

Pseudo-free action of cyclic groups

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1 Determinant of equivariant elliptic operators

$M = M^{2m}$: $2m$ -dimensional closed connected oriented Riemannian manifold

G : finite group acting on M

G -action is orientation-preserving, isometric and effective

$D : \Gamma(E) \rightarrow \Gamma(F)$: G -equivariant elliptic operator

Equivariant determinant of D is defined by

$$G \ni g \longrightarrow \det(D, g) = \frac{\det(g|_{\ker D})}{\det(g|_{\operatorname{coker} D})} \in S^1 \subset \mathbf{C}^*$$

$\det_D := \det(D, \cdot) : G \longrightarrow S^1$ is a group homomorphism

S^1 : Abelian $\implies \det_D([G, G]) = 1$

$I_D : G \longrightarrow \mathbf{R}/\mathbf{Z}$ (additive group homomorphism) is defined by

$$I_D(g) := \frac{1}{2\pi\sqrt{-1}} \log \det(D, g) \pmod{\mathbf{Z}}$$

Theorem 1 I_D is an additive group homomorphism and hence

$$(a) \ I_D(g) + I_D(h) - I_D(gh) = 0 \quad (\forall g, h \in G)$$

$$(b) \ \det(D, g)^N = 1 \iff N I_D(g) = 0$$

Proposition 1 $g^p = 1$ ($p \geq 2$) $\implies$

$$I_D(g) \equiv \frac{p-1}{2p} \text{Ind}(D) - \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - \xi_p^{-k}} \text{Ind}(D, g^k) \pmod{\mathbf{Z}}$$

where $\xi_p = e^{2\pi\sqrt{-1}/p}$: primitive p -th root of unity,

$$\text{Ind}(D, g^k) = \text{Tr}(g^k| \ker D) - \text{Tr}(g^k| \text{coker } D) \in \mathbf{C},$$

$$\text{Ind}(D) = \text{Ind}(D, 1) = \dim \ker D - \dim \text{coker } D \in \mathbf{Z}$$

$\text{Ind}(D)$, $\text{Ind}(D, h)$ ($h \in G$) and hence $I_D(h)$ is calculated by using the Atiyah-Singer index theorem.

$\text{Ind}(D)$ is a characteristic number defined by using

the principal symbol of D

$$\left\{ \begin{array}{l} M \supset \text{Fix}(g) = \{q_1, q_2, \dots, q_n\} \text{ consists of points} \\ T_{q_i} M = \oplus_{j=1}^m N(\tau_{ij}) \left(h|N(\tau) = \begin{pmatrix} \cos \frac{2\pi\tau}{p} & -\sin \frac{2\pi\tau}{p} \\ \sin \frac{2\pi\tau}{p} & \cos \frac{2\pi\tau}{p} \end{pmatrix} \right) \end{array} \right\}$$

$$\implies \text{Ind}(D, h) = \sum_{i=1}^n \frac{\text{Tr}(h|E_{q_i}) - \text{Tr}(h|F_{q_i})}{\prod_{j=1}^m (1 - \xi_p^{\tau_{ij}})(1 - \xi_p^{-\tau_{ij}})}$$

Proposition 2 Let S be the signature operator and assume that p is an odd prime number. Then we have

$$\text{Ind}(S) = \text{Sign}(M), \quad \text{Ind}(S, h) = \sum_{i=1}^n \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi\tau_{ij}}{p} \right)$$

where $\text{Sign}(M)$ is the signature of M .

$\text{Spin}(2m)$: universal covering group of $\text{SO}(2m)$ ($m \geq 2$)

$\text{Spin}^c(2m) := \text{Spin}(2m) \times S^1 / \sim \quad ((s, z) \sim (-s, -z))$

$\exists P : \text{Spin}^c\text{-structure of } M \iff w_2(TM) = \exists u \in H^2(M; \mathbf{Z}) \pmod{2}$

$\exists \text{Spin}$ or $\exists$ almost complex structure $\implies \exists \text{Spin}^c\text{-structure}$

$\eta = P \times_{\text{Spin}^c(2m)} \mathbf{C}$: associated complex line bundle over M

($\text{Spin}^c\text{-structure}$ is determined by $c_1(\eta)$ if $H^1(M; \mathbf{Z}_2) = 0$)

G -action on a Spin^c -manifold M is called a Spin^c -action

if the action lifts to an action on the Spin^c -structure of M .

