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Long-time behavior of solutions of Hamilton-Jacobi equations with convex and coercive Hamiltonians*

広島大学・大学院工学研究科 市原直幸 (Naoyuki Ichihara)[†]
Graduate School of Engineering,
Hiroshima University

概要

We establish general convergence results on the long-time behavior of viscosity solutions to Hamilton-Jacobi equations in $\mathbb{R}^n$ with convex and coercive Hamiltonians. We give three types of sufficient conditions so that the solution converges to a "steady state" as the time tends to infinity. Our approach is based on the variational representation formula for viscosity solutions of Hamilton-Jacobi equations.

1 Introduction and Preliminaries.

This paper is concerned with the Cauchy problem for the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\,\cdot\,, 0) = u_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{1}$$

where the Hamiltonian H satisfies the following conditions:

(A1) $H \in \mathrm{BUC}(\mathbb{R}^n \times B(0, R))$ for all $R > 0$, where $B(0, R) := \{x \in \mathbb{R}^n \mid |x| \leq R\}$,
(A2) $\inf\{H(x,p) \mid x \in \mathbb{R}^n,\ |p| \geq R\} \longrightarrow +\infty$ as $R \to +\infty$,
(A3) $H(x,p)$ is convex with respect to p for every $x \in \mathbb{R}^n$.

Note that the solvability of (1) in the sense of viscosity solution is well known. (See for instance Appendix A of [14] for the proof. See also [1, 7, 19] for the general theory of viscosity solutions.)

Theorem 1.1. *Assume (A1)-(A3). Then, for any $T > 0$ and $u_0 \in \mathrm{UC}(\mathbb{R}^n)$, there exists a viscosity solution $u \in \mathrm{UC}(\mathbb{R}^n \times (0,T))$ of $u_t + H(x, Du) = 0$ in $\mathbb{R}^n \times (0,T)$ satisfying $u(\,\cdot\,, 0) = u_0$ on $\mathbb{R}^n$. Moreover, the solution is unique in the class $\mathrm{UC}(\mathbb{R}^n \times [0,T])$ for every $T > 0$.*

The objective of this paper is to investigate the long-time behavior of the viscosity solution to (1). More precisely, we prove the convergence of the form

$$u(x,t) + at - \phi(x) \longrightarrow 0 \qquad \text{in } C(\mathbb{R}^n) \qquad \text{as } t \to \infty \tag{2}$$

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[†]E-mail: naoyuki@hiroshima-u.ac.jp. Supported in part by Grant-in-Aid for Young Scientists, No. 19840032, JSPS.

for some $a \in \mathbb{R}$ and $\phi \in C(\mathbb{R}^n)$, where $C(\mathbb{R}^n)$ is equipped with the topology of locally uniform convergence. Note that the function $\phi(x) - at$, called the asymptotic solution of (1), enjoys the following time-independent Hamilton-Jacobi equation in the viscosity sense:

$$H(x, D\phi) = a \qquad \text{in } \mathbb{R}^n. \tag{3}$$

We denote by $\mathcal{S}^-_{H-a}$ (resp. $\mathcal{S}^+_{H-a}$ and $\mathcal{S}_{H-a}$) the set of continuous viscosity subsolutions (resp. supersolutions and solutions) of (3). Observe here that if there exists an $a \in \mathbb{R}$ such that $\phi_0 \le u_0 \le \psi_0$ in $\mathbb{R}^n$ for some $\phi_0 \in \mathcal{S}^-_{H-a}$ and $\psi_0 \in \mathcal{S}^+_{H-a}$, then in view of the standard comparison theorem, wee see that

$$t^{-1}u(\,\cdot\,, t) \longrightarrow -a \qquad \text{in } C(\mathbb{R}^n) \qquad \text{as } t \to \infty. \tag{4}$$

Our interest is, therefore, to investigate asymptotics of the next order.

In this paper, we deal with the case where $a = 0$, namely, we assume that

(A4) there exist $\phi_0 \in \mathcal{S}^-_H$ and $\psi_0 \in \mathcal{S}^+_H$ such that $\phi_0 \le \psi_0$ in $\mathbb{R}^n$,

and prove the convergence $u(\,\cdot\,, t) \longrightarrow \phi$ in $C(\mathbb{R}^n)$ as $t \to \infty$ for any given initial function u_0 in the class

$$\Phi_0 := \{u_0 \in \mathrm{UC}(\mathbb{R}^n) \mid \phi_0 - C \le u_0 \le \psi_0 + C \text{ in } \mathbb{R}^n \text{ for some } C > 0\},$$

where ϕ may depend on the choice of u_0. Notice here that assuming $a = 0$ is not a real restriction. Indeed, once (4) is established, (2) can be reduced to the case where $a = 0$ by considering $H - a$ and $u(x, t) + at$ instead of H and $u(x, t)$, respectively.

The study on asymptotic problems of this type has been developed especially in the last decade. As one of the most typical cases, it was proved that if H satisfies (A1), (A2), and $H(x, p)$ is $\mathbb{Z}^n$-periodic with respect to x and is strictly convex with respect to p, then there exists a unique $a \in \mathbb{R}$ such that (2) is valid for every $\mathbb{Z}^n$-periodic initial function $u_0 \in \mathrm{BUC}(\mathbb{R}^n)$. We refer to the literatures [3, 5, 8, 9, 10, 20, 21, 22, 23] and references therein for more details. Remark that [3] deals with non-convex Hamiltonians whereas the others are concerned only with convex ones.

It has also been of interest in recent years on the long-time behavior of viscosity solutions to (1) that are not necessarily spatially periodic. As far as non-periodic solutions are concerned, the above (A1)-(A4) are insufficient to obtain the convergence (2) for every $u_0 \in \Phi_0$ even if we admit strict convexity for H in any sense (see [4, 14]). The papers [2, 12, 14, 17] deal with some situations in which the solution of (1) has indeed the required convergence of the form (2) for suitable (a, ϕ).

Motivated by these earlier results, we established in [16], on which this paper is based, general convergence results for the solution of (1) which, on the one hand, cover most of existing results, and, on the other hand, involve a few observations which seem to be new. The first one is concerned with strict convexity for H. As pointed out in several literatures, it is necessary in some situations to require a sort of strict convexity for H so that the solution of (1) converges to an asymptotic solution as $t \to \infty$. In the present paper, we use condition $(\mathrm{A5})_+$ or $(\mathrm{A5})_-$ which guarantees, respectively, strict convexity of $H(x, p)$ in p uniformly in the sets $\{H \ge 0\}$ or $\{H \le 0\}$ (see Section 2 for their precise definitions). We point out here that in spite of our convexity assumption (A3), the latter condition is not covered by [3] in which convergence of the type (2) is obtained in the periodic case under fairly weak assumptions on H.

The second observation is discussed in connection with our dynamical approach basing on the following classical variational formula:

$$u(x,t) = \inf\Big\{ \int_{-t}^{0} L(\eta(s),\dot\eta(s))\,ds + u_0(\eta(-t)) \;\Big|\; \eta \in \mathcal{C}([-t,0];x)\Big\}, \tag{5}$$

where $L(x,\xi) := \sup_{p\in\mathbb{R}^n}(p\cdot\xi - H(x,p))$ and $\mathcal{C}([-t,0];x) := \{\eta \in \mathrm{AC}([-t,0],\mathbb{R}^n) \mid \eta(0) = x\}$, and we denote by $\mathrm{AC}([-t,0],\mathbb{R}^n)$ the set of curves $\eta : [-t,0] \longrightarrow \mathbb{R}^n$ being absolutely continuous on $[-s,0]$ for all $0 < s \le t$. It is standard to see that the function $u(x,t)$ defined by (5) is indeed the viscosity solution of (1). It will be revealed in Section 3 that, for each $x \in \mathbb{R}^n$, solutions, say $\eta^{(t)}$, of the variational problem in the right-hand side of (5) possess a distinctive behavior as $t \to \infty$ called "swich-back", from which we obtain a new type of convergence result. As far as we know, such a motion in connection with the asymptotic behavior of solutions of (1) was not studied before.

One other novelty of this paper (and thus that of [16]) is related to Hamiltonians and initial data with "weak" periodicity. In Section 4, we give some results which particularly extend [14] studying Hamilton-Jacobi equations with semi-periodic Hamiltonians and semi-almost periodic initial data. See also [13] for some information in this direction.

In the rest of this introductory section, we briefly sketch the procedure for the proof of (2) (see also [14]). Let $(T_t)_{t\ge0}$ be the nonlinear semigroup on $\mathrm{UC}(\mathbb{R}^n)$ defined by $(T_tu_0)(x) := u(x,t)$, where $u(x,t)$ is the solution of the Cauchy problem (1). For a given $u_0 \in \Phi_0$, we set

$$u_0^-(x) := \sup\{\,\phi(x) \mid \phi \in \mathcal{S}_H^-,\ \phi \le u_0 \text{ in } \mathbb{R}^n\},\quad u_\infty(x) := \inf\{\,\psi(x) \mid \psi \in \mathcal{S}_H,\ \psi \ge u_0^- \text{ in } \mathbb{R}^n\}.$$

Then, it follows that $u_0^- \in \mathcal{S}_H^-$ and $u_\infty \in \mathcal{S}_H^+$ by standard arguments in the viscosity solution theory. It is also well known (e.g. [8, 11, 17]) that u_0^- can be represented as

$$u_0^-(x) = \inf\{d_H(x,y) + u_0(y) \mid y \in \mathbb{R}^n\},\quad x \in \mathbb{R}^n, \tag{6}$$

where d_H is defined by

$$d_H(x,y) := \sup\{\phi(x) - \phi(y) \mid \phi \in \mathcal{S}_H^-\}. \tag{7}$$

Note that $d_H(\,\cdot\,,y) \in \mathcal{S}_H^-$ for all $y \in \mathbb{R}^n$ and d_H can be written as

$$d_H(x,y) = \inf\Big\{ \int_{-t}^{0} L(\eta(s),\dot\eta(s))\,ds \;\Big|\; t>0,\ \eta \in \mathcal{C}([-t,0];x),\ \eta(-t) = y\Big\}. \tag{8}$$

Moreover, we can show the following lemma (see Lemma 4.1 of [14] for the proof).

Lemma 1.2. *Assume (A1)-(A4). Then, for every $u_0 \in \Phi_0$, one has $u_\infty \in \mathcal{S}_H$ and*

$$(T_tu_0^-)(x) = \inf_{s\ge t} u(x,s), \qquad u_\infty(x) = \liminf_{t\to\infty} u(x,t).$$

Hence, the problem is reduced to proving the convergence

$$T_tu_0 \longrightarrow u_\infty \quad \text{in } C(\mathbb{R}^n) \text{ as } t \to \infty. \tag{9}$$

Now, for a fixed $x \in \mathbb{R}^n$, we set $u^+(x) := \limsup_{t\to\infty} u(x,t)$ and choose any diverging sequence $\{t_j\}_j \subset (0,\infty)$ such that $u^+(x) = \lim_{j\to\infty} u(x,t_j)$. The rough idea of showing (9) is to find a family of curves $\mu_j \in \mathcal{C}([-t_j,0];x)$, $j \in \mathbb{N}$, such that

$$u_\infty(x) \geq \lim_{j\to\infty} \left(\int_{-t_j}^0 L(\mu_j(s),\dot\mu_j(s))\,ds + u_0(\mu_j(-t_j)) \right). \tag{10}$$

If (10) is true for some $\{\mu_j\}$, then in view of (5),

$$u^+(x) = \lim_{j\to\infty} u(x,t_j) \leq \lim_{j\to\infty} \left(\int_{-t_j}^0 L(\mu_j(s),\dot\mu_j(s))\,ds + u_0(\mu_j(-t)) \right) \leq u_\infty(x),$$

from which we conclude that $u(x,t) \longrightarrow u_\infty(x)$ as $t \to \infty$ for all $x \in \mathbb{R}^n$. We remark here that, under our assumptions (A1)-(A4), the above pointwise convergence yields locally uniform convergence (9) (e.g. [17] for its justification). Observe also that μ_j can be regarded, up to a small error, as a minimizer of the right-hand side of (5) with $t = t_j$ for each $j \in \mathbb{N}$. In the following sections, we divide our consideration into several situations according to the type of $\{\mu_j\}$.

In any case, the so-called extremal curves play an important role. Recall that for given $x \in \mathbb{R}^n$ and $\phi \in \mathcal{S}_H$, a curve $\gamma \in \mathcal{C}((-\infty,0];x)$ is said an extremal curve for ϕ at x if it satisfies

$$\phi(x) = \int_{-t}^0 L(\gamma(s),\dot\gamma(s))\,ds + \phi(\gamma(-t)) \quad \text{for all } \ t > 0. \tag{11}$$

The existence of such curves is guaranteed by Lemma 3.3 of [14]. We denote by $\mathcal{E}_x(\phi)$ the set of all extremal curves for ϕ at x. We often use the notation $\mathcal{E}_x := \mathcal{E}_x(u_\infty)$ for simplicity of notation.

