

Products of k -spaces, and questions

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As is well-known, every product of a locally compact space with a k -space is a k -space, but not every product of a metric space with a k -space is a k -space. We consider characterizations or conditions for (finite) products of k -spaces to be k -spaces, and pose related questions. For other topics on the products of k -spaces, see [T3], [T4], for example.

We assume that spaces are regular T_1 , and maps are continuous and onto.

1 Definitions and Preliminaries

Let X be a space, and let $\mathcal{P}$ be a (not necessarily open or closed) cover of X . Then X is determined by a cover $\mathcal{P}$,¹ if $U \subset X$ is open in X if and only if $U \cap P$ is relatively open in P for every $P \in \mathcal{P}$. Here, we can replace “open” by “closed”. Every space is determined by its open (or hereditarily closure-preserving closed) cover.

Let us recall that a space is a k -space (resp. sequential space) if it is determined by a cover of compact (resp. compact metric) subsets. Sequential space are k -spaces, and the converse holds if points are G_δ -sets. A space X is called a k_ω -space [M3] (resp. s_ω -space) if X is determined by a countable cover of compact (resp. compact metric) subsets.

A space X is called a bi - k -space (resp. bi -quasi- k -space) [M3] if, whenever a filter base $\mathcal{F}$ accumulates at $x \in X$, then there exists a k -sequence (resp. q -sequence) $\{A_n : n \in N\}$ such that $x \in \overline{F \cap A_n}$ for all $n \in N$ and all $F \in \mathcal{F}$. When the filter base $\mathcal{F}$ is a decreasing sequence, then such a space X is a countably bi - k -space (resp. countably bi -quasi- k -space) [M3]. Here, a k -sequence (resp. q -sequence) is a decreasing sequence $\{A_n : n \in N\}$ such that $A = \bigcap \{A_n : n \in N\}$ is compact (resp. countably compact), and any open set $U \supset A$ contains some A_n ([M3]).

Let us recall that a space X is of pointwise countable type (resp. q -space) if each point has nbds $\{V_n : n \in N\}$ which is a k -sequence (resp. q -

¹Following [GMT], we shall use “ X is determined by $\mathcal{P}$ ” instead of the usual “ X has the weak topology with respect to $\mathcal{P}$ ”.

sequence). Also, a space is an M -space if and only if it is the inverse image of a metric space under a quasi-perfect map. The following diagrams hold.

(a) Locally compact spaces, or first countable spaces $\rightarrow$ spaces of point-wise countable type $\rightarrow$ bi- k -spaces $\rightarrow$ countably bi- k -spaces $\rightarrow$ k -spaces.

(b) Locally countably compact spaces, or M -spaces $\rightarrow$ q -spaces $\rightarrow$ bi-quasi- k -spaces $\rightarrow$ countably bi-quasi- k -spaces.

A space X is called a *Tanaka space* [My2], if X satisfies the following condition (C) in [T2].

(C) Let $\{A_n : n \in N\}$ be a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in N$. Then there exist $x_n \in A_n$ such that $\{x_n : n \in N\}$ converges to some point $y \in X$. If $y = x$, then such a space X is called *countably bi-sequential* [M3] (= *strongly Fréchet* [S]).

Sequentially compact spaces, or sequential countably bi-quasi- k -spaces are Tanaka spaces. But, every Tanaka space (actually, sequentially compact space) need not be sequential, not even a k -space².

A space X is *strongly sequential* [M1] if, whenever $\{A_n : n \in N\}$ is a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in N$, then the point x belongs to the (idempotent) *sequential closure* of A , where A is the set of all limit points of convergent sequences $\{x_n : n \in N\}$ with $x_n \in A_n$. Namely, a space X is strongly sequential if and only if it is a sequential space such that if $\{A_n : n \in N\}$ is a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in N$, then the point x belongs to the (usual) closure of the above set A . Strongly Fréchet spaces are strongly sequential. Every strongly sequential space is precisely a sequential Tanaka space ([My2]).

A map $f : X \rightarrow Y$ is called *bi-quotient* [M2] if, whenever $y \in Y$ and $\mathcal{U}$ is a cover of $f^{-1}(y)$ by open subsets of X , then finitely many $f(U)$, with $U \in \mathcal{U}$, cover some nbd of y in Y . If $\mathcal{U}$ is countable, then such a map f is called *countably bi-quotient* [S]. Open maps, or perfect maps are bi-quotient. Every product of bi-quotient maps is bi-quotient, hence quotient ([M2]). A map $f : X \rightarrow Y$ is called a *compact* (resp. *s-map*) if every $f^{-1}(y)$ is compact (resp. separable).

²This is pointed out by Z. Dolecki or P. Nyikos.

