

The asymptotic behavior of multiple zeta functions at non-positive integers

TOMOKAZU ONOZUKA

1 Introduction

The Euler-Zagier multiple zeta function $\zeta_d(s_1, \dots, s_d)$ is defined by

$$\zeta_d(s_1, \dots, s_d) := \sum_{m_1=1}^{\infty} \cdots \sum_{m_d=1}^{\infty} \frac{1}{m_1^{s_1} (m_1 + m_2)^{s_2} \cdots (m_1 + \cdots + m_d)^{s_d}} \quad (1.1)$$

where s_i ($i = 1, \dots, d$) are complex variables. Matsumoto [4] proved that the series (1.1) is absolutely convergent in

$$\{(s_1, \dots, s_d) \in \mathbb{C}^d \mid \Re(s_d(d-k+1)) > k \ (k = 1, \dots, d)\}$$

where $s_d(n) = s_n + s_{n+1} + \cdots + s_d$ ($n = 1, \dots, d$). Akiyama, Egami and Tanigawa [1] and Zhao [7] proved the meromorphic continuation to the whole space independently.

The function $\zeta_d(s_1, \dots, s_d)$ has singularities on

$$\begin{cases} s_d = 1, \\ s_{d-1} + s_d = 2, 1, 0, -2, -4, \dots, \\ s_d(d-j+1) \in \mathbb{Z}_{\leq j} \ (j = 3, 4, \dots, d), \end{cases} \quad (1.2)$$

where $\mathbb{Z}_{\leq j}$ is the set of integers less than or equal to j ; $\mathbb{Z}_{\geq j}$ is defined similarly. Therefore $(-r_1, \dots, -r_d) \in \mathbb{Z}_{\leq 0}^d$ lies on the set of singularities. Moreover, it is an indeterminacy of $\zeta_d(s_1, \dots, s_d)$. For example, Sasaki [6] proved that

$$\lim_{s_3 \rightarrow 0} \lim_{s_2 \rightarrow 0} \lim_{s_1 \rightarrow 0} \zeta_3(s_1, s_2, s_3) = -\frac{3}{8}, \quad (1.3)$$

$$\lim_{s_1 \rightarrow 0} \lim_{s_2 \rightarrow 0} \lim_{s_3 \rightarrow 0} \zeta_3(s_1, s_2, s_3) = -\frac{1}{4}. \quad (1.4)$$

Since $(0, 0, 0)$ is an indeterminacy of $\zeta_3(s_1, s_2, s_3)$, (1.3) and (1.4) give different values.

Akiyama, Egami and Tanigawa [1] defined the regular values by

$$\zeta_d(-r_1, \dots, -r_d) := \lim_{s_1 \rightarrow -r_1} \cdots \lim_{s_d \rightarrow -r_d} \zeta_d(s_1, \dots, s_d),$$

and Akiyama and Tanigawa [2] considered the reverse and central values given by

$$\begin{aligned}\zeta_d^R(-r_1, \dots, -r_d) &:= \lim_{s_d \rightarrow -r_d} \cdots \lim_{s_1 \rightarrow -r_1} \zeta_d(s_1, \dots, s_d), \\ \zeta_d^C(-r_1, \dots, -r_d) &:= \lim_{\varepsilon \rightarrow 0} \zeta_d(-r_1 + \varepsilon, \dots, -r_d + \varepsilon),\end{aligned}$$

respectively. Further, Sasaki [6] generalized the regular and reverse values. He defined multiple zeta values for coordinatewise limits by

$$\zeta_d(\overset{i_1}{-r_1}, \dots, \overset{i_d}{-r_d}) := \lim_{\substack{s_j \rightarrow -r_j \\ i_j = d}} \cdots \lim_{\substack{s_j \rightarrow -r_j \\ i_j = 1}} \zeta_d(s_1, \dots, s_d),$$

where $\{i_1, \dots, i_d\} = \{1, \dots, d\}$. He obtained all multiple zeta values of depth 3 for coordinatewise limits. On the other hand, Komori [3] considered more general multiple zeta functions, and he obtained multiple zeta values at non-positive integers given by