This paper is organized as follows. In the next section, we establish a theorem which covers, as particular cases, some results of Barles-Roquejoffre [2] and Ishii [17]. At the end of Section 2, we also discuss the relationship between the long-time behavior of extremal curves and ideal boundaries studied in Ishii-Mitake [18]. In Sections 3, we treat a class of Hamiltonians that provide switch-back motions for μ_j. Section 4 is devoted to establishing some results concerning the long-time behavior of viscosity solutions of Hamilton-Jacobi equations with weak periodicity. Several examples are given in the final sention.

2 First convergence result.

Let H satisfy (A1)-(A4) and let $u_0 \in \Phi_0$. We begin this section with a few simple lemmas.

Lemma 2.1. *Suppose that for every $x \in \mathbb{R}^n$, there exists a $\gamma \in \mathcal{E}_x$ such that*

$$\lim_{t\to\infty} (u_0 - u_\infty)(\gamma(-t)) = 0. \tag{12}$$

Then, the convergence (9) holds.

Proof. Let $\gamma \in \mathcal{E}_x$ satisfy (12). By the definition of extremal curves and the variational formula (5), we see that

$$u(x,t) \le \int_{-t}^{0} L(\gamma(s),\dot\gamma(s))\,ds + u_0(\gamma(-t)) = u_\infty(x) - u_\infty(\gamma(-t)) + u_0(\gamma(-t))$$

for all $t > 0$. In view of (12) and Lemma 1.2, we conclude that

$$\limsup_{t\to\infty} u(x,t) \le u_\infty(x) + \lim_{t\to\infty}(u_0 - u_\infty)(\gamma(-t)) = u_\infty(x) = \liminf_{t\to\infty} u(x,t),$$

which implies (9). □

We next prove that if H satisfies a sort of strict convexity, then (12) is not necessarily needed for extremal curves $\gamma = \{\gamma(-t)\,|\,t>0\}$ bounded in $\mathbb{R}^n$. We set $Q := \{(x,p)\in\mathbb{R}^{2n}\,|\,H(x,p)=0\}$ and

$$S := \{(x,\xi)\in\mathbb{R}^{2n}\,|\,(x,p)\in Q,\ \ \xi\in D_2^- H(x,p) \ \text{ for some } p\in\mathbb{R}^n\},$$

where $D_2^- H(x,p)$ stands for the subdifferential of H with respect to the p-variable. In what follows, we use the following assumption:

(A5)$_+$ (resp. **(A5)$_-$**) There exists a modulus ω satisfying $\omega(r) > 0$ for $r > 0$ such that for all $(x,p)\in Q$, $\xi\in D_2^- H(x,p)$ and $q\in\mathbb{R}^n$,

$$H(x,p+q) \ge \xi\cdot q + \omega((\xi\cdot q)_+) \qquad (\text{resp.}\quad \ge \xi\cdot q + \omega((\xi\cdot q)_-)), \tag{13}$$

where $r_\pm := \max\{\pm r, 0\}$ for $r\in\mathbb{R}$.

Roughly speaking, (A5)$_+$ (resp. (A5)$_-$) means that $H(x,\cdot\,)$ is strictly convex on the set $\{p\in\mathbb{R}^n\,|\,H(x,p)\ge 0\}$ (resp. $\{p\in\mathbb{R}^n\,|\,H(x,p)\le 0\}$) uniformly in $x\in\mathbb{R}^n$. Notice here that condition (A5)$_-$ has been discussed in [15] when $n=1$. This strict convexity yields the following property for L.

Lemma 2.2. *Let H satisfy (A1)-(A4) and (A5)$_+$ (resp. (A5)$_-$). Then, there exists a constant $\delta_1 > 0$ and a modulus ω_1 such that for any $\varepsilon\in[0,\delta_1]$ (resp. $\varepsilon\in[-\delta_1,0]$) and $(x,\xi)\in S$,*

$$L(x,(1+\varepsilon)\xi) \le (1+\varepsilon)L(x,\xi) + |\varepsilon|\omega_1(|\varepsilon|). \tag{14}$$

Proof. The proof of (14) under (A5)$_+$ is exactly the same as that of Lemma 3.2 in [14]. Moreover, by a careful review of its proof, we see that (14) is also true under (A5)$_-$. □

Remark 2.3. The estimate of this type was proved first by [8] when $H(x,\cdot\,)$ is strictly convex.

Proposition 2.4. *Let H satisfy (A1)-(A4) and one of (A5)$_+$ or (A5)$_-$. Let $u_0\in\Phi_0$, $x\in\mathbb{R}^n$ and $\gamma\in\mathcal{E}_x$, and suppose that $u^+(x) = \lim_{j\to\infty} u(x,t_j)$ and $\sup_j|\gamma(-t_j)| < \infty$ for some diverging sequence $\{t_j\}\subset(0,\infty)$. Then, $u^+(x)\le u_\infty(x)$.*

Proof. Fix any $\delta > 0$ and set $x_j := \gamma(-t_j)$ for $j\in\mathbb{N}$. By taking a subsequence if necessary, we may assume that $x_j\to y$ as $j\to\infty$ for some $y\in\mathbb{R}^n$.

In view of coercivity (A2), we see that $\{u(\cdot\,,t)\,|\,t>0\}$ is equi-continuous on $\mathbb{R}^n$ and u_0^- and u_∞ are Lipschitz continuous on $\mathbb{R}^n$. In particular, there exists an $\varepsilon > 0$ such that $|x-x'|<\varepsilon$ implies

$$|u(x,t) - u(x',t)| + |u_0^-(x) - u_0^-(x')| + |u_\infty(x) - u_\infty(x')| < \delta \tag{15}$$

for every $t > 0$. In what follows, we fix such $\varepsilon > 0$ and assume that $|x_j - y| < \varepsilon$ for all $j \in \mathbb{N}$.

We first assume (A5)$_+$ and show that $u^+(x) \le u_\infty(x)$. Fix a $\tau > 0$ so that $u_0^-(y)+\delta > u(y,\tau)$. For each $j \in \mathbb{N}$, we set $\varepsilon_j := (t_j - \tau)^{-1}\tau$ and define $\gamma_j \in \mathcal{C}((-\infty,0];x)$ by $\gamma_j(s) := \gamma((1+\varepsilon_j)s)$. Then, from (5), (14) and the fact that $(\gamma(s),\dot\gamma(s)) \in S$ for a.e. $s \in (-\infty,0)$, we have

$$u(x,t_j) \le \int_{-t_j+\tau}^0 L(\gamma_j(s),\dot\gamma_j(s))\,ds + u(x_j,\tau) < u_\infty(x) - u_\infty(x_j) + t_j\varepsilon_j\,\omega_1(\varepsilon_j) + u(y,\tau) + \delta$$
$$\le u_\infty(x) - u_\infty(y) + t_j\varepsilon_j\,\omega_1(\varepsilon_j) + u_0^-(y) + 3\delta \le u_\infty(x) + t_j\varepsilon_j\,\omega_1(\varepsilon_j) + 3\delta.$$

By letting $j \to \infty$ and then $\delta \to 0$, we obtain $u^+(x) \le u_\infty(x)$.

We next assume (A5)$_-$. Observe from (5) and (15) that

$$u(x,t_j) \le \int_{-t_1}^0 L(\gamma(s),\dot\gamma(s))\,ds + u(x_1,t_j-t_1)$$
$$< u_\infty(x) - u_\infty(x_1) + u(x_2,t_j-t_1) + 2\delta < u_\infty(x) - u_\infty(y) + u(x_2,t_j-t_1) + 3\delta.$$

By renumbering $\{t_j\}$ if necessary, we may assume that $t_2 > t_1 + \tau$. For each $j \in \mathbb{N}$, we set

$$\varepsilon_j = \frac{t_2 - t_1 - \tau}{t_j - t_1 - \tau}, \qquad \gamma_j(s) = \gamma(-t_2 + (1-\varepsilon_j)s), \quad s \le 0.$$

Since $\varepsilon_j \to 0$ as $j \to 0$, we may assume that $\varepsilon_j \in (0,\delta_1)$ for all $j \in \mathbb{N}$, where δ_1 is the constant taken from Lemma 2.2. Then, in view of (15) and the fact that $u_0^-(y) + \delta > u(y,\tau)$, we see that

$$u(x_2,t_j-t_1) \le \int_{-t_j+t_1+\tau}^0 L(\gamma_j(s),\dot\gamma_j(s))\,ds + u(x_j,\tau)$$
$$< u_\infty(x_2) - u_\infty(x_j) + t_j\varepsilon_j\omega_1(\varepsilon_j) + u(y,\tau) + \delta < t_j\varepsilon_j\omega_1(\varepsilon_j) + u_0^-(y) + 4\delta.$$

Thus, we have

$$u(x,t_j) < u_\infty(x) - u_\infty(y) + u(x_2,t_j-t_1) + 3\delta$$
$$< u_\infty(x) - u_\infty(y) + t_j\varepsilon_j\omega_1(\varepsilon_j) + u_0^-(y) + 7\delta < u_\infty(x) + t_j\varepsilon_j\omega_1(\varepsilon_j) + 7\delta.$$

By letting $j \to \infty$ and then $\delta \to 0$, we get $u^+(x) \le u_\infty(x)$. □

We are now in position to state the main theorem of this section. For a given $\phi \in \mathcal{S}_H$, we define the set $\Lambda(\phi)$ by

$$\Lambda(\phi) := \{\,\{\gamma(-t_j)\}_j \subset \mathbb{R}^n \mid \gamma \in \mathcal{E}_x(\phi) \text{ and } |\gamma(-t_j)| \longrightarrow \infty \text{ as } j \to \infty\}. \tag{16}$$

In what follows, we set $\Lambda := \Lambda(u_\infty)$ if there is no confusion.

Theorem 2.5. *Let H satisfy (A1)-(A4) and one of (A5)$_+$ or (A5)$_-$, and let $u_0 \in \Phi_0$. Then, the convergence (9) holds provided that*

$$\lim_{j\to\infty}(u_0 - u_\infty)(x_j) = 0 \qquad \textit{for all } \{x_j\} \in \Lambda. \tag{17}$$

Proof. Fix any $x \in \mathbb{R}^n$ and any diverging $\{t_j\}$ such that $u^+(x) = \lim_{j\to\infty} u(x, t_j)$. We take an arbitrary $\gamma \in \mathcal{E}_x$ and set $x_j = \gamma(-t_j)$ for $j \in \mathbb{N}$. If $\lim_{j\to\infty} |x_j| = \infty$, then we get $u^+(x) \le u_\infty(x)$ by Lemma 2.1 and (17). On the other hand, if $\liminf_{j\to\infty} |x_j| < \infty$, then by taking a subsequence if necessary, we may assume that $\sup_{j\in\mathbb{N}} |x_j| < \infty$. Thus, we can apply Proposition 2.4 to get the same inequality. □

As an easy consequence of Theorem 2.5, we obtain the following convergence result which covers, as typical cases, Theorem 4.2 of [2] and (a version of) Theorem 1.3 in [17] (see also Remark 2.10 below).

Theorem 2.6. *Let H satisfy (A1)-(A4) and $u_0 \in \Phi_0$. Let $\psi \in \mathrm{Lip}(\mathbb{R}^n)$ and $\sigma \in C(\mathbb{R}^n)$ be such that*

$$H(x, D\psi(x)) \le -\sigma(x) \qquad a.e. \quad x \in \mathbb{R}^n. \tag{18}$$

Then, one has the convergence (9) provided one of the following (a) or (b) holds:
(a) $\sigma(x) > 0$ for all $x \in \mathbb{R}^n$ and condition (17),
(b) $(A5)_+$ or $(A5)_-$, and

$$\sigma \ge 0 \ \ in \ \ \mathbb{R}^n \setminus B(0,R) \ \ for\ some \ \ R > 0 \quad and \quad \lim_{|x|\to\infty} (\phi_0 - \psi)(x) = \infty.$$

Remark 2.7. Let $\mathcal{A}_H \subset \mathbb{R}^n$ be the Aubry set for H, i.e., $\mathcal{A}_H := \{y \in \mathbb{R}^n \mid d_H(\cdot, y) \in \mathcal{S}_H\}$. Then, we see that condition (a) yields $\mathcal{A}_H = \emptyset$. On the other hand, condition (b) implies that $\mathcal{A}_H$ is non-empty and compact.

Before proving Theorem 2.6, we point out the following facts.