In the following characterizations, (1) is well-known, (2) is routinely shown, and (3) is due to [M3].

Characterization: (1) X is a k -space (resp. sequential space) $\Leftrightarrow X$ is the quotient image of a locally compact (resp. locally compact, metric) space.

(2) (a) X is a k_ω -space (resp. s_ω -space) $\Leftrightarrow X$ is the quotient image of a locally compact Lindelöf (resp. locally compact, separable metric) space.

(b) X is a space determined by a point-finite cover of compact (resp. compact metric) subsets $\Leftrightarrow X$ is the quotient compact image of a locally compact paracompact (resp. locally compact metric) space. Here, we can replace “point-finite cover” by “point-countable cover”, but change “quotient compact image” to “quotient s -image”.

(3) (a) X is a bi- k -space (resp. bi-quasi- k -space) $\Leftrightarrow X$ is the bi-quotient image of a paracompact M -space (resp. M -space).

(b) X is a countably bi- k -space (resp. countably bi-quasi- k -space) $\Leftrightarrow X$ is the countably bi-quotient image of a paracompact M -space (resp. M -space).

In the following results, (1) is well-known (see [M1], for example). (2) (resp. (3)) is due to [M3] (resp. [M2]). (4) holds in view of [My1] and [M2], here note that every product of a first countable space with a strongly sequential space is strongly sequential ([M1]). (5) is due to [T1].

Result: (1) Every product of a locally compact space (resp. locally countably compact, sequential space) with a k -space (resp. sequential space) is a k -space (resp. sequential space).

(2) Every product of bi- k -spaces is a bi- k -space, hence a k -space.

(3) Every product of k_ω -spaces is a k_ω -space, hence a k -space.

(4)³ Every product of a first countable space with a sequential Tanaka space is a sequential space.

(5) For sequential spaces X and Y , $X \times Y$ is sequential if and only if

³This is an affirmative answer to the author's question (when he prepared [T2]). F. Mynard obtained this result by use of categorical method ([My1] & [My2]). The result is also proved by use of *multisequences* method ([D]), or directly shown without these methods ([L]).

it is a k -space.

2. Questions and Comments

Question 1. ([T5]) Every product of sequentially compact (or countably compact) k -spaces X and Y is a k -space ?

Comment: (1.1) Question 1 is affirmative if X or Y is sequential ([T1]). But, not every product of a countably compact first countable space with a k -space is a k -space.

(1.2) Every product of a k -and- q -space with a bi- k -space (or sequential q -space) is a k -space by (2.2) below. If Question 1 is affirmative, then every product of k -and- q -spaces is a k -space.

(1.3) Let X be sequentially compact (countably compact; q), and let Y be sequentially compact (resp. countably compact k ; q -and- k), then $X \times Y$ is sequentially compact (resp. countably compact; q). Note that every sequentially compact space need not be a k -space.

Question 2. Let X be a k -space which is bi-quasi- k . Let Y be a sequential space. Then the following are equivalent ?

- (a) $X \times Y$ is a k -space.
- (b) X is locally countably compact, or Y is a Tanaka space ?

Comment: (2.1) Question 2 is affirmative if X is a bi- k -space by (2.2) & (2.4) below.

(2.2) In Question 2, (b) $\Rightarrow$ (a) holds. In general, the following case (c₁) or (c₂) implies that $X \times Y$ is a k -space ([T5]).

(c₁) X is a k -space which is bi-quasi- k , and Y is a sequential Tanaka space (in particular, a sequential countably bi-quasi- k -space).

(c₂) X is a bi- k -space, and Y is a k -space which is countably bi-quasi- k .

(2.3) Every product of sequential countably bi- k -spaces (actually, countably bi-sequential, countable spaces) need not be a k -space (not a Tanaka space) under ($2^{\aleph_0} < 2^{\aleph_1}$) ([O]).

(2.4) In Question 2, (a) $\Rightarrow$ (b) holds if X is a first countable space ([T2]), more generally, a bi- k -space ([TS], etc.).

(2.5) Every product of sequential Tanaka spaces (actually, countably bi-sequential, countable spaces) need not be a Tanaka space (hence, not strongly sequential). (Also, cf. (2.3)). But, every product $X \times Y$ of

Tanaka spaces is a Tanaka space if X is bi-quasi- k . Thus, for sequential spaces X and Y , (c_1) or (c_2) in (2.2) implies that $X \times Y$ is a Tanaka space which is sequential by means of (2.2) and *Result* (5). In view of this and (2.3), the author has following question: For sequential spaces X and Y , if $X \times Y$ is a Tanaka space, then $X \times Y$ is sequential ?

