

Some Computations on Higher Nash Blowups of Toric Surfaces

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1 Background

Let X be a variety over $k = \bar{k}$. The **Nash blowup** (X^*, π) of X was defined in [N]. For the Nash blowup, the following problem is suggested and studied: Is X^* (or its normalization) a resolution of singularities of X ?

Recently **Higher Nash blowups** was defined by A. Oneto & E. Zetini [OZ], T. Yasuda [Y1] independently, and analogous questions are studied.

Definition. (Yasuda's construction [Y1])

Let $n > 0$ be an integer. Let $X_{\text{sm}} := X \setminus \text{Sing}(X)$ and $\gamma_n : X_{\text{sm}} \rightarrow \text{Hilb}(X)$, $P \mapsto [P^{(n)}]$ where $P^{(n)}$ is the n -th fat point whose support is P , and its graph $\Gamma_n : X_{\text{sm}} \rightarrow X \times \text{Hilb}(X)$ with 1st projection $\pi_n : X \times \text{Hilb}(X) \rightarrow X$.

Now let $\text{Nash}_n(X) := (\text{the schematic closure of } \Gamma_n(X_{\text{sm}}))$ and $\pi_n := \pi_n|_{\text{Nash}_n(X)}$. Then $(\text{Nash}_n(X), \pi_n)$ is called the **n -th Nash blowup** of X . It is isomorphic to the classical Nash blowup when $n = 1$.

Regarding resolution of singularities of X , Yasuda suggested the following problem in [Y1]:

Problem.

Is $\text{Nash}_n(X)$ non-singular for all $n \gg 0$ when $\text{char } k = 0$?

Known Results. (T. Yasuda [Y1])

1. If $\text{char } k = 2, 3$, then there exists X such that $\text{Nash}_n(X)$ is singular for all $n > 0$.
2. If $\text{char } k = 0$ and X is a curve, then the answer is YES.

2 Main Result

Let $k = \mathbb{C}$ in what follows. Yasuda stated in [Y2] that A_3 -singularity is probably a counter example for the problem, and here is my main result:

Main Result.

Let $X := (z^4 - xy = 0) \subset \mathbb{A}^3$, which is a toric surface with an A_3 -singular point $P = (0, 0, 0)$. Then $\text{Nash}_n(X)$ is singular for all $n > 0$.

3 Gröbner Bases in Subalgebras

The proof of main result is roughly explained below.

Let X be an affine normal toric variety over $\mathbb{C}$. Then $X = \text{Spec}(S)$ for some subalgebra $S = \mathbb{C}[x^{a_1}, \dots, x^{a_s}] \subset \mathbb{C}[x_1, \dots, x_n]$ where $x^{a_i} = x_1^{a_{i,1}} \cdots x_n^{a_{i,n}}$ is a monomial in $\mathbb{C}[x_1, \dots, x_n]$ for $a_i = (a_{i,1}, \dots, a_{i,n})$. A. Duarte showed the following theorem:

Theorem. (A. Duarte [D])

Let J_n be the ideal $\langle x^{a_1} - 1, \dots, x^{a_s} - 1 \rangle^{n+1}$ in S . Then the normalization $\overline{\text{Nash}_n(X)}$ of $\text{Nash}_n(X)$ is the toric variety whose fan is given by $\text{GF}(J_n)$, the **Gröbner fan** of J_n .

Note that $\text{GF}(J_n)$ is not a Gröbner fan of an ideal in the usual polynomial ring but **monomial subalgebra** S . Duarte [D] established a theory of Gröbner bases in monomial subalgebras, and an algorithm to compute a Gröbner basis of any ideal in S w. r. t. any monomial order on S .

Duarte's algorithm needs to be extended slightly in order to compute a **reduced** Gröbner basis and a Gröbner fan:

Proposition. (c.f. [D], Algorithm A.3.13.)

Let $<$ be any monomial order on $S = \mathbb{C}[x^{a_1}, \dots, x^{a_s}]$ and I be any ideal in S . Then there exists a matrix M (obtained from $a_1, \dots, a_s$ and $<$ explicitly) such that the following steps give the reduced Gröbner bases of I w. r. t. $<$:

1. Take the ring map $T : \mathbb{C}[y_1, \dots, y_s] \rightarrow S$, $y_i \mapsto x^{a_i}$ and fix any monomial order $<'$ on $\mathbb{C}[y_1, \dots, y_s]$.
2. Compute the reduced Gröbner basis G of $T^{-1}(I)$ w. r. t. **M -weighted order $<'_M$** .
3. $\widehat{G} := G \setminus \ker(T)$.
4. Take a subset $\widehat{G}_{\min}$ of $\widehat{G}$ satisfying the followings:
 - $\forall g \in \widehat{G}, \exists g' \in \widehat{G}_{\min}; \text{lm}_{<}(T(g')) \mid \text{lm}_{<}(T(g))$ in S .
 - $\forall \text{distinct } g', g'' \in \widehat{G}_{\min}, \text{lm}_{<}(T(g')) \nmid \text{lm}_{<}(T(g''))$ in S .
5. $T(\widehat{G}_{\min}) = \{T(g') \mid g' \in \widehat{G}_{\min}\}$ is the red. G. b. of I .

Using above algorithm, $\text{GF}(J_n)$ was computed by Macaulay2 for $X = (z^4 - xy = 0) = \text{Spec}(\mathbb{C}[u, u^3v^4, uv])$ and $n \leq 24$. Then certain regularity was observed, and the following proposition which Yasuda and Duarte suggested was proved by induction on n :

Proposition.

For $X = (z^4 - xy = 0)$, $\text{GF}(J_n)$ contains a **non-regular cone** for all $n > 0$.

Hence $\overline{\text{Nash}_n(X)}$ is singular for all $n > 0$, and so is $\text{Nash}_n(X)$.

References

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