

TOEPLITZ 作用素 及び HANKEL 作用素
の HYPONORMALITY について

東北大学 理学研究科
吉野 崇

以下の結果は、今年の秋の学会で既に報告したものであるが、ここでは、その証明も含めて詳しく報告する。

A bounded measurable function $\varphi \in L^\infty$ on the circle induces the multiplication operator on L^2 called the Laurent operator L_φ given by

$$L_\varphi f = \varphi f \text{ for } f \in L^2.$$

And the Laurent operator induces in a natural way twin operators on H^2 called Toeplitz operator T_φ given by

$$T_\varphi f = PL_\varphi f \text{ for } f \in H^2,$$

where P is the orthogonal projection from L^2 onto H^2 and Hankel operator H_φ given by

$$H_\varphi f = J(I - P)L_\varphi f \text{ for } f \in H^2,$$

where J is the unitary operator on L^2 defined by

$$J(z^{-n}) = z^{n-1}, \quad n = 0, \pm 1, \pm 2, \dots$$

Lemma 1. For $f \in L^2$, let $f^*(z) = \overline{f(\bar{z})}$. Then $\|f^*\|_2 = \|f\|_2$ and $f^* \in L^2$. Particularly, if $f \in H^2$, then $f^* \in H^2$ also.

Lemma 2. For $\varphi \in L^\infty$, $\|\varphi^*\|_\infty = \|\varphi\|_\infty$ and $\varphi^* \in L^\infty$. Particularly, if φ is inner, then φ^* is also inner.

Lemma 3. For $\varphi \in H^\infty$, $J(I - P)L_\varphi^* = T_\varphi^*J(I - P)$.

Concerning these twin operators, the following results are well known.

Proposition 1. ([1]) $A \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $T_z^*AT_z = A$. And, in particular, $A \in \mathcal{B}(H^2)$ is analytic Toeplitz operator (i.e., $A = T_\varphi$ for some $\varphi \in H^\infty$) if and only if $T_zA = AT_z$.

Proposition 2. ([4]) Let q be a non-constant inner function, and let Q be the orthogonal projection from L^2 onto $K = H^2 \ominus T_qH^2$. If $A \in \mathcal{B}(K)$ commutes with QL_zQ , then there is a function ψ in H^∞ such that $\|\psi\|_\infty = \|A\|$ and $A = QL_\psi Q$.

Remark 1. In Proposition 2, we may assume that q is a zero function or an inner function. Because, in the case where $q = 0$, Proposition 2 reduces to Proposition 1 and, in the case where q is a constant inner function, we may take $\psi = 0$ because $A = O$.

Proposition 3. H_φ has the following properties ;

- (1) $T_z^*H_\varphi = H_\varphi T_z$
(Hence $\mathcal{N}_{H_\varphi} = \{x \in H^2 ; H_\varphi x = o\}$ is invariant under T_z
and $\mathcal{N}_{H_\varphi} = \{o\}$ or $\mathcal{N}_{H_\varphi} = T_qH^2$, where q is inner)
- (2) $H_\varphi^* = H_{\varphi^*}$
- (3) $H_{\alpha\varphi + \beta\psi} = \alpha H_\varphi + \beta H_\psi$, $\alpha, \beta \in \mathbb{C}$
- (4) $H_\varphi = O$ if and only if $(I - P)\varphi = o$ (i.e., $\varphi \in H^\infty$)
- (5) $\|H_\varphi\| = \inf\{\|\varphi + \psi\|_\infty ; \psi \in H^\infty\}$

Now we state here the relations between these twin operators.

Proposition 4. $H_\psi^*H_\varphi = T_{\overline{\psi}\varphi} - T_{\overline{\psi}}T_\varphi$ and

$$H_{\overline{\varphi}}^*H_{\overline{\psi}} - H_\varphi^*H_\psi = T_\varphi^*T_\psi - T_\varphi T_\psi^*.$$

Proposition 5. For any $\psi \in H^\infty$, $H_\varphi T_\psi = H_{\varphi\psi}$ and $T_\psi^*H_\varphi = H_\varphi T_{\psi^*}$.

Concerning the operator inequality of Hankel operators, we have the following.

Theorem 1. The following assertions are equivalent.