Remark 1 Let G be a cyclic group and assume that $m \geq 2$. Then

if $H^1(M; \mathbf{Z}) = 0 (\implies H^1(\text{SO}(M); \mathbf{Z}) = 0)$ and $c_1(\eta)$ is G -invariant

$\implies$ any G -action lifts to the Spin^c -structure $P(\xrightarrow{S^1} \text{SO}(M))$.

(A.Hattori and T.Yoshida, Lifting compact group actions in fiber bundles, Japan. J. Math. **2**, 13-25 (1976).)

L : (arbitrary) complex G -line bundle over M

$$D_L : \Gamma(S_+ \otimes L) \xrightarrow{\nabla} \Gamma(T^*M \otimes S_+ \otimes L) \xrightarrow{cm} \Gamma(S_- \otimes L) : L\text{-valued Dirac operator}$$

(where $S_{\pm}$ are half spinor bundles and cm is the Clifford multiplication)

Proposition 3

h acts on the fibers $\eta|_{q_i}$, $L|_{q_i}$ via multiplications by $\xi_p^{\kappa_i}$, $\xi_p^{\mu_i}$ respectively

$$\begin{aligned} \implies \text{Ind}(D_L) &= e^{c_1(L)} e^{\frac{c_1(\eta)}{2}} \hat{A}(TM)[M] , \\ \text{Ind}(D_L, h) &= \sum_{i=1}^n \varepsilon_i \xi_p^{\mu_i} \xi_p^{\frac{\nu_i}{2}} \prod_{j=1}^m \frac{1}{1 - \xi_p^{-\tau_{ij}}} \end{aligned}$$

where $\hat{A}$ is the $\hat{A}$ -class, $\varepsilon_i = \pm 1$ and $\nu_i = \kappa_i - \sum_{j=1}^m \tau_{ij}$.

(ε , κ_i are not determined by the data of the action on M)

Proposition 4 $M = (M, J)$: almost complex manifold

$G \subset \text{Aut}_J(M)$: group of automorphisms of M preserving

the almost complex structure J

$h|_{T_{q_i}M} = (\xi_p^{\tau_{i1}}, \dots, \xi_p^{\tau_{im}})$: diagonal matrix with respect to J

h acts on the fiber $L|_{q_i}$ via multiplication by $\xi_p^{\mu_i}$

$$\implies \text{Ind}(D_L) = e^{c_1(L)} \text{Td}(TM)[M] , \quad \text{Ind}(D_L, h) = \sum_{i=1}^n \xi_p^{\mu_i} \prod_{j=1}^m \frac{1}{1 - \xi_p^{-\tau_{ij}}}$$

where Td is the Todd class.

2 Finite subgroup of the mapping class group

Γ_σ : mapping class group of genus σ ($\sigma \geq 2$)

Σ_σ : compact Riemann surface of genus σ ($\sigma \geq 2$)

In this section, G -action $\iff \exists J; G \subset \text{Aut}_J(\Sigma_\sigma) \xrightarrow{(*)} G \subset \Gamma_\sigma$.

(J : (integrable) almost complex structure, $(*)$: Nielsen realization)

$\mathbf{Z}_p = \langle g \rangle \subset \text{Aut}_J(\Sigma_\sigma)$

$\pi : \Sigma_\sigma \longrightarrow \Sigma_\sigma / \mathbf{Z}_p$: branched covering with b branch point

$y_1, \dots, y_b \in \Sigma_\sigma / \mathbf{Z}_p$ of order $(n_1, \dots, n_b)$

$\pi^{-1}(y_i) = \{q_i, g \cdot q_i, \dots, g^{r_i-1} \cdot q_i\}$: consists of $r_i := p/n_i$ points

