

Some Applications of Fourier Analysis to Uniform Distribution Mod 1.

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We say that a sequence $(\lambda_n)_{n=1}^{\infty}$ of real numbers is uniformly distributed mod 1 (u.d. mod 1) if we have for any $0 \leq \alpha < \beta \leq 1$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n 1 = \beta - \alpha,$$

$$\alpha \leq \{\lambda_k\} \leq \beta$$

where $\{\lambda_k\}$ denotes the fractional part of λ_k . Then Weyl proved that $(\lambda_n)_{n=1}^{\infty}$ is u.d. mod 1 iff

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{2\pi i h \lambda_k} = 0,$$

for all fixed natural numbers h . Among other important results, he gave the following "metric result".

Theorem A. If $(a_n)_{n=1}^{\infty}$ is a sequence of distinct natural numbers, then $(x a_n)_{n=1}^{\infty}$ is u.d. mod 1 for almost all fixed x .

In fact, behind his proof of this theorem, a much more general principle was laid. For instance, the following generalization is possible [3].

Theorem B. If $(\lambda_n)_{n=1}^{\infty}$ is a sequence of real numbers such that

$$\inf_{n \neq m} |\lambda_n - \lambda_m| > 0,$$

then the sequence $(x\lambda_n)_{n=1}^{\infty}$ is u.d. mod 1 for almost all real numbers x .

The main object of this article is to indicate another approach so that we can improve these results to some extent.

Let

$$(1) \quad \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series of $f \in L^1$.

In 1920 around, Russian mathematician Lusin made the conjecture that for all $f \in L^2$, (1) converges for almost all x . About ten years later, Kolmogorov constructed an $f \in L^1$ such that (1) diverges for all x . However, in 1966 L. Carleson [1] has proved the Lusin conjecture affirmatively. A version of his theorem is

Theorem C. If $\sum_{n=1}^{\infty} (a_n^2 + b_n^2) < \infty$, then (1) converges for almost all x .

It will be worth stating here an open problem due to Orlicz [4]:

Give an example of (1) divergent for all x and such that

$$\sum_{n=1}^{\infty} (a_n^{2+\varepsilon} + b_n^{2+\varepsilon}) < \infty$$

for every $\varepsilon > 0$.

We may now apply Theorem C to the Fourier integral

$$\int_0^{\infty} f(t) e^{ixt} dt$$

and obtain the following

Theorem 1. If $\lambda_{k+1} - \lambda_k \geq \kappa > 0$ ($k=1,2,\dots$) and

$$\sum_{k=1}^{\infty} |a_k|^2 < \infty,$$

then

$$(2) \quad \sum_{k=1}^{\infty} a_k e^{i\lambda_k x}$$

converges for almost all x .

Corollary 1. Under the assumption of Theorem 1,

$$\prod_{k=1}^{\infty} (1 + a_k e^{i\lambda_k x})$$

converges for almost all x .

Corollary 2. If $\lambda_{k+1} - \lambda_k \geq \kappa > 0$ ($k=1,2,\dots$), then

$$\sum_{k=1}^N e^{i\lambda_k x h} = o(\sqrt{N \log N} (\log \log N)^{\delta})$$

for almost all x , where $h \in \mathbb{N}$ is fixed and $\delta > 1/2$ an arbitrary constant.

More generally, we can in fact prove

Theorem 2. If

$$(3) \quad \lambda_{k+1} - \lambda_k \geq \mu_k > 0, \quad (k=1,2,\dots)$$

$$\lambda_k \rightarrow \infty \quad (k \rightarrow \infty),$$

and for almost all x

$$(4) \quad \sum_{k=1}^{\infty} \frac{\mu_k}{\sin^2(\mu_k x/2)} |a_k|^2 < \infty,$$

then (2) converges for almost all x .

Corollary 3. The Dirichlet series

$$\sum_{n=1}^{\infty} \frac{c_n}{n^s} \quad (s = \sigma + it)$$

converges on $\sigma = 1/2$ for almost all t if

$$\sum_{n=1}^{\infty} |c_n|^2 < \infty.$$

From Theorem 2 we obtain the following

Theorem 3. If (3) holds and for almost all x

$$(5) \quad \sum_{k=1}^{\infty} \frac{\mu_k}{k^2 \sin^2(\mu_k x/2)} < \infty,$$

then the sequence $(x\lambda_k)_1^{\infty}$ is u.d. mod 1 for almost all x .

If we denote by p_n the n th prime ($p_1=2$), then for sufficiently large n we have

$$(\log p_{n+1})^{\delta} - (\log p_n)^{\delta} \gg \frac{(\log n)^{\delta-2}}{n}, \quad (\delta > 0)$$

by the prime number theorem and Hoheisel's theorem. Thus the following interesting result is immediately deduced from Theorem 3.

Corollary 4. The sequence $(x(\log p_n)^{\delta})_1^{\infty}$ is u.d. mod 1 for almost all x , provided $\delta > 3$.

Our method cannot afford anything if $0 < \delta \leq 3$, and we are tempted to conjecture that Corollary 4 is still true if $\delta > 1$. We can prove, however, that $(x(\log p_n)^{\delta})_1^{\infty}$ is $(M, (\log n)^{\delta-1}/n)$ - u.d. mod 1 for almost all x if $\delta > 1$ (cf. Corollary 5 below). A sequence $(\lambda_n)_1^{\infty}$ of real numbers is said to be (M, α_n) - u.d. mod 1 if there exists a sequence of positive numbers such that

$$\alpha_1 > \alpha_2 > \dots > \alpha_n > \dots$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty$$

and

$$\sum_{n=1}^N \alpha_n e^{2\pi i h \lambda_n} = 0 \quad \left(\sum_{n=1}^N \alpha_n \right)$$

for any fixed $h \in \mathbb{N}$.

It is easy to see that any u.d. sequence is (M, α_n) - u.d. for any α_n .

By a similar argument due to Cossar [2], we obtain

Theorem 4. If (3) holds and for almost all x

$$\sum_{k=1}^{\infty} \left| \frac{c_k}{\lambda_k} \right|^p \cdot \frac{\mu_k}{|\sin(\mu_k x/2)|^p} < \infty,$$

for some $1 < p \leq 2$, then

$$\sum_{k=1}^n c_k e^{i\lambda_k x} = 0 \quad (\lambda_{n+1})$$

for almost all x .

Corollary 5. If $\lambda_n \ll \sum_{k=1}^n c_k$ and

$$\lambda_{k+1} - \lambda_k \gg 1/k(\log k)^\alpha$$

for some $0 \leq \alpha < 1$, then $(x\lambda_k)_1^\infty$ is (M, c_n) - u.d. mod 1 for almost all x .

We remark that our argument breaks down if $\alpha = 1$. So it does not follow that $(x \log p_n)_1^\infty$ is $(M, 1/n)$ - u.d. mod 1 for almost all x .

References

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- [4] Mauldin, R. D. (Ed.) : The Scottish Book, Birkhäuser, 1981.