

Some vector-valued theta series on $U(2, 2)$ and $Sp(1, 1)$

Atsuo YAMAUCHI

This is an overview of the paper [5], which is a joint work with Hiro-aki NARITA.

In this talk, we will first construct vector-valued (holomorphic) singular modular forms on $U(2, 2)$, which can be regarded as generalizations of theta series. The restrictions of these singular forms to $Sp(1, 1)$ are modular forms which generate quaternionic discrete series.

1 The construction of vector-valued singular forms on $U(2, 2)$

First of all, let us construct vector-valued singular modular forms on $U(2, 2)$. Put

$$U(2, 2) = \left\{ g \in GL(4, \mathbb{C}) \mid g \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix} \bar{g} = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix} \right\},$$

$$\mathfrak{H}_{U(2,2)} = \{ z \in M_2(\mathbb{C}) \mid \sqrt{-1}(\bar{t}z - z) > 0 \}.$$

As is well-known, the group $U(2, 2)$ acts on $\mathfrak{H}_{U(2,2)}$ as $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} (z) = (\alpha_1 z + \alpha_2)(\alpha_3 z + \alpha_4)^{-1}$, and we put $\mu_1 \left(\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, z \right) = \alpha_3 z + \alpha_4$.

For any non-negative integer κ , we denote by V_κ the space of homogeneous polynomials of degree κ with two variables. (Note that $\dim_{\mathbb{C}} V_\kappa = \kappa + 1$.) We define a map $\sigma_\kappa : M_2(\mathbb{C}) \rightarrow \text{End}(V_\kappa)$ by

$$(\sigma_\kappa(h)v)(X, Y) = v(aX + cY, bX + dY) \quad \left(h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) \right).$$

This induces an irreducible rational representation $(\sigma_\kappa, V_\kappa)$ of $GL_2(\mathbb{C})$. The space V_κ has a basis $\{v_i = v_i(X, Y) := X^i Y^{\kappa-i} \mid 0 \leq i \leq \kappa\}$. The vector v_κ is a highest weight vector.

Take an imaginary quadratic field K and fix it. For any $r, s \in K^2$ and any $\mathbb{Z}$ -lattice L in K^2 , we define a V_κ -valued holomorphic function $\theta^{(\kappa)}(z; L, r, s)$ on $z \in \mathfrak{H}_{U(2,2)}$ as

$$\theta^{(\kappa)}(z; L, r, s) = \sum_{x \in L+r} \exp(\pi \sqrt{-1} \text{Tr}_{K/\mathbb{Q}}(\bar{t}xs)) \exp(\pi \sqrt{-1} \bar{x}zx) \sigma_\kappa((x, *)) \cdot v_\kappa.$$

Here $(x, *)$ denotes an arbitrary 2×2 -matrix whose first column is x , since $\sigma_\kappa((x, *)) \cdot v_\kappa$ does not depend on $*$.

Theorem 1.1. *The function $\theta^{(\kappa)}(z; L, r, s)$ is a modular form of weight $\sigma_\kappa \otimes \det$. It means, there exists a congruence subgroup Γ of $U(2, 2 : K)(:= U(2, 2) \cap GL_4(K))$ so that*

$$\theta^{(\kappa)}(\gamma(z); L, r, s) = \det(\mu_1(\gamma, z))\sigma_\kappa(\mu_1(\gamma, z))\theta^{(\kappa)}(z; L, r, s)$$

for any $\gamma \in \Gamma$.

Remark 1. Note that $\theta^{(\kappa)}(z; L, r, s)$ is so-called a singular form since all of its Fourier coefficients at non-degenerate indices are 0.

Remark 2. In case $\kappa = 0$, the scalar valued function $\theta^{(0)}(z; L, r, s)$ is a pull-back of a theta series on $Sp(4, \mathbb{Q})$, and a modular form of weight 1.

This theorem can be proved in the same way as the proof of automorphy of scalar-valued theta series.