For $1 \leq i \leq b$, assume that $g^{r_i} | T_{\pi^{-1}(y_i)} \Sigma_\sigma = \xi_{n_i}^{t_i}$ where $1 \leq t_i \leq n_i - 1$

and t_i is prime to n_i

$D_\ell : \otimes^\ell T\Sigma_\sigma$ -valued Dirac operator on Σ_σ

Theorem 2

$$I_{D_\ell}(g^z) - z I_{D_\ell}(g) = 0 \iff \varphi_{\ell,z}(t_1, \dots, t_b) \in \mathbf{Z},$$

$$N I_{D_\ell}(g^z) = 0 \iff N \psi_{\ell,z}(t_1, \dots, t_b) \in \mathbf{Z}$$

for any z ($1 \leq z < p$) which is prime to p and for any ℓ ($0 \leq \ell < p$) where

$$\begin{aligned} & \varphi_{\ell,z}(t_1, \dots, t_b) \\ &= (1-z) \frac{p-1}{2p} (1-\sigma)(2\ell+1) \\ & \quad - \sum_{i=1}^b \frac{1}{n_i} \sum_{j=1}^{n_i-1} \frac{1}{1-\xi_{n_i}^{-j}} \left(\frac{\xi_{n_i}^{jzt_i\ell}}{1-\xi_{n_i}^{-jzt_i}} - z \frac{\xi_{n_i}^{jt_i\ell}}{1-\xi_{n_i}^{-jt_i}} \right), \\ & I_{D_\ell}(g^k) \stackrel{\text{mod. } \mathbf{Z}}{\equiv} \psi_{\ell,z}(t_1, \dots, t_b) \\ &= \frac{p-1}{2p} (1-\sigma)(2\ell+1) - \sum_{i=1}^b \frac{1}{n_i} \sum_{j=1}^{n_i-1} \frac{\xi_{n_i}^{jzt_i\ell}}{(1-\xi_{n_i}^{-j})(1-\xi_{n_i}^{-jzt_i})} \end{aligned}$$

and N is a natural number such that $\det(D_\ell, g)^N = 1$.

Precise values of φ and ψ are obtained by using the following proposition.

Proposition 5

$12p I_{D_\ell}(g^z)$ is an integer and we have

$$\begin{aligned}
 & 12p I_{D_\ell}(g^z) \\
 & \equiv 6(p-1)(1-\sigma)(2\ell+1) \\
 & + \sum_{i=1}^b r_i \left\{ zt_i(n_i-1)(7n_i-11) + 6 \sum_{j=\left[\frac{(\ell+1)zt_i}{n_i}\right]+1}^{\left[\frac{(\ell+n_i+1)zt_i}{n_i}\right]} f_{n_i} \left(\left[\frac{jn_i-1}{zt_i} \right] - \ell - 1 \right) \right\} \\
 & \pmod{12p}
 \end{aligned}$$

where $f_{n_i}(x) = x^2 - (n_i-2)x - (n_i-1)^2$ and $[\]$ is the Gauss's symbol.

Riemann-Hurwitz equation

$$2\sigma - 2 = p(2\bar{\sigma} - 2) + \sum_{i=1}^b (p - r_i) \quad \left(r_i = \frac{p}{n_i} \right)$$

where $\bar{\sigma}$ is the genus of $\Sigma_\sigma/\mathbf{Z}_p$.

Example 1 The necessary and sufficient condition of (σ, p) for $\mathbf{Z}_p \subset \Gamma_\sigma$ is known. (W.J.Harvey, Cyclic groups of automorphisms of a compact Riemann surface, Quart. J. Math. **17**, 86-97 (1966))

We consider one hundred cases where $2 \leq \sigma, p \leq 11$.

Then $\mathbf{Z}_p \not\subset \Gamma_\sigma$ if and only if

$$\begin{aligned}
 & (\sigma, p) = (2, 7), (2, 11), (3, 11), (4, 11), (5, 7), (7, 11), (8, 11), (9, 11) \\
 & \implies \text{R-H equation is not satisfied for any } \bar{\sigma}, b, r_i
 \end{aligned}$$

or

$$(*1) \quad (\sigma, p) = (2, 9), (3, 5), (3, 10), (4, 7), (5, 9), (6, 11), (11, 7)$$

(Results of Harvey)

Using the R-H equation and Theorem 2, we can show that $\mathbf{Z}_p \not\subset \Gamma_\sigma$ for (σ, p) in (*1).