$$\begin{aligned}\zeta_d(\overset{w}{-\mathbf{r}}) &= \lim_{z_{w^{-1}(d)} \rightarrow -r_{w^{-1}(d)}} \cdots \lim_{z_{w^{-1}(1)} \rightarrow -r_{w^{-1}(1)}} \zeta_d(z_1, \dots, z_d), \\ \zeta_d(\overset{\boldsymbol{\theta}}{-\mathbf{r}}) &= \zeta_d(\overset{\theta_1}{-r_1}, \dots, \overset{\theta_d}{-r_d}) = \lim_{\delta \rightarrow 0} \zeta_d(-r_1 + \delta\theta_1, \dots, -r_d + \delta\theta_d),\end{aligned}$$

where $-\mathbf{r} = (-r_1, \dots, -r_d) \in \mathbb{Z}_{\leq 0}^d$, $w \in \mathfrak{S}_d$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in \mathbb{C}^d$. To obtain these values by Komori's method, we need to compute generalized multiple Bernoulli numbers.

In the present paper, we calculate the asymptotic behavior of multiple zeta functions at non-positive integers. By using that result, we can evaluate the limit values of multiple zeta functions at non-positive integers. For example,

$$\lim_{\varepsilon \rightarrow 0} \zeta_3(\varepsilon^2, \varepsilon, \varepsilon) = -\frac{1}{3}. \quad (1.5)$$

This limit value is not contained in the above 2 kinds of values, however by the result, we can compute this value.

2 Main Theorem

In this section, we state the main theorem.

Let B_m be the m th Bernoulli number, and $B(x, y)$ be the beta function. For $(m_1, \dots, m_d) \in \mathbb{Z}_{\geq 0}^d$, $(p_1, \dots, p_d) \in \mathbb{Z}_{\geq 0}^d$ and $(\varepsilon_1, \dots, \varepsilon_d) \in \mathbb{C}^d$, let $m_d(n)$, $p_d(n)$ and $\varepsilon_d(n)$ be $m_n + m_{n+1} + \dots + m_d$, $p_n + p_{n+1} + \dots + p_d$ and $\varepsilon_n + \varepsilon_{n+1} + \dots + \varepsilon_d$ respectively. In addition, the Pochhammer symbol $(a)_n$ is defined by $(a)_n := \Gamma(a+n)/\Gamma(a)$.

Theorem 1. *Suppose that $\varepsilon_j \neq 0$, $\varepsilon_d(j) \neq 0$ ($j = 1, \dots, d$), $|\varepsilon_1| + \dots + |\varepsilon_d| \leq \frac{1}{2}$ and $|\varepsilon_k/\varepsilon_d(j)| \ll 1$ as $(\varepsilon_1, \dots, \varepsilon_d) \rightarrow (0, \dots, 0)$ ($j = 1, \dots, d$, $k = j, \dots, d$). Then for*

$m_j \in \mathbb{Z}_{\geq 0}$ ($j = 1, \dots, d$), we have

$$\begin{aligned} \zeta_d(-m_1 + \varepsilon_1, \dots, -m_d + \varepsilon_d) = & (-1)^{m_d} m_d! \sum_{\substack{p_1 + \dots + p_d = d+M \\ p_1, \dots, p_d \geq 0 \\ -m_d(j) - d + j + p_d(j) < 2 \text{ or} \\ -m_d(j-1) - d + j + p_d(j) \geq 2 \ (2 \leq j \leq d)}} \frac{B_{p_1} \dots B_{p_d}}{p_1! \dots p_d!} \times \\ & \times \prod_{j=2}^d \frac{[\varepsilon_d(j)]_{-m_d(j) - d + j + p_d(j) - 1}}{[\varepsilon_d(j-1)]_{-m_d(j-1) - d + j + p_d(j) - 1}} + \sum_{j=1}^d O(\varepsilon_j) \end{aligned}$$

as $(\varepsilon_1, \dots, \varepsilon_d) \rightarrow (0, \dots, 0)$, where

$$\begin{aligned} M &:= m_1 + \dots + m_d, \\ [a]_n &:= \begin{cases} a(n-1)! & (n \geq 1), \\ (-1)^n (-n)!^{-1} & (n < 1). \end{cases} \end{aligned}$$