Lemma 2.8. *Let H satisfy (A1)-(A4) and $u_0 \in \Phi_0$. Let $D \subset \mathbb{R}^n$ be an open set and suppose that there exist $\delta > 0$ and $\psi \in \mathcal{S}_H^-$ such that $\sup_D |\psi - \phi_0| < \infty$ and*

$$H(x, D\psi(x)) \le -\delta \qquad a.e.\ x \in D. \tag{19}$$

Then, for any $\varepsilon > 0$, $x \in D$ and $\gamma \in \mathcal{E}_x$, there exists a $\tau > 0$ such that $\gamma(-t) \notin D_\varepsilon$ for all $t \ge \tau$, where $D_\varepsilon := \{x \in D \mid \mathrm{dist}(x, D^c) > \varepsilon\}$.

Proof. Fix any $\varepsilon > 0$, $x \in D$ and $\gamma \in \mathcal{E}_x$. Observe that $\sup_{t>0} |(u_\infty - \phi_0)(\gamma(-t))| < \infty$. Indeed, for every $t > s \ge 0$, we have

$$\phi_0(\gamma(-s)) - \phi_0(\gamma(-t)) \le \int_{-t}^{-s} L(\gamma(r), \dot\gamma(r))\, dr = u_\infty(\gamma(-s)) - u_\infty(\gamma(-t)),$$

which implies that the function $t \mapsto (u_\infty - \phi_0)(\gamma(-t))$ is non-increasing on $[0, \infty)$. Since $\inf_{\mathbb{R}^n}(u_\infty - \phi_0) > -\infty$, we conclude that $\sup_{t>0} |(u_\infty - \phi_0)(\gamma(-t))| < \infty$.

Next, we claim that for any $s > 0$, there exists a $t > s$ such that $\gamma(-t) \notin D$. Indeed, suppose that $\gamma(-t) \in D$ for all $t > s$. Then, in view of (19), for every $t > s$,

$$\psi(\gamma(-s)) - \psi(\gamma(-t)) + \int_{-t}^{-s} \delta\, dr \le \int_{-t}^{-s} L(\gamma(r), \dot\gamma(r))\, dr = u_\infty(\gamma(-s)) - u_\infty(\gamma(-t)).$$

Since $\sup_D |\psi - \phi_0| < \infty$ by assumption, we have

$$\delta(t - s) \le 2 \sup_{r>0} |(u_\infty - \phi_0)(\gamma(-r))| + 2 \sup_{y\in D} |(\phi_0 - \psi)(y)| \qquad \text{for all } \ t > s.$$

By letting $t \to \infty$, we get the contradiction. Thus, we can choose a diverging $\{t_j^+\} \subset (0,\infty)$ such that $\gamma(-t_j^+) \notin D$ for all $j \in \mathbb{N}$.

We now show that there exists a $\tau > 0$ such that $\gamma(-t) \notin D_\varepsilon$ for all $t \geq \tau$. We argue by contradiction. Suppose that there exists a diverging $\{t_j^-\} \subset (0,\infty)$ such that $\gamma(-t_j^-) \in D_\varepsilon$ for all $j \in \mathbb{N}$. By renumbering $\{t_j^+\}$ and $\{t_j^-\}$ if necessary, we may assume that $t_j^- < t_j^+ < t_{j+1}^-$ for all $j \in \mathbb{N}$.

We take any $A > 0$. Then, there exists a $C_A > 0$ such that

$$L(x,\xi) - q \cdot \xi \geq A|\xi| - C_A \qquad \text{for all } (x,\xi) \in \mathbb{R}^{2n} \text{ and } q \in B(0,A). \tag{20}$$

Indeed, by setting $C_A := \sup\{|H(x,p)| \mid x \in \mathbb{R}^n,\ p \in B(0,2A)\}$, we have

$$L(x,\xi) = \sup_{p\in\mathbb{R}^n} \{\xi \cdot p - H(x,p)\} \geq \xi \cdot (q + A|\xi|^{-1}\xi) - H(x, q + A|\xi|^{-1}\xi) \geq q \cdot \xi + A|\xi| - C_A$$

for every $x \in \mathbb{R}^n$, $\xi \neq 0$ and $q \in B(0,A)$. On the other hand, we observe that

$$\psi(\gamma(-s)) - \psi(\gamma(-t)) = \int_{-t}^{-s} q(r) \cdot \dot\gamma(r)\, dr \qquad \text{for all } t > s \geq 0 \tag{21}$$

for some $q \in L^\infty(-\infty,0;\mathbb{R}^n)$ satisfying $q(r) \in \partial_c\psi(\gamma(r))$ for a.e. $r \in (-\infty,0]$, where $\partial_c\psi(z)$ stands for the Clarke differential of ψ at $z \in \mathbb{R}^n$, namely,

$$\partial_c\psi(z) := \bigcap_{r>0} \overline{\mathrm{co}}\{\, D\psi(y) \mid y \in B(z,r),\ \phi \text{ is differentiable at } y\,\}.$$

In view of (20) and (21), we obtain

$$\begin{aligned}\int_{-t}^{-s} (A|\dot\gamma(r)| - C_A)\, dr &\leq \int_{-t}^{-s} L(\gamma(r),\dot\gamma(r))\, dr - (\psi(\gamma(-s)) - \psi(\gamma(-t))) \\ &= (u_\infty - \psi)(\gamma(-s)) - (u_\infty - \psi)(\gamma(-t)).\end{aligned}$$

Now, for each $j \in \mathbb{N}$, we set $\tau_j^- := \inf\{t > t_j^- \mid \gamma(-t) \notin D\}$, $\tau_j^+ := \sup\{t < t_{j+1}^- \mid \gamma(-t) \notin D\}$, and choose any a, $b > 0$ such that $(a,b) \subset (-\tau_j^-, -t_j^-)$ or $(a,b) \subset (-t_{j+1}^-, -\tau_j^+)$ for some $j \in \mathbb{N}$. Since $\gamma((a,b)) \subset D$, we see that

$$\int_a^b |\dot\gamma(s)|\, ds \leq A^{-1}C_A(b-a) + 2A^{-1}\sup_D |u_\infty - \psi|.$$

Fix an $A > 0$ so large that $2A^{-1}\sup_D |u_\infty - \psi| < \varepsilon/2$. Then, we see that for all $j \in \mathbb{N}$,

$$\varepsilon \leq \int_{-\tau_j^-}^{-t_j^-} |\dot\gamma(s)|\, ds \leq \frac{\varepsilon}{2} + A^{-1}C_A(\tau_j^- - t_j^-), \quad \varepsilon \leq \int_{-t_{j+1}^-}^{-\tau_j^+} |\dot\gamma(s)|\, ds \leq \frac{\varepsilon}{2} + A^{-1}C_A(t_{j+1}^- - \tau_j^+).$$

From these estimates, for any $N \in \mathbb{N}$, we have

$$\begin{aligned}2\sup_D |u_\infty - \psi| &\geq (u_\infty - \psi)(\gamma(-t_1^-)) + (u_\infty - \psi)(\gamma(-t_{N+1}^-)) \\ &\geq \sum_{j=1}^N \left(\int_{-\tau_j^-}^{-t_j^-} + \int_{-t_{j+1}^-}^{-\tau_j^+} \right) \delta\, ds \geq \delta A C_A^{-1} \varepsilon N.\end{aligned}$$

By letting $N \to \infty$, we get the contradiction. Hence, we conclude that $\gamma(-t) \notin D_\varepsilon$ for all $t \geq \tau$ for some $\tau > 0$. □

Lemma 2.9. *Assume (A1)-(A4) and let $u_0 \in \Phi_0$. Assume also (b) in Theorem 2.6. Then, the set $\{\gamma(-t) \mid t > 0\}$ is bounded in $\mathbb{R}^n$ for every $\gamma \in \mathcal{E}_x$.*

Proof. Observe first that $u_\infty \geq \phi_0 - C$ in $\mathbb{R}^n$ for some $C > 0$. Then, in view of (18), we see that for every $t > 0$,

$$\psi(x) - \psi(\gamma(-t)) + \int_{-t}^{0} \sigma(\gamma(s))\, ds \leq \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds \leq u_\infty(x) - \phi_0(\gamma(-t)) + C.$$

Thus,

$$(\phi_0 - \psi)(\gamma(-t)) + \int_{-t}^{0} \sigma(\gamma(s))\, ds \leq (u_\infty - \psi)(x) + C \qquad \text{for all } \ t > 0.$$

From this and property (b), we conclude that the set $\{\gamma(-t) \mid t > 0\}$ is bounded. □

Proof of Theorem 2.6. We assume (a). Notice from Lemma 2.8 that $|\gamma(-t)| \longrightarrow \infty$ as $t \to \infty$ for every $\gamma \in \mathcal{E}_x$. Thus, in view of (17) and Lemma 2.1, we get the convergence (9).

Assume next that (b) holds. Then, by Lemma 2.9, $\sup_{t>0} |\gamma(-t)| < \infty$ for any $\gamma \in \mathcal{E}_x$. Thus, we can apply Proposition 2.4 to obtain the convergence (9). □

Remark 2.10. Theorem 2.6 with (a) generalizes Theorem 4.2 of Barles-Roquejoffre [2]. In our context, their assumption is equivalent to say that the function σ in (18) satisfies $\sigma \geq \delta$ in $\mathbb{R}^n$ for some $\delta > 0$ and

$$\lim_{|x|\to\infty} (u_0 - u_\infty)(x) = 0. \tag{22}$$

Remark that (22) is strictly stronger than (17). We discuss this point in Example 5.1.

Another remark is that Theorem 2.6 with (b) is a version of Theorem 1.3 of [17] in which the following condition is imposed in addition to the whole strict convexity of H:
There exist $\phi_i \in C^{0+1}(\mathbb{R}^n)$ and $\sigma_i \in C(\mathbb{R}^n)$ with $i = 0, 1$ such that for $i = 0, 1$,

$$H(x, D\phi_i(x)) \leq -\sigma_i(x) \quad \text{a.e. } x, \qquad \lim_{|x|\to\infty} \sigma_i(x) = \infty, \qquad \lim_{|x|\to\infty} (\phi_0 - \phi_1)(x) = \infty. \tag{23}$$

Notice here that the second condition in (23) can be replaced with $\sigma_i \geq 0$ in $\mathbb{R}^n$ once we have shown $t^{-1}u(x,t) \longrightarrow 0$ as $t \to \infty$.

Remark 2.11. In Theorem 2.6, the family of minimizing curves $\{\mu_j\}$ in the right-hand side of (5) with $t = t_j$ for each $j \in \mathbb{N}$ can be constructed as follows. We first consider (a). In this case, it suffices to set $\mu_j(s) = \gamma(s)$, $s \in [-t_j, 0]$, for each $j \in \mathbb{N}$. In particular, we find that $|\mu_j(-t_j)| = |\gamma(-t_j)| \longrightarrow \infty$ as $j \to \infty$.

We next consider (b). For simplicity, we only deal with the case where $(A5)_+$ holds. For $j \in \mathbb{N}$, we choose $\eta_j \in \mathcal{C}([-\tau, 0]; x_j)$ such that

$$u(x_j, \tau) + \delta > \int_{-\tau}^{0} L(\eta_j(s), \dot\eta_j(s))\, ds + u_0(\eta_j(-\tau)),$$

where $\tau > 0$ is the number taken in Theorem 2.4. Then, the curve $\mu_j \in \mathcal{C}([-t_j, 0]; x)$ can be constructed as

$$\mu_j(s) = \begin{cases} \gamma((1+\varepsilon_j)s) & \text{if } \ s \in [-t_j + \tau, 0], \\ \eta_j(s + t_j - \tau) & \text{if } \ s \in [-t_j, -t_j + \tau], \end{cases} \tag{24}$$

where $\varepsilon_j := (t_j - \tau)^{-1}\tau$. From this and the boundedness of $\{\gamma(-t) \mid t > 0\}$, we easily see that there exists an $R > 0$ such that $\{\mu_j(s) \mid s \in [-t_j, 0]\} \subset B(0, R)$ for all $j \in \mathbb{N}$.

Before closing this section, we discuss the relationship between the set Λ and the ideal boundary in the sense of Ishii-Mitake [18]. For this purpose, we recall the notation used in Sections 4 and 5 of [18].

We denote by $\mathcal{A}_H$ the Aubry set for H and set $\Omega_0 := \mathbb{R}^n \setminus \mathcal{A}_H$. Let $\pi : \phi \mapsto \{\phi + c \,|\, c \in \mathbb{R}\}$ be the projection from $C(\mathbb{R}^n)$ to the quotient space $C(\mathbb{R}^n)/\mathbb{R}$, and let $d^\pi : \Omega_0 \longrightarrow C(\mathbb{R}^n)/\mathbb{R}$ be the mapping defined by $d^\pi(y) := \pi(d_H(\,\cdot\,, y))$. We set $\mathcal{D}_0 := d^\pi(\Omega_0)$. Note that d^π is bijective in view of Lemma 4.2 of [18] and the definition of $\mathcal{D}_0$.