Example 2

G_p : finite non-Abelian group generated by $g, h_1, \dots, h_s$

$$\text{order}(g) = p, \text{order}(h_j) = q_j \ (1 \leq j \leq s)$$

Assume that $[G_p, G_p] \ni \exists \gamma$ such that γ is expressed as the product

$$\text{of } m \text{ } g\text{'s } (0 < m < p) \text{ and } \mu_j \text{ } h_j\text{'s } (0 \leq \mu_j < q_j, 1 \leq j \leq s)$$

$$\text{which satisfies that } d := \text{g.c.d.}(p, m\beta) < p$$

(where $\beta := \ell.c.m.(q_1^{\varepsilon_1} \dots q_s^{\varepsilon_s})$, $\varepsilon_j = 1$ if $\mu_j > 0$, $\varepsilon_j = 0$ if $\mu_j = 0$)

Assume moreover that $\text{Aut}_J(\Sigma_\sigma) \supset G_p$ ($2 \leq \sigma \leq 11$, $3 \leq p \leq 11$)

$$\implies \text{Aut}_J(\Sigma_\sigma) \supset \mathbf{Z}_p = \langle g \rangle$$

$\Sigma_\sigma \longrightarrow \Sigma_\sigma/\mathbf{Z}_p$: branched covering with b branch points

$$y_1, \dots, y_b \text{ of order } (n_1, \dots, n_b)$$

Then, $[G_p, G_p] \subset \ker \det(D_\ell, \cdot) \implies$

$$\begin{aligned} 1 = \det(D_\ell, \gamma^\beta) &= \det(D_\ell, g)^{m\beta} \iff \det(D_\ell, g)^d = 1 \\ \iff 0 &= dI_{D_\ell}(g^z) \equiv d\psi_{\ell,z}(t_1, \dots, t_b) \pmod{\mathbf{Z}} \\ &\left(\begin{array}{l} \exists t_i : \text{prime to } n_i \ (1 \leq i \leq b), \\ \forall z \ (1 \leq z < p, \ z : \text{prime to } p), \ \forall \ell \ (0 \leq \ell < p) \end{array} \right) \end{aligned}$$

R-H equation, results of Harvey $\implies$

$$\begin{aligned} (\sigma, p) = (2, 5) &\implies (b, \{n_1, \dots, n_b\}) = (3, \{5, 5, 5\}) \\ (\sigma, p) = (7, 5) &\implies (b, \{n_1, \dots, n_b\}) = (3, \{5, 5, 5\}) \\ (\sigma, p) = (3, 9) &\implies (b, \{n_1, \dots, n_b\}) = (3, \{3, 9, 9\}) \\ (\sigma, p) = (4, 9) &\implies (b, \{n_1, \dots, n_b\}) = (3, \{9, 9, 9\}) \\ (\sigma, p) = (11, 9) &\implies (b, \{n_1, \dots, n_b\}) = (5, \{3, 9, 9, 9, 9\}) \\ (\sigma, p) = (7, 10) &\implies (b, \{n_1, \dots, n_b\}) = (4, \{2, 10, 10, 10\}), (5, \{2, 2, 2, 5, 10\}) \\ (\sigma, p) = (5, 11) &\implies (b, \{n_1, \dots, n_b\}) = (3, \{11, 11, 11\}). \end{aligned}$$

Using Theorem 2, we have the following results.

$$\underline{p = 5, \sigma = 2, 7 \implies (b, \{n_1, \dots, n_b\}) = (3, \{5, 5, 5\})}$$

$$\psi_{1,1}(t_1, t_2, t_3) \notin \mathbf{Z} \ (\forall t_1, t_2, t_3 = 1, 2, 3, 4) \implies G_5 \not\subset \text{Aut}_J(\Sigma_2), \text{Aut}_J(\Sigma_7)$$

$$\underline{p = 9, \sigma = 3 \implies (b, \{n_1, \dots, n_b\}) = (3, \{3, 9, 9\})}$$

$$3\psi_{1,1}(t_1, t_2, t_3) \notin \mathbf{Z} \ (\forall t_1 = 1, 2, \forall t_2, t_3 = 1, 2, 4, 5, 7, 8) \implies G_9 \not\subset \text{Aut}_J(\Sigma_3)$$

$$\underline{p = 9, \sigma = 4 \implies (b, \{n_1, \dots, n_b\}) = (3, \{9, 9, 9\})}$$

$$3\psi_{1,1}(t_1, t_2, t_3) \notin \mathbf{Z} \ (\forall t_1, t_2, t_3 = 1, 2, 4, 5, 7, 8) \implies G_9 \not\subset \text{Aut}_J(\Sigma_4)$$

