$, so is the left hand side, and

$$\begin{aligned}
S'(t, \omega) &= -\frac{1}{2} \left(f(1, \omega) W_1 - \int_0^1 W_t f'(t, \omega) dt \right) (-1 + 2t) \\
&+ \left(\int_0^1 \left(\int_0^t W_s f'(s, \omega) ds \right) dt - \int_0^1 W_t f(t, \omega) dt \right) \\
&- \int_0^t W_u f'(u, \omega) du + W_t f(t, \omega).
\end{aligned}$$

We note that

$$\left| \int_s^t W_u f'(u, \omega) du \right| \leq \sqrt{|t-s|} \sup_{u \in [0,1]} |W_u| \sqrt{\int_0^1 |f'(u, \omega)|^2 du}.$$

Hence if we fix $s \in (0, 1)$ arbitrary then we have

$$\limsup_{t \downarrow s} \frac{S'(t, \omega) - S'(s, \omega)}{\sqrt{2(t-s) \log \log \frac{1}{t-s}}} = \limsup_{t \downarrow s} \frac{W_t f(t, \omega) - W_s f(s, \omega)}{\sqrt{2(t-s) \log \log \frac{1}{t-s}}} = f(s, \omega) \quad a.s.$$

from the law of iterated logarithm of the Brownian motion, recalling that we assume $f(t, \omega)$ is nonnegative. Set $\mathbb{S}$ be a countable dense subset of $(0, 1)$. Then we have the following theorem:

Theorem 1. *Let $f(t, \omega)$ be a nonnegative function which is differentiable for almost all ω satisfying $\int_0^1 f(t, \omega) dt \in L^2(\Omega, dP)$ and $f'(t, \omega) \in L^2([0, 1] \times \Omega, dt dP)$. Then we have*

$$P \left(\limsup_{t \downarrow s} \frac{S'(t, \omega) - S'(s, \omega)}{\sqrt{2(t-s) \log \log \frac{1}{t-s}}} = f(s, \omega) \quad \text{for all } s \in \mathbb{S} \right) = 1.$$

Remark 1. Since we assume $f(t, \omega)$ is continuous, the theorem above is sufficient to identify $f(t, \omega)$ for all $t \in [0, 1]$.

Remark 2. Note that 0th SFC $\tilde{f}_0(\omega)$ does not appear in the construction of $S'(t, \omega)$, that is, $\tilde{f}_0(\omega)$ is not necessary to identify $f(t, \omega)$.

Moreover, since $\frac{1}{-4\pi^2 n^2} \tilde{f}_n(\omega) e_n(t)$ is a continuously differentiable function, we get the following corollary:

Corollary 1. Let $f(t, \omega)$ be a nonnegative function which is differentiable for almost all ω satisfying $\int_0^1 f(t, \omega) dt \in L^2(\Omega, dP)$ and $f'(t, \omega) \in L^2([0, 1] \times \Omega, dt dP)$. Let Λ be a finite subset of $\mathbb{Z}$ containing 0. Set

$$S_\Lambda(t, \omega) = \sum_{n \in \Lambda^c} \frac{1}{-4\pi^2 n^2} \tilde{f}_n(\omega) e_n(t).$$

Then we have

$$P \left(\limsup_{t \downarrow s} \frac{S'_\Lambda(t, \omega) - S'_\Lambda(s, \omega)}{\sqrt{2(t-s) \log \log \frac{1}{t-s}}} = f(s, \omega) \text{ for all } s \in \mathbb{S} \right) = 1.$$

Remark 3. Corollary 1 does not mean that $\tilde{f}_n(\omega)$, $n \in \Lambda$, is reconstructed from $\{\tilde{f}_n(\omega), n \in \Lambda^c\}$ by deterministic procedures. Indeed, if $f(t, \omega) = 1$, then $\tilde{f}_0(\omega) = W_1$ is independent of $\{\tilde{f}_n(\omega) = \int_0^1 \overline{e_n(t)} dW_t, n \in \Lambda^c\}$.

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