We fix a standard complete metric ρ on $C(\mathbb{R}^n)$ which defines the topology of locally uniform convergence. We denote by ρ^π the induced metric on $C(\mathbb{R}^n)/\mathbb{R}$, that is,

$$\rho^\pi(\xi_1, \xi_2) := \inf\{\rho(\phi_1, \phi_2) \,|\, \phi_1 \in \xi_1,\ \phi_2 \in \xi_2\}, \qquad \xi_1, \xi_2 \in C(\mathbb{R}^n)/\mathbb{R}.$$

Then, we can define the metric ρ_0 on Ω_0 by $\rho_0(x, y) := \rho^\pi(d^\pi(x), d^\pi(y))$. Observe from Proposition 4.3 of [18] that the identity map $x \mapsto x$ is a homeomorphism from (Ω_0, ρ_0) to (Ω_0, ρ_E), where ρ_E stands for the Euclidean distance.

Let $(\hat{\Omega}_0, \rho_0)$ be the completion of (Ω_0, ρ_0). Since $d^\pi : (\Omega_0, \rho_0) \longrightarrow (\mathcal{D}_0/\mathbb{R}, \rho^\pi)$ is isometric by the definition of ρ_0, d^π can be extended to the isomorphism $(\hat{\Omega}_0, \rho_0) \longrightarrow (\overline{\mathcal{D}_0/\mathbb{R}}, \rho^\pi)$, where $\overline{\mathcal{D}_0/\mathbb{R}}$ denotes the closure of $\mathcal{D}_0/\mathbb{R}$ in $C(\mathbb{R}^n)/\mathbb{R}$ with respect to ρ^π. Following the paper [18], we call the set $\Delta_0 := \hat{\Omega}_0 \setminus \Omega_0$ the ideal boundary of Ω_0. We also denote by Δ_0^* the totality of points $y \in \Delta_0$ such that for some sequence $\{y_j\} \subset \Omega_0$,

$$\phi(y_j) + d_H(\,\cdot\,, y_j) \longrightarrow \phi \quad \text{in } C(\mathbb{R}^n) \text{ as } j \to \infty \text{ for all } \phi \in d^\pi(y). \tag{25}$$

Now, let $\{x_j\} \in \Lambda(\psi)$ for a given $\psi \in \mathcal{S}_H$, where $\Lambda(\psi)$ is defined by (16). Then, by mimicking the arguments in Section 5 of [18], we easily see that there exist a subsequence $\{y_j\} \subset \{x_j\}$ and a $y \in \Delta_0$ such that $\rho_0(y_j, y) \longrightarrow 0$ as $j \to \infty$ and (25) holds. In particular, $y \in \Delta_0^*$. We set

$$\Lambda_0(\psi) := \{y \in \Delta_0^* \,|\, \lim_{j\to\infty} \rho_0(x_j, y) = 0 \quad \text{for some } \{x_j\} \in \Lambda(\psi)\}. \tag{26}$$

Then by definition, $\Lambda_0(\psi) \subset \Delta_0^* \setminus \mathcal{A}_H$ for all $\psi \in \mathcal{S}_H$. In what follows, we use the notation $\Lambda_0 := \Lambda_0(u_\infty)$.

Similarly as in [18], for given $u \in \mathrm{UC}(\mathbb{R}^n)$ and $y \in \Delta_0^*$, we define the function $g(u, y) : \mathbb{R}^n \longrightarrow (-\infty, \infty]$ by

$$g(u, y)(x) := \phi(x) + \lim_{r\to 0} \sup\{(u - \phi)(\xi) \,|\, \xi \in \Omega_0,\ \rho_0(\xi, y) < r\},$$

where ϕ is any element of $d^\pi(y)$ and remark that $g(u, y)(x)$ does not depend on the choice of $\phi \in d^\pi(y)$. If $g(u, y) = g(v, y)$ for some $y \in \Delta_0^*$ and u, $v \in \mathrm{UC}(\mathbb{R}^n)$, then $\lim_{j\to\infty}(u - v)(x_j) = 0$ for every $\{x_j\} \subset \mathbb{R}^n$ such that $\lim_{j\to\infty} \rho_0(x_j, y) = 0$.

Taking into account these observations, we reformulate Theorem 2.5 as follows.

Theorem 2.12. *Let H satisfy (A1)-(A4) and one of $(A5)_+$ or $(A5)_-$. Let $u_0 \in \Phi_0$. Then, the convergence (9) holds provided that*

$$g(u_\infty, y) = g(u_0, y) \quad \text{in } \mathbb{R}^n \quad \text{for all } y \in \Lambda_0.$$

We next try to obtain a representation formula for u_∞ in terms of the ideal boundary. For $u \in \mathrm{UC}(\mathbb{R}^n)$ and $y \in \mathcal{A}_H$, we set $g(u, y) := d_H(\,\cdot\,, y) + u(y)$. Recall first the following theorem.

Theorem 2.13 (Theorem 5.4 of [18]). *Let $u \in \mathcal{S}_H$. Then,*

$$u(x) = \inf\{g(u,y)(x) \mid y \in \Delta_0^* \cup \mathcal{A}_H\}. \tag{27}$$

By using this theorem, we have the following representation formula for u_∞ which is a natural generalization of the usual ones (e.g. Theorem 5.7 of [8] and Theorem 8.1 of [17]).

Proposition 2.14. *Let H satisfy (A1)-(A4) and let $u_0 \in \Phi_0$. Then,*

$$u_\infty(x) = \inf\{g(u_0^-, y)(x) \mid y \in \Lambda_0 \cup \mathcal{A}_H\}.$$

To show this proposition, we use the following lemma.

Lemma 2.15. *Let H satisfy (A1)-(A4) and let $u_0 \in \Phi_0$. Then, for every $x \in \mathbb{R}^n$ and $\gamma \in \mathcal{E}_x$,*

$$\lim_{t\to\infty} (u_\infty - u_0^-)(\gamma(-t)) = 0. \tag{28}$$

Proof. Let $(T_t)_{t\geq 0}$ be the semigroup defined in Section 1. Then, from the variational formula (5) with u_0^- in place of u_0, we observe that for every $t > 0$,

$$(T_t u_0^-)(x) \leq \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + u_0^-(\gamma(-t)) = u_\infty(x) - u_\infty(\gamma(-t)) + u_0^-(\gamma(-t)).$$

Since $(T_t u_0^-)(x) \longrightarrow u_\infty(x)$ as $t \to \infty$ by Lemma 1.2, we have $\limsup_{t\to\infty} (u_\infty - u_0^-)(\gamma(-t)) \leq 0$. Noting that $u_\infty \geq u_0^-$ in $\mathbb{R}^n$ by definition, we obtain (28). □

Proof of Proposition 2.14. Remark first that, by a careful review of the original proof of Theorem 5.4 in [18], the representation formula (27) can be rewritten as

$$u(x) = \inf\{g(u,y)(x) \mid y \in \Lambda_0(u) \cup \mathcal{A}_H\}. \tag{29}$$

We also observe from Lemma 2.15 and the definition of $g(u,y)$ that $g(u_\infty, y) = g(u_0^-, y)$ for all $y \in \Lambda_0 \cup \mathcal{A}_H$. Hence, the proof is complete by setting $u = u_\infty$ in (29). □

3 Second convergence result.

In this section, we deal with Hamiltonians that provide another type of motions for $\{\mu_j\}$ which we call in this paper "switch-back". In order to explain the meaning of this word, we begin with a simple example.

Let $n = 1$ and consider the Cauchy problem

$$\begin{cases} u_t + |Du| - e^{-|x|} = 0 & \text{in } \mathbb{R} \times (0, +\infty), \\ u(\,\cdot\,, 0) = \min\{|x| - 2, 0\} & \text{on } \mathbb{R}. \end{cases}$$

Clearly, the Hamiltonian $H(x,p) := |p| - e^{-|x|}$ satisfies (A1)-(A3). Since $e^{-|x|} \in \mathcal{S}_H$, H enjoys (A4) with $\phi_0 = \psi_0 = e^{-|x|}$, and the initial function $u_0(x) := \min\{|x| - 2, 0\}$ belongs to $\Phi_0 = \mathrm{BUC}(\mathbb{R})$. We see moreover that $u_0^-(x) = -e^{-|x|} - 1$ and $u_\infty(x) = e^{-|x|} - 1$.

Let $L(x,\xi)$ be the Lagrangian associated with H, that is, $L(x,\xi) = \chi_{[-1,1]}(\xi) + e^{-|x|}$, where $\chi_{[-1,1]}(\xi) := 0$ for $|\xi| \le 1$ and $\chi_{[-1,1]}(\xi) := +\infty$ for $|\xi| > 1$. For a given $x \in \mathbb{R}$, we define $\gamma \in \mathcal{C}((-\infty,0];x)$ by $\gamma(s) := x - \mathrm{sgn}(x)\,s$ for $s \in (-\infty,0]$, where we have set $\mathrm{sgn}(x) := 1$ for $x \ge 0$ and $\mathrm{sgn}(x) = -1$ for $x < 0$. Then, it is easy to see that $\gamma \in \mathcal{E}_x$ and $|\gamma(-t)| \longrightarrow \infty$ as $t \to \infty$. We choose a diverging $\{t_j\} \subset (0,\infty)$ such that $u^+(x) = \lim_{j\to\infty} u(x,t_j)$ and $|x| < t_j$ for all $j \in \mathbb{N}$.

We next define $\mu_j \in \mathcal{C}([-t_j,0];x)$, $j \in \mathbb{N}$, by

$$\mu_j(s) := \begin{cases} \gamma(s) & \text{for } -\dfrac{t_j - |x|}{2} \le s \le 0, \\ \mathrm{sgn}(x)\,(s+t_j) & \text{for } -t_j \le s \le -\dfrac{t_j-|x|}{2}. \end{cases}$$

Note that $u_0(\mu_j(-t_j)) = u_0(0) = -2$ for all $j \in \mathbb{N}$. Then, we see that

$$u(x,t_j) \le \int_{-t_j}^{0} L(\mu_j(s),\dot\mu_j(s))\,ds + u_0(\mu_j(-t_j)) = e^{-|x|} - 1 - 2e^{-\frac{t_j+|x|}{2}} \underset{j\to\infty}{\longrightarrow} u_\infty(x).$$

Thus, (9) is valid. We remark here that if t_j is sufficiently large, then $\mu_j(-t)$ goes toward ∞ or $-\infty$ along the curve γ up to the time $t = (t_j - |x|)/2$ and then it turns back to the origin. This motion explains well the word "switch-back".

It is also worth mentioning that the condition (17) in Theorem 2.5 does not hold in this case. Indeed, since $\lim_{t\to\infty}|\gamma(-t)| = \infty$, we have $\lim_{t\to\infty}(u_0 - u_\infty)(\gamma(-t)) = 1 > 0$.

We now consider a more general situation. In the rest of this section, we assume the following:

(A6) $H(x,0) \le 0$ for all $x \in \mathbb{R}^n$ and there exists a $\lambda \ge 1$ such that

$$H(x,-\lambda p) \ge H(x,p) \quad \text{for all } (x,p) \in \mathbb{R}^{2n}. \tag{30}$$

Note that (A6) implies

$$L(x,-\lambda^{-1}\xi) \le L(x,\xi) \quad \text{for all } (x,\xi) \in \mathbb{R}^{2n}. \tag{31}$$

Theorem 3.1. *Let H satisfy (A1)-(A3), (A4) with $\phi_0 = 0$ and (A6). Then, the convergence (9) holds for every $u_0 \in \Phi_0$.*

Remark 3.2. Assumption (A6) can be relaxed as

(A6)$'$ There exists a $\lambda \ge 1$ such that for every $(x,p) \in Q$, $\xi \in D_2^- H(x,p)$, $q \in \mathbb{R}^n$ and $q' \in \partial_c\phi_0(x)$,

$$H(x,q' - \lambda q) \ge \xi \cdot (q' + q - p), \tag{32}$$

where $\phi_0 \in \mathcal{S}_H^-$ is taken from (A4) and $\partial_c\phi_0(x)$ denotes the Clarke derivative of ϕ_0 at $x \in \mathbb{R}^n$.

Assumption (A6) is a particular case where $\phi_0 = 0$ in (A6)$'$. See [16] for details.