- (1) $H_{\varphi_1} H_{\varphi_1}^* \leq \lambda^2 H_{\varphi_2} H_{\varphi_2}^*$ for some $\lambda \geq 0$.
- (2) There exists a function $h \in H^\infty$ such that $\|h\|_\infty \leq \lambda$ for some $\lambda \geq 0$ and that $H_{\varphi_1} = H_{\varphi_2} T_h$.

To prove this theorem, we need the following.

Lemma 4. ([3]) For $A, B \in \mathcal{B}(\mathcal{H})$, the following assertions are equivalent.

- (1) $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$.
- (2) There exists a $C \in \mathcal{B}(\mathcal{H})$ uniquely such that $A = BC$ and that

$$\begin{aligned} (a) \quad & \|C\|^2 = \inf\{\mu; AA^* \leq \mu BB^*\} \\ (b) \quad & \mathcal{N}_A = \mathcal{N}_C \quad \text{and} \quad (c) \quad C\mathcal{H} \subseteq [B^*\mathcal{H}]^\sim. \end{aligned}$$

Proof of Theorem 1. If $H_{\varphi_1} H_{\varphi_1}^* \leq \lambda^2 H_{\varphi_2} H_{\varphi_2}^*$ for some $\lambda \geq 0$, then, by Lemma 4, there exists a $A \in \mathcal{B}(H^2)$ uniquely such that $H_{\varphi_1} = H_{\varphi_2} A$ and that

$$\begin{aligned} (a) \quad & \|A\|^2 = \inf\{\mu: H_{\varphi_1} H_{\varphi_1}^* \leq \mu H_{\varphi_2} H_{\varphi_2}^*\} \leq \lambda^2 \\ (b) \quad & \mathcal{N}_{H_{\varphi_1}} = \mathcal{N}_A \quad \text{and} \quad (c) \quad AH^2 \subseteq [H_{\varphi_2}^* H^2]^\sim^{L^2}. \end{aligned}$$

And then, by Proposition 3 (1), $\mathcal{N}_{H_{\varphi_2}} = T_q H^2$, where q is a zero function or an inner function and, by Proposition 5, we have, for any $\psi \in H^\infty$,

$$\begin{aligned} A^* T_\psi^* H_{\varphi_2}^* &= A^* H_{\varphi_2}^* T_\psi^* = H_{\varphi_1}^* T_\psi^* \\ &= T_\psi^* H_{\varphi_1}^* = T_\psi^* A^* H_{\varphi_2}^* \end{aligned}$$

and hence

$$(A^* T_\psi^* - T_\psi^* A^*)[H_{\varphi_2}^* H^2]^\sim^{L^2} = \{o\}. \quad (i)$$

Since

$$\langle (T_q A - A T_q) H^2, H_{\varphi_2}^* H^2 \rangle = \langle H^2, (T_q A - A T_q)^* H_{\varphi_2}^* H^2 \rangle = 0 \quad \text{by (i),}$$

$(T_q A - A T_q) H^2 \subseteq \mathcal{N}_{H_{\varphi_2}} = T_q H^2$ and $\mathcal{N}_{H_{\varphi_2}}$ is invariant under A and hence $[H_{\varphi_2}^* H^2]^\sim^{L^2}$ is invariant under A^* . Since $[H_{\varphi_2}^* H^2]^\sim^{L^2}$ is invariant under T_z^* by Proposition 3 (2) and (1) and since

$$(A^* T_z^* - T_z^* A^*)[H_{\varphi_2}^* H^2]^\sim^{L^2} = \{o\} \quad \text{by (i),}$$

$A^*[[H_{\varphi_2}^* H^2]^{\sim L^2}]$ commutes with $T_z^*[[H_{\varphi_2}^* H^2]^{\sim L^2}]$ and hence $(A^*[[H_{\varphi_2}^* H^2]^{\sim L^2}])^*$ commutes with $QL_zQ = (T_z^*[[H_{\varphi_2}^* H^2]^{\sim L^2}])^*$, where Q is the orthogonal projection from L^2 onto $[H_{\varphi_2}^* H^2]^{\sim L^2}$. And, by Proposition 2 and Remark 1, there is a function h in H^∞ such that

$$\|h\|_\infty = \|(A^*[[H_{\varphi_2}^* H^2]^{\sim L^2}])^*\| = \|A^*[[H_{\varphi_2}^* H^2]^{\sim L^2}]\| \leq \|A^*\| = \|A\| \leq \lambda$$

and $(A^*[[H_{\varphi_2}^* H^2]^{\sim L^2}])^* = QL_hQ$. And then, for any $f \in H^2$, we have

$$\begin{aligned} H_{\varphi_1}^* f &= A^* H_{\varphi_2}^* f = QL_h^* H_{\varphi_2}^* f = QT_h^* H_{\varphi_2}^* f \\ &= H_{\varphi_2}^* T_h^* f = T_h^* H_{\varphi_2}^* f \quad \text{by Proposition 5} \end{aligned}$$

and $H_{\varphi_1}^* = T_h^* H_{\varphi_2}^*$ and hence $H_{\varphi_1} = H_{\varphi_2} T_h$.