Let $S = \{\infty\} \cup \{p_n : n \in N\} \cup \{p_{nm} : n, m \in N\}$ be an infinite countable space such that each p_{nm} is isolated in S , $K = \{p_n : n \in N\}$ converges to $\infty \notin K$, and each $L_n = \{p_{nm} : m \in N\}$ converges to $p_n \notin L_n$. We recall the following canonical spaces; the *Arens' space* S_2 , and the *sequential fan* S_ω . S_2 is not Fréchet, but S_ω is Fréchet.

$S_2 = S$, but $\bigcup \{F_n : n \in N\}$ is closed in S for every finite $F_n \subset L_n$ ($n \in N$).

$S_\omega = S_2 / (K \cup \{\infty\})$ (i.e., the space obtained from the topological sum of countably many convergent sequences by identifying all the limit points).

Question 3. ([TS]) Let X be a bi- k -space, and let Y be a sequential space. Then the following are equivalent ?

- (a) $X \times Y$ is a k -space.
- (b) X is locally countably compact, or Y contains no (closed) copy of S_ω , and no (closed) copy of S_2 ?

Let us recall that a cover $\mathcal{P}$ of a space X is a k -network for X if, for any compact subset K , and any open set V with $K \subset V$, $K \subset \bigcup \mathcal{F} \subset V$ for some finite $\mathcal{F} \subset \mathcal{P}$. If K is a single point, then such a cover $\mathcal{P}$ is called a *network*. Bases are k -networks, and k -networks are networks. Quotient s -images (or closed images) of metric spaces have point-countable k -networks. Paracompact M -spaces with point-countable k -networks are metrizable ([GMT]).

Comment: (3.1) In Question 3, (a) $\Rightarrow$ (b) holds ([TS]).

(3.2) Question 3 is reduced to the following question in view of (2.1): For a sequential space X , X is a Tanaka space if and only if it contains no (closed) copy of S_ω , and no S_2 ? (The “only if” part holds).

(3.3) Question 3 is affirmative if the sequential space Y is one of the following spaces ([TS]).

- (A₁) Fréchet space.
- (A₂) Space in which every point is a G_δ -set.
- (A₃) Hereditarily normal space.
- (A₄) Space having a point-countable k -network.
- (A₅) Closed image of a countably bi- k -space.
- (A₆) Closed image of an M -space.

(3.4) The author does not know whether Question 3 is affirmative when the sequential space Y is the quotient s -image of a paracompact (countably) bi- k -space ([TS]). Question 3 is affirmative if the domain is metric by (A₄).

Question 4. ([T6]) For a k -space X , X is locally countably compact if and only if $X \times Y$ is a k -space for every quotient compact image Y of a *locally compact* metric space ?

Let us recall that a space X is called *symmetric* if there exists a real valued, non-negative function d defined on $X \times X$ such that (a) $d(x, y) = 0$ iff $x = y$, (b) $d(x, y) = d(y, x)$, and (c) $F \subset X$ is closed in X iff $d(x, F) > 0$ for any $x \in X - F$. If we replace (c) by " $d(x, F) = 0$ iff $x \in \overline{F}$ ", then such a space X is called *semi-metric*. Semi-metric spaces, or quotient compact images of metric spaces (e.g., the space S_2) are symmetric. Symmetric spaces are sequential. Symmetric M -spaces are metrizable ([N]).

Comment: (4.1) In Question 4, the "only if" part holds.

(4.2) Question 4 is affirmative if X is one of the following spaces. For (B₁), see (5.2) below. For (B₄), we can replace " k -space" by "symmetric space" in Question 4.

(B₁) Bi- k -space.

(B₂) Space having character $\leq 2^\omega$ (in particular, locally separable space).

(B₃) Space having a point-countable k -network.

(B₄) Symmetric space.

(4.3) Question 4 is affirmative if we omit the locally compactness of the metric domain. Question 4 is also affirmative if we replace "metric space" by "Fréchet space"; or "quotient compact image" by "closed image".

(4.4) A k -space X is locally compact if and only if $X \times Y$ is a k -space for every quotient compact image Y of a locally compact, paracompact

space. Here, we can replace “quotient compact image” by “closed image”.

Question 5. ([T6]) For a k -space X , X is a locally k_ω -space if and only if $X \times Y$ is a k -space for every k_ω -space Y ?

Comment: (5.1) In Question 5, the “only if” part holds by *Result* (3).

(5.2) If we replace “ k_ω -space” Y by “ s_ω -space” Y , then Question 5 is negative under $(MA + \neg CH)$.

(5.3) A bi- k -space X is locally compact (resp. locally countably compact) if and only if $X \times Y$ is a k -space for every k_ω -space (resp. s_ω -space) Y . Here, the space Y can be chosen to be the quotient compact (or closed) image of a locally compact Lindelöf (resp. locally compact separable metric) space.

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