Proof of Theorem 3.1. Fix any $u_0 \in \Phi_0$, $x \in \mathbb{R}^n$ and $\gamma \in \mathcal{E}_x$. Since $\phi_0 = 0$ by assumption, we see that $u_\infty \ge -C$ in $\mathbb{R}^n$ for some $C > 0$. We also observe that $L \ge 0$ in $\mathbb{R}^{2n}$ in view of the assumption $H(\cdot,0) \le 0$ in $\mathbb{R}^n$. In particular, the function $t \mapsto \int_{-t}^{0} L(\gamma(s),\dot\gamma(s))\,ds$ is non-decreasing and

$$0 \le \int_{-t}^{0} L(\gamma(s),\dot\gamma(s))\,ds = u_\infty(x) - u_\infty(\gamma(-t)) \le u_\infty(x) + C \quad \text{for all } t \ge 0.$$

Fix an arbitrary $\varepsilon > 0$. Then, there exists a $t_0 > 0$ such that

$$\int_{-t_0-\theta}^{-t_0} L(\gamma(s), \dot\gamma(s))\, ds < \varepsilon \qquad \text{for all } \theta > 0. \tag{33}$$

We next choose a $\tau > 0$ such that

$$u_0^-(\gamma(-t_0)) + \varepsilon > u(\gamma(-t_0), \tau). \tag{34}$$

Now, we fix any diverging $\{t_j\} \subset (0,\infty)$ so that $u^+(x) = \lim_{j\to\infty} u(x,t_j)$ and then take $\{\theta_j\} \subset (0,\infty)$ such that $t_j = t_0 + (1+\lambda)\theta_j + \tau$ for all $j \in \mathbb{N}$, where $\lambda \geq 1$ is the constant taken from (A6). Note that $\theta_j \to \infty$ as $j \to \infty$.

For each $j \in \mathbb{N}$, we set $t_{1j} := t_0 + \theta_j$ and $t_{2j} := t_{1j} + \lambda\theta_j$, and we define $\gamma_j \in \mathcal{C}([-t_{2j}, 0]; x)$ by

$$\gamma_j(s) := \begin{cases} \gamma(s) & \text{if } s \in [-t_{1j}, 0], \\ \gamma(-\lambda^{-1}s - (1+\lambda^{-1})t_{1j}) & \text{if } s \in [-t_{2j}, -t_{1j}]. \end{cases} \tag{35}$$

Note that $\gamma_j(-t_0) = \gamma_j(-t_{2j}) = \gamma(-t_0)$. Then, in view of (31) and (33), we see that

$$\int_{-t_{2j}}^{-t_{1j}} L(\gamma_j(s), \dot\gamma_j(s))\, ds = \lambda \int_{-t_{1j}}^{-t_0} L(\gamma(s), -\lambda^{-1}\dot\gamma(s))\, ds \leq \lambda \int_{-t_0-\theta_j}^{-t_0} L(\gamma(s), \dot\gamma(s))\, ds < \lambda\varepsilon.$$

On the other hand, in view of (34) and the inequality $u_\infty \geq u_0^-$ in $\mathbb{R}^n$,

$$u_\infty(x) = \int_{-t_0}^{0} L(\gamma(s), \dot\gamma(s))\, ds + u_\infty(\gamma(-t_0)) \geq \int_{-t_0}^{0} L(\gamma(s), \dot\gamma(s))\, ds + u(\gamma(-t_0), \tau) - \varepsilon.$$

In combination with these estimates, we obtain

$$\begin{aligned} u_\infty(x) + (2+\lambda)\varepsilon &> \int_{-t_0}^{0} L(\gamma, \dot\gamma)\, ds + \int_{-t_{1j}}^{-t_0} L(\gamma, \dot\gamma)\, ds + \int_{-t_{2j}}^{-t_{1j}} L(\gamma_j, \dot\gamma_j)\, ds + u(\gamma(-t_0), \tau) \\ &= \int_{-t_{2j}}^{0} L(\gamma_j(s), \dot\gamma_j(s))\, ds + u(\gamma_j(-t_{2j}), \tau) \geq u(x, t_j). \end{aligned}$$

By letting $j \to \infty$, we have $u^+(x) = \lim_{j\to\infty} u(x,t_j) \leq u_\infty(x) + (2+\lambda)\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain $u^+(x) \leq u_\infty(x)$. □

We give in Example 5.2 a more concrete example which satisfies (A6).

Remark 3.3. Suppose in addition to (A6) that $H(x,0) < 0$ for all $x \in \mathbb{R}^n$. Then, in view of Lemma 2.8, we have $|\gamma(-t)| \longrightarrow \infty$ as $t \to \infty$ for any $\gamma \in \mathcal{E}_x$. We now fix a diverging $\{t_j\}_j \subset (0,\infty)$ such that $u^+(x) = \lim_{j\to\infty} u(x,t_j)$ and choose $\eta \in \mathcal{C}([-\tau, 0]; \gamma(-t_0))$ such that

$$u(\gamma(-t_0), \tau) + \varepsilon > \int_{-\tau}^{0} L(\eta(s), \dot\eta(s))\, ds + u_0(\eta(-\tau)).$$

If we define $\mu_j \in \mathcal{C}([-t_j, 0]; x)$, $j \in \mathbb{N}$, by

$$\mu_j(s) := \begin{cases} \gamma_j(s) & \text{if } s \in [-t_{2j}, 0], \\ \eta(s + t_{2j}) & \text{if } s \in [-t_j, -t_{2j}], \end{cases}$$

then we observe the switch-back of μ_j as in the previous example. In particular, we have neither (a) $\mu_j = \gamma$ for all $j \in \mathbb{N}$, nor (b) μ_j is bounded uniformly in $j \in \mathbb{N}$. In this sense, the switch-back motion presents a striking contrast to the curves in Section 2.

4 Third convergence result.

This section is concerned with the Cauchy problem (1) with Hamiltonian and initial function having "weak" periodicity. In this case, one other type of motions for $\{\mu_j\}$ takes place. In the rest of this section, we always assume that H satisfies (A1)-(A3), (A4) with $\phi_0 = \psi_0 = \phi$ for some fixed $\phi \in \mathcal{S}_H$. The class of initial data Φ_0 is, therefore, written as

$$\Phi_0 = \{u_0 \in \mathrm{UC}(\mathbb{R}^n) \mid \phi - C \le u_0 \le \phi + C \quad \text{in } \mathbb{R}^n \text{ for some } C > 0\}.$$

Fix an arbitrary $u_0 \in \Phi_0$. Then, there exists a $C > 0$ such that

$$u_0 - 2C \le \phi - C \le u_0^- \le u_\infty \le \phi + C \le u_0 + 2C \qquad \text{in } \mathbb{R}^n.$$

Let $\{y_j\} \subset \mathbb{R}^n$ be any sequence. By taking a subsequence if necessary, we may assume in view of (A1) and the Ascoli-Arzela theorem that

$$H(\,\cdot\, + y_j, \,\cdot\,) \longrightarrow G \qquad \text{in } C(\mathbb{R}^{2n}) \quad \text{as } j \to \infty, \tag{36}$$

$$u_0(\,\cdot\, + y_j) - u_0(y_j) \longrightarrow v_0 \qquad \text{in } C(\mathbb{R}^n) \quad \text{as } j \to \infty, \tag{37}$$

for some $G \in C(\mathbb{R}^{2n})$ and $v_0 \in \mathrm{UC}(\mathbb{R}^n)$. Note that G satisfies (A1)-(A3) with G in place of H. We denote by $\mathcal{S}_G^-$ (resp. $\mathcal{S}_G^+$, $\mathcal{S}_G$) the set of all continuous viscosity subsolutions (resp. supersolutions, solutions) of

$$G(x, D\phi) = 0 \qquad \text{in } \mathbb{R}^n. \tag{38}$$

Since the family $\{u_\infty(\,\cdot\, + y_j) - u_0(y_j)\}_j$ is uniformly bounded and equi-continuous on any compact subset of $\mathbb{R}^n$, there exist a function $\overline{u}_\infty \in C(\mathbb{R}^n)$ and a subsequence of $\{y_j\}$, which we denote by the same $\{y_j\}$, such that

$$u_\infty(\,\cdot\, + y_j) - u_0(y_j) \longrightarrow \overline{u}_\infty \qquad \text{in } C(\mathbb{R}^n) \quad \text{as } j \to \infty. \tag{39}$$

Remark that $\overline{u}_\infty \in \mathcal{S}_G$ by virtue of the stability property of viscosity solutions. We see moreover that $v_0 - 2C \le \overline{u}_\infty \le v_0 + 2C$ in $\mathbb{R}^n$. Thus, the functions

$$v_0^-(x) := \sup\{\, \phi(x) \mid \phi \in \mathcal{S}_G^-, \quad \phi \le v_0 \ \text{ in } \mathbb{R}^n\} \in \mathcal{S}_G^-,$$
$$v_\infty(x) := \inf\{\, \psi(x) \mid \psi \in \mathcal{S}_G, \quad \psi \ge v_0^- \ \text{ in } \mathbb{R}^n\} \in \mathcal{S}_G$$

are well-defined and satisfy

$$v_0 - 4C \le v_0^- \le v_\infty \le v_0 + 4C \qquad \text{in } \mathbb{R}^n. \tag{40}$$

We next consider the Cauchy problem

$$\begin{cases} v_t + G(x, Dv) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ v(\,\cdot\,, 0) = v_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{41}$$

and let $v(x,t)$ be the solution of (41). Remark here that $\liminf_{t\to\infty} v(x,t) = v_\infty(x)$ in view of Lemma 1.2. Moreover, by (36), (37) and the stability property for viscosity solutions of (41), we observe that $u(\,\cdot\, + y_j, \,\cdot\,) - u_0(y_j) \longrightarrow v$ in $C(\mathbb{R}^{2n})$ as $j \to \infty$. Taking into account these observations, we claim the following.

Theorem 4.1. *Let H satisfy (A1)-(A3), (A4) with $\phi_0 = \psi_0 = \phi$ for some $\phi \in \mathcal{S}_H$, and $(A5)_+$. Let $u_0 \in \Phi_0$. Then, the convergence (9) holds provided that for any sequence $\{y_j\} \subset \mathbb{R}^n$ satisfying (37) for some $v_0 \in \mathrm{UC}(\mathbb{R}^n)$, there exists a subsequence, which we denote by the same $\{y_j\}$, such that*

$$\limsup_{j\to\infty} (u_\infty(y_j) - u_0(y_j)) \ge v_\infty(0). \tag{42}$$

Moreover, condition $(A5)_+$ can be replaced by $(A5)_-$ if the following holds true in addition to (42):

$$u(y_j, \cdot) - u_0(y_j) \longrightarrow v(0, \cdot) \qquad \textit{uniformly in } [0,\infty) \textit{ as } j \to \infty. \tag{43}$$

Proof. Fix any $x \in \mathbb{R}^n$ and any diverging sequence $\{t_j\} \subset (0,\infty)$ such that $u^+(x) = \lim_{j\to\infty} u(x,t_j)$. We also fix a $\gamma \in \mathcal{S}_x$ and set $y_j := \gamma(-t_j)$ for $j \in \mathbb{N}$. Then, there exists a subsequence of $\{y_j\}$ such that (36) and (37) hold for some $G \in C(\mathbb{R}^{2n})$ and $v_0 \in \mathrm{UC}(\mathbb{R}^n)$, respectively. In what follows, we fix an arbitrary $\delta > 0$ and choose a $\tau > 0$ so that $v(0,\tau) - v_\infty(0) < \delta$, where v is the unique viscosity solution of (41).

We first assume $(A5)_+$ and (42). For each $j \in \mathbb{N}$, we set $\varepsilon_j := (t_j - \tau)^{-1}\tau$ and define $\gamma_j \in \mathcal{C}([-t_j+\tau, 0]; x)$ by $\gamma_j(s) = \gamma((1+\varepsilon_j)s)$. Note that $\gamma_j(-t_j+\tau) = \gamma(-t_j) = y_j$ for all $j \in \mathbb{N}$. By renumbering $j \in \mathbb{N}$, we may assume that $\varepsilon_j \in (0,\delta_1)$ for all $j \in \mathbb{N}$, where δ_1 is the constant taken from Lemma 2.2. Then, in view of (14), we see that

$$\begin{aligned} u(x,t_j) &\le \int_{-t_j+\tau}^0 L(\gamma_j, \dot\gamma_j)\, ds + u(\gamma_j(-t_j+\tau), \tau) \\ &\le \int_{-t_j}^0 L(\gamma, \dot\gamma)\, ds + t_j\varepsilon_j\, \omega_1(\varepsilon_j) + u(y_j,\tau) = u_\infty(x) - u_\infty(y_j) + t_j\varepsilon_j\,\omega_1(\varepsilon_j) + u(y_j,\tau). \end{aligned}$$

Since $v(0,\tau) - v_\infty(0) < \delta$ and $u(y_j,\tau) - u_0(y_j) \longrightarrow v(0,\tau)$ as $j \to \infty$, we conclude in combination with (42) that

$$\begin{aligned} u^+(x) - u_\infty(x) &\le -\limsup_{j\to\infty} (u_\infty(y_j) - u_0(y_j)) + \lim_{j\to\infty} (u(y_j,\tau) - u_0(y_j)) \\ &\le -v_\infty(0) + v(0,\tau) < \delta. \end{aligned}$$

Hence, letting $\delta \to 0$ yields $u^+(x) \le u_\infty(x)$.