As a special case of Theorem 1, we have the following.

Theorem 2. H_φ is hyponormal (i.e., $H_\varphi H_\varphi^* \leq H_\varphi^* H_\varphi$) if and only if $H_\varphi = H_\varphi^* T_h$ for some $h \in H^\infty$ such that $\|h\|_\infty \leq 1$.

Proof. Since $H_\varphi^* H_\varphi = H_\varphi^* H_\varphi^*$ by Proposition 3 (2), the hyponormality of H_φ is equivalent that there exists a function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$ and that $H_\varphi = H_\varphi^* T_h = H_\varphi^* T_h$ by Theorem 1 and by Proposition 3 (2).

Corollary 1. Every hyponormal Hankel operator is normal.

Proof. If H_φ is hyponormal, then $H_\varphi = H_\varphi^* T_h$ for some $h \in H^\infty$ such that $\|h\|_\infty \leq 1$ by Theorem 2 and, by Propositions 3 (2) and 5,

$$H_{\varphi^*} = H_\varphi^* = T_h^* H_\varphi = H_\varphi T_h^* = H_{\varphi^*} T_h^*.$$

Since $h^* \in H^\infty$ and $\|h^*\|_\infty = \|h\|_\infty$ by Lemmas 1 and 2, $H_{\varphi^*} = H_{\varphi^*}$ is also hyponormal by Theorem 2. Therefore H_φ is normal.

By Proposition 4, T_φ is hyponormal if and only if $H_\varphi^* H_\varphi \leq H_{\overline{\varphi}}^* H_{\overline{\varphi}}$ and, by Proposition 3 (2), $H_\varphi^* H_{\varphi^*} \leq H_{\overline{\varphi}}^* H_{\overline{\varphi}^*}$ and hence, by Theorem 1,

$$H_{\varphi^*} = H_{\overline{\varphi}}^* T_h$$

for some function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$ and, by using Proposition 3 (2) again, we have the following result.

Theorem 3. T_φ is hyponormal if and only if $H_\varphi = T_h^* H_{\bar{\varphi}}$ for some function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$.

Corollary 2. If T_φ is hyponormal, then T_{φ^*} is also hyponormal.

Proof. If T_φ is hyponormal, then, by Theorems 3 and by Proposition 5,

$$H_\varphi = T_h^* H_{\bar{\varphi}} = H_{\bar{\varphi}} T_h^*$$

for some function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$ and, by Proposition 3 (2),

$$H_{\varphi^*} = H_\varphi^* = T_h^* H_{\bar{\varphi}}^* = T_h^* H_{\bar{\varphi}^*} = T_h^* H_{\overline{\varphi^*}}$$

and hence, by Theorem 3, T_{φ^*} is also hyponormal because $h^* \in H^\infty$ and $\|h^*\|_\infty = \|h\|_\infty \leq 1$ by Lemmas 1 and 2.

For φ in L^2 , we can define the **generalised Hankel operator** H_φ as follows ;

$$H_\varphi f = J(I - P)L_\varphi f \quad \text{for } f \in \mathcal{D}(H_\varphi),$$

where $\mathcal{D}(H_\varphi) = \{f \in H^2 : \varphi f \in L^2\}.$

H_φ is generally unbounded and, for its definition domain $\mathcal{D}(H_\varphi)$,

$$H^\infty \subseteq \mathcal{D}(H_\varphi)$$

and we have the following.

Theorem 4. For $\varphi \in L^\infty$, let

$$\varphi = f + \varphi(0) + \bar{g},$$

where f and g in H_0^2 . Then, for any $\psi \in H^\infty$, we have

$$H_\varphi \psi = H_{\bar{g}} \psi.$$

Proof. $H_\varphi\psi = J(I - P)(f\psi + \varphi(0)\psi + \bar{g}\psi) = J(I - P)(\bar{g}\psi) = H_{\bar{g}}\psi.$

Remark 2. It is known that

$$L^\infty \neq H^\infty \oplus \overline{H_0^\infty}.$$

By Theorem 5.18, H_φ is a bounded extension of $H_{\bar{g}}|H^\infty$. Moreover we see that it is also a bounded extension of $H_{\bar{g}}$.