$$\underline{p = 9, \sigma = 11 \implies (b, \{n_1, \dots, n_b\}) = (5, \{3, 9, 9, 9, 9\})}$$

$$\begin{aligned} 3\psi_{1,1}(t_1, t_2, t_3, t_4, t_5) \notin \mathbf{Z} \ (\forall t_1 = 1, 2, \ , \forall t_2, t_3, t_4, t_5 = 1, 2, 3, 4) \\ \implies G_9 \not\subset \text{Aut}_J(\Sigma_{11}) \end{aligned}$$

$$\underline{p = 10, \sigma = 7 \implies (b, \{n_1, \dots, n_b\}) = (4, \{2, 10, 10, 10\}), (5, \{2, 2, 2, 5, 10\})}$$

$$\begin{aligned} \left\{ \begin{array}{l} k\psi_{1,1}(1, t_2, t_3, t_4) \notin \mathbf{Z} \ (k = 2, 5) \ (\forall t_2, t_3, t_4 = 1, 3, 7, 9) \\ k\psi_{1,1}(1, 1, 1, t_4, t_5) \notin \mathbf{Z} \ (k = 2, 5) \ (\forall t_4 = 1, 2, 3, 4, \forall t_5 = 1, 3, 7, 9) \end{array} \right. \\ \implies G_{10} \not\subset \text{Aut}_J(\Sigma_7) \end{aligned}$$

$$\underline{p = 11, \sigma = 5 \implies (b, \{n_1, \dots, n_b\}) = (3, \{11, 11, 11\})}$$

$$\begin{aligned} \{(t_1, t_2, t_3) \mid \psi_{1,1}(t_1, t_2, t_3) \in \mathbf{Z}\} \cap \{(t_1, t_2, t_3) \mid \psi_{2,1}(t_1, t_2, t_3) \in \mathbf{Z}\} = \emptyset \\ \implies G_{11} \not\subset \text{Aut}_J(\Sigma_5) \end{aligned}$$

Hence $G_5 \not\subset \Gamma_2, \Gamma_7$, $G_9 \not\subset \Gamma_3, \Gamma_4, \Gamma_{11}$, $G_{10} \not\subset \Gamma_7$, $G_{11} \not\subset \Gamma_5$

In particular,

$G_p = D(2p)$: dihedral group generated by g, h with $g^{-1}h^{-1}gh = g^{p-2}$

$\text{g.c.d}(p, p-2) < p \implies$

$D(2p) \not\subset \Gamma_\sigma$ if $(\sigma, p) = (2, 5), (7, 5), (3, 9), (4, 9), (11, 9), (7, 10), (5, 11)$

Note that $\sigma \equiv 0, 1 \pmod{p} \implies D(2p) \subset \Gamma_\sigma$.

3 0-pseudofree action of cyclic groups of prime order

Definition 1 An action is called “0-pseudofree” iff the exceptional orbits are isolated points. (E. Laitinen and P. Traczyk, Pseudofree representations and 2-pseudofree actions on spheres, Proc. Amer. Math. Soc. **97**, 151-157 (1986))
Here 0-pseudofree is simply called “pseudofree”.

$\text{Pf}(M)$: set of orientation-preserving isometries g on M

such that the fixed point set $\text{Fix}(g)$ consists of points

When $M = M^{2m}$ has a Spin^c -structure,

$\text{Pf}^c(M) := \{g \in \text{Pf}(M) \mid g \text{ preserves the } \text{Spin}^c\text{-structure}\}$

If $\mathbf{Z}_p = \langle g \rangle$, p : prime number

$\mathbf{Z}_p$ -action is pseudo-free $\iff \mathbf{Z}_p \subset \text{Pf}(M) \iff \text{Fix}(g) = \{q_1, q_2, \dots, q_n\}$: points