We next assume $(A5)_-$, (42) and (43). In view of (39) and (43), and by renumbering $\{t_j\}$ if necessary, we may assume that for every $j \in \mathbb{N}$ and $t > 0$,

$$|u(y_j,t) - u_0(y_j) - v(0,t)| + |u_\infty(y_j) - u_0(y_j) - \overline{u}_\infty(0)| < \delta. \tag{44}$$

Hereafter, we always use the same $\{t_j\}$ to denote its subsequence. Then, we observe that

$$\begin{aligned} u(x,t_j) &\le \int_{-t_1}^0 L(\gamma(s), \dot\gamma(s))\, ds + u(y_1, t_j - t_1) = u_\infty(x) - u_\infty(y_1) + u(y_1, t_j - t_1) \\ &< u_\infty(x) - \overline{u}_\infty(0) + u(y_2, t_j - t_1) - u_0(y_2) + 3\delta. \end{aligned}$$

We may assume without loss of generality that $t_2 > t_1 + \tau$. For each $j \ge 2$, we set

$$\varepsilon_j = \frac{t_2 - t_1 - \tau}{t_j - t_1 - \tau}, \qquad \gamma_j(s) = \gamma(-t_2 + (1-\varepsilon_j)s), \quad s \le 0.$$

Note that $\varepsilon_j \to 0$ as $j \to \infty$ and $\gamma_j((1-\varepsilon_j)(-t_j+t_1+\tau)) = \gamma(-t_j) = y_j$ for all $j \geq 2$. Then, in view of (14) and (44),

$$\begin{aligned} u(y_2, t_j - t_1) &\leq \int_{-t_j+t_1+\tau}^{0} L(\gamma_j(s), \dot{\gamma}_j(s))\, ds + u(y_j, \tau) \\ &< u_\infty(y_2) - u_\infty(y_j) + t_j\varepsilon_j\omega_1(\varepsilon_j) + v(0,\tau) + u_0(y_j) + \delta. \end{aligned}$$

Thus, we have

$$\begin{aligned} u(x,t_j) - u_\infty(x) &< u_\infty(y_2) - u_0(y_2) - \overline{u}_\infty(0) + t_j\varepsilon_j\omega_1(\varepsilon_j) + v(0,\tau) - u_\infty(y_j) + u_0(y_j) + 4\delta \\ &< v_\infty(0) - u_\infty(y_j) + u_0(y_j) + t_j\varepsilon_j\omega_1(\varepsilon_j) + 6\delta. \end{aligned}$$

Taking into account (42) and letting $j \to \infty$ and then $\delta \to 0$, we get $u^+(x) \leq u_\infty(x)$. □

Corollary 4.2. *Let H satisfy (A1)-(A3), (A4) with $\phi_0 = \psi_0 = \phi$ for some $\phi \in \mathcal{S}_H$, and $(A5)_+$. Let $u_0 \in \Phi_0$. Then, the convergence (9) holds provided that for any sequence $\{y_j\} \subset \mathbb{R}^n$ satisfying (37) for some $v_0 \in \mathrm{UC}(\mathbb{R}^n)$, there exists a subsequence such that*

$$u_0^-(\,\cdot\, + y_j) - u_0(y_j) \longrightarrow v_0^- \qquad in\ \ C(\mathbb{R}^n) \quad as\ \ j \to \infty. \tag{45}$$

Proof. It suffices to check (42). Observe first that

$$u_\infty(\,\cdot\, + y_j) - u_0(y_j) \geq u_0^-(\,\cdot\, + y_j) - u_0(y_j) \quad \text{in } \mathbb{R}^n \text{ for all } j \in \mathbb{N}.$$

In view of (39) and (45), for a suitable subsequence of $\{y_j\}$, we see that

$$\overline{u}_\infty(x) = \lim_{j\to\infty}(u_\infty(x+y_j) - u_0(y_j)) \geq v_0^-(x) \quad \text{for all } \ x \in \mathbb{R}^n.$$

Since $\overline{u}_\infty \in \mathcal{S}_G$, we have $\overline{u}_\infty(x) \geq v_\infty(x) \geq v_0^-(x)$ for all $x \in \mathbb{R}^n$. Thus, (42) is valid by setting $x = 0$. □

We point out here that Theorem 4.1 covers, as a particular case, Theorem 2.2 of [14] dealing with upper semi-periodic Hamiltonians and obliquely lower semi-almost periodic initial data. Here, we recall that H is upper (resp. lower) semi-periodic if for any sequence $\{y_j'\} \subset \mathbb{R}^n$, there exist a subsequence $\{y_j\} \subset \{y_j'\}$, a function $G \in C(\mathbb{R}^{2n})$ and a sequence $\{\xi_j\} \subset \mathbb{R}^n$ converging to zero as $j \to \infty$ such that $H(\,\cdot\, + y_j, \,\cdot\,)$ converges to G in $C(\mathbb{R}^{2n})$ as $j \to \infty$ and

$$H(\,\cdot\, + y_j + \xi_j, \,\cdot\,) \leq G \quad (\text{resp.} \geq G) \qquad \text{in } \ \mathbb{R}^{2n} \quad \text{for all } \ j \in \mathbb{N}. \tag{46}$$

We say that $u_0 \in \mathrm{UC}(\mathbb{R}^n)$ is obliquely lower (resp. upper) semi-almost periodic if for any $\varepsilon > 0$ and any sequence $\{y_j'\} \subset \mathbb{R}^n$, there exist a subsequence $\{y_j\} \subset \{y_j'\}$ and a function $v_0 \in \mathrm{UC}(\mathbb{R}^n)$ such that $u_0(\,\cdot\, + y_j) - u_0(y_j)$ converges to v_0 in $C(\mathbb{R}^n)$ as $j \to \infty$ and

$$u_0(\,\cdot\, + y_j) - u_0(y_j) - v_0(\,\cdot\,) > -\varepsilon \quad (\text{resp.} < \varepsilon) \qquad \text{in } \ \mathbb{R}^n \quad \text{for all } \ j \in \mathbb{N}. \tag{47}$$

If u_0 is both obliquely lower and upper semi-almost periodic, we say that u_0 is obliquely almost periodic.

Theorem 4.3 (cf. Theorem 2.2 of [14]). *Let H satisfy (A1)-(A3), (A4) with $\phi_0 = \psi_0 = \phi$ for some $\phi \in \mathcal{S}_H$, and (A5)$_+$. Let $u_0 \in \Phi_0$ and assume that H and u_0 are, respectively, upper semi-periodic and obliquely lower semi-almost periodic. Then, the convergence (9) holds.*

Proof. We check (45) in Corollary 4.2. Since the family $\{u_0^-(\cdot + y_j) - u_0(y_j) \mid j \in \mathbb{N}\}$ is precompact in $C(\mathbb{R}^n)$, we can extract a subsequence of $\{y_j\}$, which we denote by $\{y_j\}$ again, such that $u_0^-(\cdot + y_j) - u_0(y_j) \longrightarrow w$ in $C(\mathbb{R}^n)$ as $j \to \infty$ for some $w \in \mathrm{UC}(\mathbb{R}^n)$. It suffices to show that $w = v_0^-$ in $\mathbb{R}^n$. Note that $w \in \mathcal{S}_G^-$ in view of the stability of viscosity property.

Observe first that upper semi-periodicity (46) together with the Lipschitz continuity of $d_H(\cdot, \cdot)$ in both variables ensure that for any $\varepsilon > 0$ and $x \in \mathbb{R}^n$, there exists a $j_0 \in \mathbb{N}$ such that

$$d_H(x + y_j, \cdot + y_j) \geq d_G(x, \cdot) - \varepsilon \qquad \text{in } \mathbb{R}^n \quad \text{for all } j \geq j_0. \tag{48}$$

From this and obliquely lower semi-almost periodicity (47), we obtain

$$\begin{aligned} u_0^-(x + y_j) - u_0(y_j) &= \inf_{z\in\mathbb{R}^n} (d_H(x + y_j, z + y_j) + u_0(z + y_j)) - u_0(y_j) \\ &> \inf_{z\in\mathbb{R}^n} (d_G(x, z) + v_0(z)) - 2\varepsilon = v_0^-(x) - 2\varepsilon. \end{aligned}$$

On the other hand, since $u_0^- \leq u_0$ in $\mathbb{R}^n$, we have

$$u_0^-(\cdot + y_j) - u_0(y_j) \leq u_0(\cdot + y_j) - u_0(y_j) \quad \text{in } \mathbb{R}^n.$$

By taking the limit $j \to \infty$ in the last two inequalities and then letting $\varepsilon \to 0$, we get $v_0^- \leq w \leq v_0$ in $\mathbb{R}^n$. Hence, we conclude that $w = v_0^-$ in $\mathbb{R}^n$. □

Remark 4.4. If $H(x,p)$ is $\mathbb{Z}^n$-periodic with respect to x for all $p \in \mathbb{R}^n$, then (48) is obvious from the identity $d_H(\cdot + k, \cdot + k) = d_H$ in $\mathbb{R}^{2n}$ for all $k \in \mathbb{Z}^n$. Notice here that Theorem 4.1 does not require, a priori, any periodicity for H and u_0. We give in Section 5 an example having neither upper semi-periodicity for H nor obliquely lower semi-almost periodicity for u_0, but enjoying the conditions required in Theorem 4.1.

Concerning the latter part of Theorem 4.1, we have the following result.

Theorem 4.5. *Let H satisfy (A1)-(A3), (A4) with $\phi_0 = \psi_0 = \phi$ for some $\phi \in \mathcal{S}_H$, and (A5)$_-$. Let $u_0 \in \Phi_0$ and assume that $H(x,p)$ is $\mathbb{Z}^n$-periodic with respect to x for all $p \in \mathbb{R}^n$ and u0 is obliquely almost periodic. Then, the convergence (9) holds.*

Proof. It suffices to check (43). Let $\{y_j\} \subset \mathbb{R}^n$ be any sequence. We first observe from the obliquely almost periodicity for u_0 that along a subsequence of $\{y_j\}$,

$$u_0(\cdot + y_j) - u_0(y_j) \longrightarrow v_0 \qquad \text{uniformly in } \mathbb{R}^n \quad \text{as } j \to \infty. \tag{49}$$

Observe also from the $\mathbb{Z}^n$-periodicity for H that there exists a bounded $\{\xi_j\} \subset \mathbb{R}^n$ converging to some $\xi \in \mathbb{R}^n$ as $j \to \infty$ such that $H(x + y_j, p) = H(x + \xi_j, p)$ for all $(x,p) \in \mathbb{R}^{2n}$ and $j \in \mathbb{N}$, and $H(x + \xi_j, p) \longrightarrow H(x + \xi, p)$ uniformly in $\mathbb{R}^n \times B(0, R)$ as $j \to \infty$ for all $R > 0$.

We now set $G(x,p) := H(x + \xi, p)$ and let $v_j(x,t) \in C(\mathbb{R}^n \times [0,\infty))$, $j \in \mathbb{N}$, be the solution of

$$v_t + G(x, Dv) = 0 \qquad \text{in } \mathbb{R}^n \times (0, \infty) \tag{50}$$

satisfying $v_j(\cdot\,,0) = u_0(\cdot\,+y_j) - u_0(y_j)$ in $\mathbb{R}^n$. Note that by uniqueness,

$$u(x+y_j,t) - u_0(y_j) = v_j(x+\xi_j-\xi,t) \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [0,\infty) \text{ and } j \in \mathbb{N}.$$

Then, by using the nonexpansive property for solutions of (50) and the equi-continuity on $\mathbb{R}^n$ for $\{v_j(\cdot\,,t) \mid t > 0,\ j \in \mathbb{N}\}$, we have

$$\begin{aligned} |u(x+y_j,t) - u_0(y_j) - v(x,t)| &\le |v_j(x+\xi_j-\xi,t) - v_j(x,t)| + |v_j(x,t) - v(x,t)| \\ &\le \omega(|\xi_j - \xi|) + |u_0(x+y_j) - u_0(y_j) - v_0(x)|, \end{aligned}$$

where ω is a modulus. Thus, in view of (49) and letting $j \to \infty$, we obtain (43). □

Remark 4.6. We now discuss the construction of $\{\mu_j\}$ corresponding to Theorem 4.1. For simplicity, we only consider the case where $(A5)_+$ holds. Let $\tau > 0$ be the number taken in the proof of Theorem 4.1. For each $j \in \mathbb{N}$, we choose an $\eta_j \in \mathcal{C}([-\tau,0];y_j)$ such that

$$u(y_j,\tau) + \delta > \int_{-\tau}^{0} L(\eta_j(s),\dot\eta_j(s))\,ds + u_0(\eta_j(-\tau)).$$

We then define $\mu_j \in \mathcal{C}([-t_j,0];x)$, $j \in \mathbb{N}$, by

$$\mu_j(s) = \begin{cases} \gamma_j(s) & \text{if } \ s \in [-t_j+\tau,0], \\ \eta_j(s+t_j-\tau) & \text{if } \ s \in [-t_j,-t_j+\tau]. \end{cases}$$

Suppose that $\sup_{t>0} |\gamma(-t)| < \infty$. Then, $\{\mu_j\}$ is nothing but the one discussed in Remark 2.11. On the contrary, if $\{\gamma(-t) \mid t > 0\}$ is unbounded, then we have one other type of motions for $\{\mu_j\}$ which ensures the convergence (9). Notice here that condition (17) does not hold in general.