2 Restriction to $Sp(1, 1)$

Let B be a definite quaternion algebra over $\mathbb{Q}$, and $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$ be the Hamilton quaternion algebra. Since $B \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}$, we regard B as a dense subset of $\mathbb{H}$. The main involution of $\mathbb{H}$ is written as $\tau \rightarrow \bar{\tau}$ for $\tau \in \mathbb{H}$, and the reduced norm and the reduced trace are defined as $N(\tau) = \tau\bar{\tau}$ and $\text{tr}(\tau) = \tau + \bar{\tau}$. We write $\mathbb{H}^- = \{\tau \in \mathbb{H} \mid \text{tr}(\tau) = 0\}$ and $B^- = B \cap \mathbb{H}^-$.

Put

$$Sp(1, 1)_{\mathbb{R}} := \left\{ g \in M_2(\mathbb{H}) \mid g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t \bar{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

$$Sp(1, 1)_{\mathbb{Q}} = Sp(1, 1)_{\mathbb{R}} \cap M_2(B).$$

The canonical maximal compact subgroup of $Sp(1, 1)_{\mathbb{R}}$ is given as

$$K_{Sp(1, 1)} = \left\{ g \in Sp(1, 1)_{\mathbb{R}} \mid g = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \text{ with } a, b \in \mathbb{H} \right\}.$$

Hence the symmetric space of $Sp(1, 1)_{\mathbb{R}}$ is

$$\mathfrak{H}_{Sp(1, 1)} = \{\tau \in \mathbb{H} \mid \text{tr}(\tau) > 0\} \cong Sp(1, 1)_{\mathbb{R}} / K_{Sp(1, 1)},$$

and the factor of automorphy $\mu_2(g, \tau)$ is given as

$$\mu_2 \left(\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \tau \right) = \alpha_3 \tau + \alpha_4 \quad \text{for } \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in Sp(1, 1)_{\mathbb{R}} \text{ and } \tau \in \mathfrak{H}_{Sp(1, 1)}.$$

For given integer $\kappa \geq 2$, $\xi \in B^-$, $p, q \in B$ and $\mathbb{Z}$ -lattice $\Lambda \subset B$, we define a V_κ -valued function $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ on $g \in Sp(1, 1)_{\mathbb{R}}$ as

$$\begin{aligned} & \theta_\xi^{(\kappa)} \left(\begin{pmatrix} \sqrt{y} & x(\sqrt{y})^{-1} \\ 0 & (\sqrt{y})^{-1} \end{pmatrix} k_\infty; \Lambda, p, q \right) \\ &:= \sum_{\lambda \in \Lambda + p} \exp(\pi \sqrt{-1} \operatorname{tr}(\bar{\lambda} q)) y^{\frac{\kappa}{2}+1} \exp \left(-4\pi \sqrt{N(\lambda \xi \bar{\lambda})} y \right) \\ & \quad \exp(2\pi \sqrt{-1} \operatorname{tr}(\lambda \xi \bar{\lambda} x)) \sigma_\kappa(\iota(\mu_2(k_\infty, 1)))^{-1} \sigma_\kappa(\iota(\lambda u_\xi)) v_\kappa, \end{aligned}$$

where $x \in \mathbb{H}^-$, $y \in \mathbb{R}_+$ and $k_\infty \in K_{Sp(1,1)}$. The symbol ι denotes the canonical embedding of $\mathbb{H}$ into $M_2(\mathbb{C})$, and $u_\xi \in \mathbb{H}$ is as $u_\xi i \bar{u}_\xi = \frac{\xi}{\sqrt{N(\xi)}}$.

Theorem 2.1. *The function $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ is a modular form of weight σ_κ with respect to some congruence subgroup of $Sp(1, 1)_{\mathbb{Q}}$, generating a quaternionic discrete series.*

Sketch of proof. The automorphy of $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ comes from the fact that it is a pull-back of $\theta^{(\kappa)}(z; L, r, s)$ on $U(2, 2 : K)$ with $K := \mathbb{Q}(\sqrt{-N(\xi)}) \cong \mathbb{Q}(\xi)$ and suitable L, r, s .

Remark1. Note that the Fourier coefficients of $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ are 0 outside the single-orbit of indices $\{\lambda \xi \bar{\lambda} \mid \lambda \in B^\times\}$, (and so are those of $\theta_\xi^{(\kappa)}(\alpha g; \Lambda, p, q)$ for any $\alpha \in Sp(1, 1)_{\mathbb{Q}}$.) We should add that the subalgebra $\mathbb{Q}(\lambda \xi \bar{\lambda})$ is isomorphic to $\mathbb{Q}(\xi)$ ($\cong K$) for any $\lambda \in B^\times$.