In fact, since $u \in \mathcal{D}(H_{\bar{g}})$ implies $\bar{g}u \in L^2$,

$$fu = \varphi u - \varphi(0)u - \bar{g}u \in L^2$$

because $\varphi \in L^\infty$ and hence $fu \in H^2$. Therefore

$$H_{\bar{g}}u = H_\varphi u \quad \text{for } u \in \mathcal{D}(H_{\bar{g}})$$

and so H_φ is a bounded extension of $H_{\bar{g}}$.

By the same reason, $H_{\bar{\varphi}}$ is a bounded extension of $H_{\bar{f}}$.

As a special case of Theorem 1, we have the following.

Theorem 5. For $\varphi = f + \varphi(0) + \bar{g} \in L^\infty$, where f and g in H_0^2 and for some $\lambda \geq 0$, the following assertions are equivalent.

- (1) $H_\varphi^* H_\varphi \leq \lambda^2 H_{\bar{\varphi}}^* H_{\bar{\varphi}}$.
- (2) $g = T_h^* f + c$ for some constant c and some function $h \in H^\infty$ such that $\|h\|_\infty \leq \lambda$.

Proof. If $g = T_h^* f + c$ for some constant c and some function $h \in H^\infty$ such that $\|h\|_\infty \leq \lambda$, then

$$c = g - T_h^* f = P(g - \overline{h^* f}) = P(\overline{g - h^* f})$$

and $\bar{g} - h^* \bar{f} \in H^2$ and hence, by Theorem 4, for any $\psi \in H^\infty$,

$$\begin{aligned} \|H_\varphi\psi\| &= \|H_{\bar{g}}\psi\| = \|J(I - P)L_{h^*}\bar{f}\psi\| = \|T_h^* J(I - P)\bar{f}\psi\| \quad \text{by Lemma 3} \\ &\leq \|T_h^*\| \|J(I - P)\bar{f}\psi\| = \|h\|_\infty \|H_{\bar{f}}\psi\| = \|h\|_\infty \|H_{\bar{\varphi}}\psi\|. \end{aligned}$$

And since $[H^\infty]^{\sim L^2} = H^2$,

$$H_\varphi^* H_\varphi \leq \|h\|_\infty^2 H_{\bar{\varphi}}^* H_{\bar{\varphi}} \leq \lambda^2 H_{\bar{\varphi}}^* H_{\bar{\varphi}}.$$

Conversely, if $H_\varphi^* H_\varphi \leq \lambda^2 H_{\bar{\varphi}}^* H_{\bar{\varphi}}$, then, by Theorem 1 and by Proposition 3 (2), there exists a function h in H^∞ such that $\|h\|_\infty \leq \lambda$ and that $\varphi^* - \bar{\varphi}^* h \in H^\infty$ and hence $\varphi - \bar{\varphi} h^* \in H^\infty$ by Lemmas 1 and 2. Since

$$\varphi - \bar{\varphi} h^* = (f + \varphi(0) - \overline{\varphi(0)} h^* - g h^*) + (\bar{g} - \bar{f} h^*),$$

we have $\bar{g} - \bar{f} h^* \in H^2$ because $h^* \in H^\infty$. And then $\overline{\bar{g} - h^* \bar{f}} \in [H_0^2]^\perp$ and $P(\overline{\bar{g} - h^* \bar{f}}) = c$ (constant) and hence

$$c = P(g - \overline{h^* f}) = g - PL_{h^*}^* f = g - T_{h^*}^* f.$$

Corollary 3. ([2]) For $\varphi = f + \varphi(0) + \bar{g} \in L^\infty$, where f and g in H_0^2 , the following assertions are equivalent.

- (1) T_φ is hyponormal.
- (2) $g = T_{h^*}^* f + c$ for some constant c and some function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$.

Proof. Since T_φ is hyponormal if and only if $H_\varphi^* H_\varphi \leq H_{\bar{\varphi}}^* H_{\bar{\varphi}}$ by Proposition 4, we have the conclusion by setting $\lambda = 1$ in Theorem 5.

References

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