5 Examples.

We begin with an example concerning condition (a) of Theorem 2.6.

Example 5.1. Fix any $p_0 \in \mathbb{R}^n$ such that $|p_0| < 1$ and define H by $H = H(p) := |p - p_0| - 1$ for $p \in \mathbb{R}^n$. Note that the corresponding Lagrangian is $L(\xi) = p_0 \cdot \xi + 1 + \chi_{B(0,1)}(\xi)$, where $\chi_{B(0,1)}(\xi) := 0$ on $B(0,1)$ and $\chi_{B(0,1)}(\xi) := \infty$ on $\mathbb{R}^n \setminus B(0,1)$. It is easy to check that H enjoys (A1)-(A3) as well as the first part of condition (a) in Theorem 2.6. We also see by Lemma 2.8 that any extremal curve γ is diverging, namely, $|\gamma(-t)| \longrightarrow \infty$ as $t \to \infty$.

We first identify the ideal boundary Δ_0 for H. Let d_H be the function defined by (7). Observe in view of (7) or (8) that $d_H(x,y) = |x-y| + p_0 \cdot (x-y)$, $x, y \in \mathbb{R}^n$. We take any diverging sequence $\{y_j\} \subset \mathbb{R}^n$. Since

$$d_H(x,y_j) - d_H(0,y_j) = |x-y_j| - |y_j| + p_0 \cdot x = \frac{|x|^2 - 2y_j \cdot x}{|x-y_j| + |y_j|} + p_0 \cdot x$$

for all $j \in \mathbb{N}$, we see that $\{d_H(\cdot\,,y_j) - d_H(0,y_j)\}_j$ converges in $C(\mathbb{R}^n)$ to some function if and only if $\dfrac{y_j}{|y_j|} \longrightarrow \hat y$ as $j \to \infty$ for some $\hat y \in \partial B(0,1)$ in which case we have

$$d_H(x,y_j) - d_H(0,y_j) \longrightarrow -\hat y \cdot x + p_0 \cdot x = (p_0 - \hat y) \cdot x \quad \text{as } \ j \to \infty.$$

This implies that the sequence $\{d^\pi(y_j)\}_j$ converges in $(C(\mathbb{R}^n)/\mathbb{R}, \rho^\pi)$ to $\pi((p_0-\hat{y})\cdot x)$ as $j \to \infty$. Thus, in view of the fact that $\mathcal{A}_H = \emptyset$, we may identify Δ_0 with $\partial B(0,1)$ through the mapping

$$\partial B(0,1) \ni \hat{y} \mapsto \pi((p_0-\hat{y})\cdot x) \in \Delta_0 = (\overline{\mathcal{D}_0/\mathbb{R}}) \setminus (\mathcal{D}_0/\mathbb{R}).$$

We now fix any $q_0 \in \partial B(0,1)$ and set $\phi(x) := (p_0+q_0)\cdot x$ for $x \in \mathbb{R}^n$. Note that $\phi \in \mathcal{S}_H$. We try to identify the set $\Lambda_0(\phi)$ defined by (26). Observe first that γ is an extremal curve for ϕ at some $x \in \mathbb{R}^n$ if and only if

$$\phi(x) - \phi(\gamma(-t)) = \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\,ds = d_H(x, \gamma(-t)) \qquad \text{for all } \ t > 0.$$

From this and the explicit forms of ϕ, L and d_H, we see that

$$(p_0+q_0)\cdot(x-\gamma(-t)) = p_0\cdot(x-\gamma(-t)) + t = |x-\gamma(-t)| + p_0\cdot(x-\gamma(-t)),$$

from which we deduce after some computations that $\gamma(-t) = x - tq_0$ for all $t \geq 0$. Let $\{t_j\} \subset (0,\infty)$ be any diverging sequence and set $y_j := \gamma(-t_j)$. Then as $j \to \infty$,

$$\frac{y_j}{|y_j|} = \frac{x-t_jq_0}{|x-t_jq_0|} \longrightarrow -\frac{q_0}{|q_0|} =: -q_0 \in \partial B(0,1),$$

from which we conclude that $\Lambda_0(\phi) = \{-q_0\}$.

We now set $\phi_0(x) := \min\{(p_0+q_0)\cdot x,\, 0\}$, $x \in \mathbb{R}^n$. Notice that $\phi_0 \in \mathcal{S}_H^-$ in view of (A3), and that (A4) is valid with the above ϕ_0 and $\psi_0(x) := \phi(x) = (p_0+q_0)\cdot x \in \mathcal{S}_H$. Let $u_0 \in \Phi_0$ be any initial function satisfying

$$\lim_{\lambda\to\infty}(u_0-\phi_0)(x-\lambda q_0) = 0 \quad \text{for all } x \in \mathbb{R}^n.$$

Then, we can see that $u_\infty(x) = \phi(x)$ for $x \in \mathbb{R}^n$, and therefore $\Lambda_0 = \{-q_0\}$ and (17) holds. Hence, by Theorem 2.5, we have the convergence (9). We remark here that if we choose $u_0 := \phi_0$, then, $\lim_{j\to\infty}(u_0-u_\infty)(x_j) = -\infty$ for any $\{x_j\}$ such that $\lim_{j\to\infty} u_\infty(x_j) = \infty$. This example shows that (22) is strictly stronger than (17).

On the other hand, if we set $\phi(x) := \inf\{(p_0+q)\cdot x \mid q \in \partial B(0,1)\}$, $x \in \mathbb{R}^n$, then $\phi \in \mathcal{S}_H$ in view of (A3). Since $\phi = -d_H(0,\cdot\,)$ in $\mathbb{R}^n$, we observe that $\gamma \in \mathcal{E}_x(\phi)$ for $x \neq 0$ if and only if

$$\gamma(-t) = x + t\frac{x}{|x|} \qquad \text{for all } \ t \geq 0.$$

We conclude in particular that $\Lambda_0(\phi) = \partial B(0,1)$. Hence, $\{x_j\} \in \Lambda(\phi)$ if and only if $\lim_{j\to\infty}|x_j| = \infty$.

We now choose $\phi_0 = \psi_0 = \phi$ in (A4) and let $u_0 \in \Phi_0$ be any initial function such that $\lim_{|x|\to\infty}(u_0-\phi)(x) = 0$. Then, we easily see that $u_\infty = \phi$ in $\mathbb{R}^n$. Thus, two conditions (17) and (22) are equivalent in this case.

The next example is concerned with Theorem 3.1.

Example 5.2. Let H satisfy (A1)-(A3) and $H(x,0) \leq 0$ for all $x \in \mathbb{R}^n$. By setting $H_0 := H - H(\cdot\,,0)$ and $\sigma := -H(\cdot\,,0)$, H can be written as

$$H(x,p) = H_0(x,p) - \sigma(x), \quad \sigma(x) \geq 0, \qquad (x,p) \in \mathbb{R}^{2n}.$$

Note that $H_0(x,0) = 0$ for all $x \in \mathbb{R}^n$.

We assume here that there exist $\alpha > 0$, $\beta \geq 1$, $\gamma > 1$ and $C_0 > 0$ such that

$$\alpha|p|^\beta \leq H_0(x,p) \leq \alpha^{-1}|p|^\beta, \quad \sigma(x) \leq C_0(1+|x|)^{-\beta\gamma}, \quad \text{for all } (x,p) \in \mathbb{R}^{2n}. \tag{51}$$

Next, we define $\psi_0 \in \mathrm{Lip}(\mathbb{R}^n)$ by $\psi_0(x) := -\alpha^{-1}C_0\int_0^{|x|}(1+r)^{-\gamma}dr + C_1$, $x \in \mathbb{R}^n$, where $C_1 > 0$ is taken so that $\psi_0 \geq 0$ in $\mathbb{R}^n$. Then, for $x \neq 0$,

$$H(x, D\psi_0(x)) \geq \alpha|D\psi_0(x)|^\beta - \sigma(x) = C_0(1+|x|)^{-\beta\gamma} - \sigma(x) \geq 0,$$

which implies that $\psi_0 \in \mathcal{S}_H^+$. In particular, H satisfies (A4) with $\phi_0 = 0$ and the above ψ_0.

We now claim that H satisfies property (A6). Let $\lambda > 0$ be a constant which will be specified later. Observe that

$$H_0(x, -\lambda p) \geq \alpha|\lambda p|^\beta \geq \alpha^2\lambda^\beta \cdot \alpha^{-1}|p|^\beta = \alpha^2\lambda^\beta H_0(x,p) \quad \text{for all } (x,p) \in \mathbb{R}^{2n}.$$

Since $H_0 \geq 0$ in $\mathbb{R}^{2n}$ in view of the first condition of (51), by choosing λ so that $\alpha^2\lambda^\beta \geq 1$, we get $H(x,-\lambda p) \geq H(x,p)$ for all $(x,p) \in \mathbb{R}^{2n}$. Hence, H satisfies (A6). In this case, we have $\Phi_0 = \mathrm{BUC}(\mathbb{R}^n)$.

We give here an example of Theorem 4.1.

Example 5.3. Let $n = 1$, and let $f \in \mathrm{BUC}(\mathbb{R})$ be any function such that $f \geq 0$ in $\mathbb{R}$. We set $F(x) := \int_0^x f(y)\,dy$ for $x \in \mathbb{R}$ and define $H \in C(\mathbb{R}^2)$ and $\phi \in \mathrm{UC}(\mathbb{R})$ by

$$H(x,p) := p^2 - f(x)^2, \quad \phi(x) := \min\{F(x), -F(x)\}, \quad (x,p) \in \mathbb{R}^2.$$

Note that H satisfies (A1)-(A3) and (A5)$_\pm$. Moreover, since F, $-F \in \mathcal{S}_H$, we see in view of convexity (A3) that $\phi \in \mathcal{S}_H$. Thus, assumption (A4) is also fulfilled with $\phi_0 = \psi_0 = \phi$.

Now, let $p_0 \in \mathrm{BUC}(\mathbb{R})$ be any function satisfying the following property: for any $\varepsilon > 0$, there exists an $l > 0$ such that

$$\min_{|y|\leq l} p_0(x+y) < \inf_{\mathbb{R}} p_0 + \varepsilon \qquad \text{for all } x \in \mathbb{R}. \tag{52}$$

Remark that (52) is valid for any (lower semi-) almost periodic function.

We set $u_0 := \phi + p_0 \in \Phi_0$ and let $u(x,t)$ be the solution of the Cauchy problem (1) with H and u_0 defined above. What we prove is the following convergence:

$$u(\,\cdot\,,t) \longrightarrow \phi + \inf_{\mathbb{R}}(u_0 - \phi) \qquad \text{in } C(\mathbb{R}) \quad \text{as } t \to \infty. \tag{53}$$

In what follows, we only consider the case where $\inf_{\mathbb{R}}(u_0 - \phi) = \inf_{\mathbb{R}} p_0 = 0$ (which does not lose any generality). In this case, we have $u_\infty = \phi$ in $\mathbb{R}$. Note also that condition (17) of Theorem 2.5 does not hold in general.