Remark2. We can take $p \in B$ so that $\theta_\xi^{(\kappa)}(g; \Lambda, p, q) \neq 0$ for given ξ, Λ and q . In fact, there is $p \in B \setminus \{0\}$ which satisfies $N(p) < \frac{1}{4}N(l)$ for any $l \in \Lambda \setminus \{0\}$. Then the Fourier coefficient indexed by $p \xi \bar{p}$ is $\exp(\pi \sqrt{-1} \operatorname{tr}(\bar{p} q)) \sigma_\kappa(\iota(p u_\xi)) v_\kappa \neq 0$ since $N(p) < N(p + l)$ holds for any $l \in \Lambda \setminus \{0\}$.

In the end, we should note that the Fourier coefficients of $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ are essentially algebraic. Take any $0 \neq \eta \in B^-$ such that $\xi \eta \in B^-$. Then the set $\{1, \xi, \eta, \xi \eta\}$ is a basis of B over $\mathbb{Q}$. We take $u_{\xi, \eta} \in \mathbb{H}$ which satisfies $u_{\xi, \eta} i \bar{u}_{\xi, \eta} = \frac{\xi}{\sqrt{N(\xi)}}$ and $u_{\xi, \eta} j \bar{u}_{\xi, \eta} = \frac{\eta}{\sqrt{N(\eta)}}$. Note that such $u_{\xi, \eta}$ exists and is uniquely determined up to the multiple of $\{\pm 1\}$. From now on, we replace u_ξ by $u_{\xi, \eta}$ in the definition of $\theta_\xi^{(\kappa)}$. (By this replacement, the function $\theta_\xi^{(\kappa)}$ changes just a constant multiple.) We put $V_\kappa(\bar{\mathbb{Q}}) = \sum_{i=0}^{\kappa} \bar{\mathbb{Q}} v_i$ and $V_{\kappa, B}(\bar{\mathbb{Q}}) = \sigma_\kappa(\iota(u_{\xi, \eta})) V_\kappa(\bar{\mathbb{Q}})$. It can be verified that $V_{\kappa, B}(\bar{\mathbb{Q}})$ does not depend on the choice of (ξ, η) . Then we have the following.

Corollary 2.2. *If we write the Fourier expansion of $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ as*

$$\begin{aligned} & \theta_\xi^{(\kappa)} \left(\begin{pmatrix} \sqrt{y} & x(\sqrt{y})^{-1} \\ 0 & (\sqrt{y})^{-1} \end{pmatrix} k_\infty; \Lambda, p, q \right) \\ &= \sum_{\delta \in B^-} \sigma_\kappa(\iota(\mu_2(k_\infty, 1)))^{-1} C_\delta y^{\frac{\kappa}{2}+1} \exp(-4\pi \sqrt{N(\delta)} y) \exp(2\pi \sqrt{-1} \operatorname{tr}(\delta x)), \end{aligned}$$

with $C_\delta \in V_\kappa$, then $C_\delta \in V_{\kappa,B}(\overline{\mathbb{Q}})$ for any $\delta \in B^-$.

References

- [1] T.Arakawa, On automorphic forms of a quaternion unitary group of degree two, J. Fac. Sci. Univ. Tokyo Sect.IA **28** (1982), 547-566.
- [2] T.Arakawa, On certain automorphic forms of $Sp(1, q)$, in: Automorphic Forms of Several Variables, Taniguchi Symposium, Katata (1983), 1-48.
- [3] H.Narita, Fourier-Jacobi expansion of automorphic forms on $Sp(1, q)$ generating quaternionic discrete series, J. Funct. Anal., **239** (2006), 638-682.
- [4] H.Narita, Theta lifting from elliptic cusp forms to automorphic forms on $Sp(1, q)$, to appear in Math. Z.
- [5] A.Yamauchi and H.Narita, Some vector-valued singular automorphic forms on $U(2, 2)$ and their restriction to $Sp(1, 1)$, preprint.

Atsuo Yamauchi
 Graduate school of mathematics, Nagoya university,
 Furou-chou, Chikusa, Nagoya 464-8602, Japan
 e-mail: yamauchi@math.nagoya-u.ac.jp