Theorem 3 Assume that $\mathbf{Z}_p \subset \text{Pf}(M)$ and suppose that $T_{q_i}M = \bigoplus_{j=1}^m N(\tau_{ij})$ ($1 \leq i \leq n$). Then using $pI_D(g) = 0$ (D : signature operator), we have

$$\sum_{k=1}^{\frac{p-1}{2}} \sum_{i=1}^n \prod_{j=1}^{2s} \cot \frac{\pi k \tau_{ij}}{p} \equiv 0 \pmod{\mathbf{Z}} \text{ if } m = 2s$$

$$\sum_{k=1}^{\frac{p-1}{2}} \cot \frac{\pi k}{p} \sum_{i=1}^n \prod_{j=1}^{2s-1} \cot \frac{\pi k \tau_{ij}}{p} \equiv 0 \pmod{\mathbf{Z}} \text{ if } m = 2s - 1$$

Corollary 1

$$\mathbf{Z}_3 \subset \text{Pf}(M) \implies n : \text{even} \quad \text{or} \quad n \geq 3^{[(m+1)/2]}$$

Assume that M has a Spin^c -structure and $\mathbf{Z}_p \subset \text{Pf}^c(M)$

L : (arbitrary) $\mathbf{Z}_p$ -line bundle, D_L : L -valued Dirac operator

Theorem 4 When $p = 2$, $n \geq 2^m$ unless n and $\text{Ind}(D_L) (\forall L)$ are even.

Theorem 5 When $p = 3$ or $p = 5$,

$$\begin{aligned} \gamma &= \min_{s \in \mathbf{Z}} \left| \frac{p-1}{2} \text{Ind}(D_L) - ps \right| : \text{distance from } \frac{p-1}{2} \text{Ind}(D_L) \text{ to } p\mathbf{Z} \\ \implies n &\geq \frac{\gamma}{3(p-1)} \left(2 \sin \frac{\pi}{p} \right)^{m+1} \\ \det(D_L, g) = 1 &\implies n \geq \frac{\gamma}{p-1} \left(2 \sin \frac{\pi}{p} \right)^{m+1} \end{aligned}$$

Example 4

$$M_{pk} = \Sigma_{pk} \times \overbrace{S^2 \times \cdots \times S^2}^{m-1 \text{ times}} \quad (p : \text{prime})$$

has a Spin^c -structure with

$$\begin{aligned} c_1(\eta) &= (2s + 2 - 2pk)y + \sum_{j=1}^{m-1} (2t_j + 2)z_j \\ &\in H^2(M_{pk}; \mathbf{Z}) \cong H^2(\Sigma_{pk}; \mathbf{Z}) \oplus \oplus_{j=1}^{m-1} H^2(S^2; \mathbf{Z}) \quad (s, t_j \in \mathbf{Z}) \end{aligned}$$

M_{pk} admits an pseudo-free Spin^c -action of $\mathbf{Z}_p$ with 2^m fixed points if the Spin^c -structure comes from the almost complex structure.

$\implies s, t_j$'s equal 0.

Since we can show that

$$\begin{aligned} \text{Ind}(D) &= (s + 1 - pk) \prod_{j=1}^{m-1} (t_j + 1) \equiv (s + 1) \prod_{j=1}^{m-1} (t_j + 1) \pmod{p}, \\ s, t_j \text{'s are even and } \mathbf{Z}_2 &\subset \text{Pf}^c(M_{2k}) \xrightarrow{\text{Theorem 4}} n \geq 2^m \\ (s + 1) \prod_{j=1}^{m-1} (t_j + 1) &\equiv 1, 2 \pmod{3}, \mathbf{Z}_3 \subset \text{Pf}^c(M_{3k}) \\ \implies \gamma = 1 &\xrightarrow{\text{Theorem 5}} n \geq \frac{1}{6} 3^{\frac{m+1}{2}} (\sim 0.866^m \cdot 2^m) \\ (s + 1) \prod_{j=1}^{m-1} (t_j + 1) &\equiv 1, 4 \pmod{5}, \mathbf{Z}_5 \subset \text{Pf}^c(M_{5k}) \\ \implies \gamma = 2 &\xrightarrow{\text{Theorem 5}} n \geq \frac{1}{6} \left(\frac{5 - \sqrt{5}}{8} \right)^{\frac{m+1}{2}} (\sim 0.294^m \cdot 2^m) \\ (s + 1) \prod_{j=1}^{m-1} (t_j + 1) &\equiv 2, 3 \pmod{5}, \mathbf{Z}_5 \subset \text{Pf}^c(M_{5k}) \\ \implies \gamma = 1 &\xrightarrow{\text{Theorem 5}} n \geq \frac{1}{12} \left(\frac{5 - \sqrt{5}}{8} \right)^{\frac{m+1}{2}} (\sim 0.294^m \cdot 2^m) \end{aligned}$$