To show the convergence (53), we check (42) in Theorem 4.1. Notice that Theorem 2.2 of [14] cannot be applied to this example since both H and u_0 do not satisfy semi- or semi-almost periodicity assumptions. Fix any $x \in \mathbb{R}$, $\gamma \in \mathcal{E}_x$, and choose any diverging $\{t_j\} \subset (0,\infty)$ such that $u^+(x) = \lim_{j\to\infty} u(x,t_j)$. We set $y_j := \gamma(-t_j)$ for $j \in \mathbb{N}$. By taking a subsequence of $\{y_j\}$ if necessary, we have either $\sup_j |y_j| < \infty$ or $\lim_{j\to\infty}|y_j| = \infty$. Since the former case can be reduced to Theorem 2.5, it suffices to consider the latter case. In what follows, we assume that $\lim_{j\to\infty} y_j = \infty$ (the case where $\lim_{j\to\infty} y_j = -\infty$ can be treated in a similar way), and any subsequence of $\{y_j\}$ will be denoted by the same $\{y_j\}$.

Since $\{f(\cdot + y_j)\}_j$, $\{p_0(\cdot + y_j)\}_j$ and $\{u_0(\cdot + y_j) - u_0(y_j)\}_j$ are pre-compact in $C(\mathbb{R})$, there exist f_+, $q_0 \in \mathrm{BUC}(\mathbb{R})$ and $v_0 \in \mathrm{UC}(\mathbb{R})$ such that

$$f(\cdot + y_j) \longrightarrow f_+ \quad \text{and} \quad p_0(\cdot + y_j) \longrightarrow q_0 \qquad \text{in } C(\mathbb{R}) \quad \text{as } j \to \infty \tag{54}$$

and $u_0(\cdot + y_j) - u_0(y_j) \longrightarrow v_0$ in $C(\mathbb{R})$ as $j \to \infty$. Remark here that q_0 inherits property (52). Indeed, fix any $\varepsilon > 0$ and choose an $l > 0$ so that (52) holds. Observe that $\inf_{\mathbb{R}} q_0 = 0$ by the second convergence in (54) and the fact that $\inf_{\mathbb{R}} p_0 = \inf_{\mathbb{R}}(u_0 - \phi) = 0$. For each $j \in \mathbb{N}$, we choose a $z_j \in (-l, l)$ such that $p_0(x + y_j + z_j) = \min_{|y|\le l} p_0(x + y_j + y) < \varepsilon$. Since $\sup_j |z_j| \le l$, we may assume that $\lim_{j\to\infty} z_j = z$ for some $z \in (-l, l)$. Thus,

$$\min_{|y|\le l} q_0(x + y) \le q_0(x + z) = \lim_{j\to\infty} p_0(x + y_j + z_j) < \varepsilon,$$

which shows that (52) is valid with q_0 in place of p_0.

We now set $F_+(x) := \int_0^x f_+(y)\, dy$ for $x \in \mathbb{R}$. Then, we see that

$$\phi(\cdot + y_j) - \phi(y_j) \longrightarrow -F_+ \quad \text{in } C(\mathbb{R}) \quad \text{as } j \to \infty. \tag{55}$$

It is also not difficult to check that $v_0 = -F_+ + q_0 - q_0(0)$ in $\mathbb{R}$. We set $G(x,p) := p^2 - f_+(x)^2$ and define $d_G \in C(\mathbb{R}^2)$ by (7) with G instead of H. Observe that

$$d_G(x, y) = \max\{F_+(x) - F_+(y), F_+(y) - F_+(x)\}, \qquad x,\, y \in \mathbb{R}.$$

Since F_+ is non-decreasing on $\mathbb{R}$, we have

$$\begin{aligned} v_0^-(x) \le \inf_{y\ge x}\{d_G(x,y) + v_0(y)\} &= \inf_{y\ge x}\{F_+(y) - F_+(x) - F_+(y) + q_0(y) - q_0(0)\} \\ &= -F_+(x) - q_0(0) + \inf_{y\ge x} q_0(y). \end{aligned}$$

In view of property (52) for q_0, we obtain $v_0^- \le -F_+ - q_0(0)$ in $\mathbb{R}$. On the other hand, observing that $v_0(x) \ge -F_+(x) - q_0(0) \in \mathcal{S}_H$, we have $v_0^- \ge -F_+ - q_0(0)$ in $\mathbb{R}$. Thus, $v_0^- = -F_+ - q_0(0)$ in $\mathbb{R}$. This implies that $v_\infty = v_0^-$ in $\mathbb{R}$. Since $v_\infty(0) = -F_+(0) - q_0(0) = -q_0(0)$, we find that

$$\limsup_{j\to\infty}\,(u_\infty - u_0)(y_j) = -\liminf_{j\to\infty}\,(u_0 - \phi)(y_j) = -q_0(0) = v_\infty(0),$$

which is (42).

The following can be regarded as a generalization of the previous example to multi-dimensional cases.

Example 5.4. For each $i = 1, \ldots, n$, let $f_i \in \mathrm{BUC}(\mathbb{R}^n)$, $i = 1, \ldots, n$, be such that $\inf_{\mathbb{R}^n} f_i \ge 0$ or $\sup_{\mathbb{R}^n} f_i \le 0$. We set

$$H(x,p) = \max_{1\le i\le n} \{p_i^2 - f_i(x)p_i\}, \qquad x \in \mathbb{R}^n, \quad p = (p_1, \ldots, p_n) \in \mathbb{R}^n.$$

Clearly, $H(x,0) = 0$ for all $x \in \mathbb{R}^n$ and H satisfies (A1)-(A3), (A4) with $\phi_0 = \psi_0 = 0$, and $(A5)_+$. Observe here that $\Phi_0 = \mathrm{BUC}(\mathbb{R}^n)$. We choose any $u_0 \in \Phi_0$ satisfying the following property: for any $\varepsilon > 0$, there exists an $l > 0$ such that

$$\min_{|y|\le l} u_0(x+y) < \inf_{\mathbb{R}^n} u_0 + \varepsilon \qquad \text{for all } \ x \in \mathbb{R}^n. \tag{56}$$

Let $u(x,t)$ be the solution of the Cauchy problem (1) with H and u_0 defined above. We claim here that (53) holds with $\phi = 0$, that is,

$$u(\,\cdot\,,t) \longrightarrow \inf_{\mathbb{R}^n} u_0 \qquad \text{in } \ C(\mathbb{R}^n) \quad \text{as } \ t \to \infty. \tag{57}$$

To prove this, we check (45) in Corollary 4.2. For this purpose, we may assume without loss of generality that $\inf_{\mathbb{R}^n} u_0 = 0$. Then, $u_0 \ge u_0^- \ge 0$ in $\mathbb{R}^n$. We also observe from the assumption on f_i that, for any $\phi \in \mathcal{S}_H^-$, $\phi(x)$ is non-increasing or non-decreasing with respect to the k-th component of x for every $1 \le k \le n$. This and (56) implies that $u_0^- = 0$ in $\mathbb{R}^n$.

Let $\{y_j\} \subset \mathbb{R}^n$ be any sequence such that $u_0(\,\cdot\, + y_j) \longrightarrow v_0$ in $C(\mathbb{R}^n)$ as $j \to \infty$ for some $v_0 \in \mathrm{BUC}(\mathbb{R}^n)$. Remark that $\inf_{\mathbb{R}^n} v_0 = 0$ and v_0 inherits property (56). By taking a subsequence of $\{y_j\}$ if necessary, we may assume that $f_i(\,\cdot\, + y_j) \longrightarrow g_i$ in $C(\mathbb{R}^n)$ as $j \to \infty$ for each $i = 1, \dots, n$ for some $g_i \in \mathrm{BUC}(\mathbb{R}^n)$, $i = 1, \dots, n$. Then, we have $\inf_{\mathbb{R}^n} g_i \ge 0$ or $\sup_{\mathbb{R}^n} g_i \le 0$ according to the sign of f_i for each $i = 1, \dots, n$.

Now, we set $G(x,p) = \max_{1\le i\le n} \{p_i^2 - g_i(x)p_i\}$, $x \in \mathbb{R}^n$, $p = (p_1, \dots, p_n) \in \mathbb{R}^n$. Then, for any $\phi \in \mathcal{S}_G^-$, $\phi(x)$ is non-increasing or non-decreasing with respect to the k-th component of x for every $1 \le k \le n$. This fact together with property (56) for v_0 ensure that $v_0^- = 0$ in $\mathbb{R}^n$. Hence, we conclude that (45) is valid.

Remark 5.5. The Hamiltonian in Example 5.4 can be generalized in the following way. Let H satisfy (A1)-(A3), $(A5)_+$ and $H(x,0) = 0$ for all $x \in \mathbb{R}^n$. We set $\phi_0 = \psi_0 = 0$ in (A4) and choose any $u_0 \in \Phi_0$ satisfying (56). We set $K_H(x) = \{p \in \mathbb{R}^n \,|\, H(x,p) \le 0\}$ for $x \in \mathbb{R}^n$ and denote by $K_H^*(x)$ the polar cone of $K_H(x)$, i.e.,

$$K_H^*(x) := \{\xi \in \mathbb{R}^n \,|\, \xi \cdot p \le 0 \quad \text{for all } \ p \in K_H(x)\}.$$

Fix any $x \in \mathbb{R}^n$, $\gamma \in \mathcal{E}_x$ and any diverging $\{t_j\} \subset (0,\infty)$ and set $y_j := \gamma(-t_j)$ for $j \in \mathbb{N}$. Let $G \in C(\mathbb{R}^{2n})$ and $v_0 \in C(\mathbb{R}^n)$ be the functions satisfying, respectively, $H(\,\cdot\, + y_j, \,\cdot\,) \longrightarrow G$ in $C(\mathbb{R}^{2n})$ as $j \to \infty$, and $u_0(\,\cdot\, + y_j) \longrightarrow v_0$ in $C(\mathbb{R}^n)$ as $j \to \infty$. We define $K_G(x)$ and $K_G^*(x)$ similarly as $K_H(x)$ and $K_H^*(x)$, respectively. Now, we assume the following:

(H) There exists a cone $K \subset \mathbb{R}^n$ with vertex 0 such that

$$\mathrm{Int}(K) \ne \emptyset \quad \text{and} \quad K \subset K_H^*(x),\ K_G^*(x) \quad \text{for all } x \in \mathbb{R}^n.$$

We claim that the convergence (57) still holds under (H). Note that H in Example 5.4 satisfies property (H).

To check the claim, we first observe that $d_H(x,y) = 0$ if $x - y \in K$. Indeed, for $\xi \in K$ and $t > 0$, there exists a $q \in L^\infty(0,t;\mathbb{R}^n)$ such that $q(s) \in (\partial_c d_H(\,\cdot\,,y))(y + s\xi) \subset K_H(y + s\xi)$ a.e. $s \in [0,t]$, and

$$d_H(y + t\xi, y) = \int_0^t q(s) \cdot \xi \, ds \le 0,$$

from which we obtain $d_H(y+\xi,y)=0$ for all $y\in\mathbb{R}^n$ and $\xi\in K$. Similarly, we have $d_G(y+\xi,y)=0$ for all $y\in\mathbb{R}^n$ and $\xi\in K$.

Now, fix any $x\in\mathbb{R}^n$. Then, in view of (56), for any $\varepsilon>0$, there exists a sequence $\{z_j\}\subset\mathbb{R}^n$ such that $x-z_j\in K$ and $u_0(z_j)\le\inf_{\mathbb{R}^n}u_0+\varepsilon$ for all $j\in\mathbb{N}$. Thus, $u_0^-(x)\le d_H(x,z_j)+u_0(z_j)\le\inf_{\mathbb{R}^n}u_0+\varepsilon$, which implies that $u_0^-=\inf_{\mathbb{R}^n}u_0$ in $\mathbb{R}^n$. Similarly, we see that $v_0^-=\inf_{\mathbb{R}^n}v_0$ in $\mathbb{R}^n$.

We now show that $\inf_{\mathbb{R}^n}u_0=\inf_{\mathbb{R}^n}v_0$, from which we obviously obtain (45) in Corollary 4.2 and therefore (57). In view of property (56), we can choose a sequence $\{z_j\}\subset\mathbb{R}^n$ such that $u_0(z_j)\le\inf_{\mathbb{R}^n}u_0+\varepsilon$ and $|y_j-z_j|\le l$ for all $j\in\mathbb{N}$. We may assume by taking a subsequence of $\{z_j\}$ that $y_j-z_j\longrightarrow z$ for some $z\in\mathbb{R}^n$ as $j\to\infty$. Then,$u_0(z_j)=u_0(z_j-y_j+y_j)\longrightarrow v_0(z)$ as $j\to\infty$, which implies that $\inf_{\mathbb{R}^n}v_0\le v_0(z)\le\inf_{\mathbb{R}^n}u_0+\varepsilon$. Thus, we have $\inf_{\mathbb{R}^n}v_0\le\inf_{\mathbb{R}^n}u_0$. Since the opposite inequality is obvious by definition, we conclude that $\inf_{\mathbb{R}^n}v_0=\inf_{\mathbb{R}^n}u_0$. Hence, the proof is complete.

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