Moreover $\mathbf{Z}_3 \subset \text{Pf}(M_{pk}) \xrightarrow{\text{Corollary 1}} n \neq 1, 3, 5, \dots, 3^{[(m+1)/2]} - 2$

Example 5

$$M = \mathbf{CP}^2 \times \mathbf{CP}^k \ (k \geq 3)$$

Assume that $\text{Pf}(M) \supset \mathbf{Z}_p = \langle g \rangle$ (p : odd prime)

$$H^2(M; \mathbf{Z}) = H^2(\mathbf{CP}^2; \mathbf{Z}) \oplus H^2(\mathbf{CP}^k; \mathbf{Z}) = \{\lambda x + \mu y \mid \lambda, \mu \in \mathbf{Z}\} = \mathbf{Z} \oplus \mathbf{Z}$$

where $H^2(\mathbf{CP}^2; \mathbf{Z}) \cong \mathbf{Z} = \langle x \rangle$ and $H^2(\mathbf{CP}^k; \mathbf{Z}) \cong \mathbf{Z} = \langle y \rangle$

$$g^*(x^2 y^k) = x^2 y^k \xLeftrightarrow{(*1)} g^* x = x, \ g^* y = y \xRightarrow{(*2)} \# \text{Fix}(g) = 3(k+1)$$

((*1) : not trivial, (*2) : Lefschetz fixed point theorem)

$$\left(\begin{array}{l} \text{For example, if } k < p, \\ \text{pseudo-free action of } \mathbf{Z}_p \text{ on } \mathbf{CP}^j \ (j = 2 \text{ or } k) \text{ defined by} \\ \quad g : [z_0 : z_1 : z_2 : \dots : z_j] \longrightarrow [z_0 : \xi_p z_1 : \xi_p^2 z_2 : \dots : \xi_p^j z_j] \\ \text{has } j+1 \text{ fixed points} \\ \implies \text{diagonal action of } \mathbf{Z}_p \text{ on } M \text{ has } 3(k+1) \text{ fixed points} \end{array} \right)$$

Almost complex structure J of $M \implies \text{Spin}^c$ -structure P_J of M

$$\implies c_1(\eta) = 3x + (k+1)y \implies g^* c_1(\eta) = c_1(\eta)$$

$H^1(M; \mathbf{Z}) = 0 \implies \mathbf{Z}_p$ -action lifts to an action on the Spin^c -structure P_J

We can show that $\text{Ind}(D) = 1$, and hence

$$\begin{aligned} & \gamma = 1 \ (p = 3), \ \gamma = 2 \ (p = 5) \text{ in Theorem 5} \implies \\ & \left\{ \begin{array}{l} 3(3-1) \cdot 3(k+1) \geq \left(2 \sin \frac{\pi}{3}\right)^{2+k+1} \iff k \leq 5 \quad (p = 3) \\ 3(5-1) \cdot 3(k+1) \geq 2 \left(2 \sin \frac{\pi}{5}\right)^{2+k+1} \iff k \leq 37 \quad (p = 5) \end{array} \right. \end{aligned}$$

$$k \geq 3 \implies 3(k+1) < 3^{[(3+k+1)/2]} \xrightarrow{\text{Corollary 1}} \mathbf{Z}_3 \not\subset \text{Pf}(M) \text{ if } k \text{ is even}$$

If $G_p \subset \text{Pf}^c(M)$ (p : odd prime), then $\det(D, g) = 1$ and hence

$$\begin{aligned} & \gamma = 1 \ (p = 3), \ \gamma = 2 \ (p = 5) \text{ in Theorem 5} \implies \\ & \left\{ \begin{array}{l} (3-1) \cdot 3(k+1) \geq \left(2 \sin \frac{\pi}{3}\right)^{2+k+1} \iff \nexists k \quad (p = 3) \\ (5-1) \cdot 3(k+1) \geq 2 \left(2 \sin \frac{\pi}{5}\right)^{2+k+1} \iff k \leq 29 \quad (p = 5) \end{array} \right. \end{aligned}$$