In the theorem, ε_j ($j = 1, \dots, d$) should satisfy $|\varepsilon_k/\varepsilon_d(j)| \ll 1$ ($j = 1, \dots, d$, $k = j, \dots, d$). Let us consider this condition. If $|\varepsilon_k/\varepsilon_d(j)| \rightarrow \infty$, then $\varepsilon_d(j)$ tends to 0 rapidly. By (1.2), $s_j + \dots + s_d = -M$ is a singular locus. Therefore, when $|\varepsilon_k/\varepsilon_d(j)| \rightarrow \infty$, the point $(-m_1 + \varepsilon_1, \dots, -m_d + \varepsilon_d)$ approximates asymptotically to the singular locus. Hence, $|\varepsilon_k/\varepsilon_d(j)| \ll 1$ means geometrically that $(-m_1 + \varepsilon_1, \dots, -m_d + \varepsilon_d)$ does not approximate asymptotically to the singular locus.

3 Examples

By the main theorem, we can compute various multiple zeta values at non-positive integers. Let us see some examples.

In the case $d = 2$, we have

$$\begin{aligned} \zeta_2(\varepsilon_1, \varepsilon_2) &= \frac{1}{3} + \frac{1}{12} \cdot \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} + \sum_{j=1}^2 O(\varepsilon_j), \\ \zeta_2(-1 + \varepsilon_1, \varepsilon_2) &= \frac{1}{24} + \sum_{j=1}^2 O(\varepsilon_j), \\ \zeta_2(\varepsilon_1, -1 + \varepsilon_2) &= \frac{1}{12} + \sum_{j=1}^2 O(\varepsilon_j), \\ \zeta_2(-1 + \varepsilon_1, -1 + \varepsilon_2) &= \frac{1}{360} + \frac{1}{720} \cdot \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} + \sum_{j=1}^2 O(\varepsilon_j). \end{aligned}$$

In the case $d = 3$, we have

$$\begin{aligned}\zeta_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= -\frac{1}{4} - \frac{1}{24} \cdot \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} - \frac{1}{24} \cdot \frac{\varepsilon_2 + 2\varepsilon_3}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^3 O(\varepsilon_j), \\ \zeta_3(-1 + \varepsilon_1, \varepsilon_2, \varepsilon_3) &= -\frac{17}{720} - \frac{1}{144} \cdot \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} + \frac{1}{720} \cdot \frac{-\varepsilon_2 + 3\varepsilon_3}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^3 O(\varepsilon_j), \\ \zeta_3(\varepsilon_1, -1 + \varepsilon_2, \varepsilon_3) &= -\frac{19}{360} + \frac{1}{360} \cdot \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^3 O(\varepsilon_j), \\ \zeta_3(\varepsilon_1, \varepsilon_2, -1 + \varepsilon_3) &= -\frac{3}{40} - \frac{1}{720} \cdot \frac{4\varepsilon_2 + 3\varepsilon_3}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^3 O(\varepsilon_j).\end{aligned}$$

Note that the example (1.5) comes from the first example of the above, taking $\varepsilon_1 = \varepsilon^2$ and $\varepsilon_2 = \varepsilon_3 = \varepsilon$.

In the case $d = 4$, we have

$$\begin{aligned}\zeta_4(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) &= \frac{1}{5} + \frac{1}{36} \cdot \frac{\varepsilon_4}{\varepsilon_3 + \varepsilon_4} + \frac{1}{48} \cdot \frac{\varepsilon_3 + 2\varepsilon_4}{\varepsilon_2 + \varepsilon_3 + \varepsilon_4} \\ &\quad + \frac{1}{720} \cdot \frac{19\varepsilon_2 + 33\varepsilon_3 + 52\varepsilon_4}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4} + \frac{1}{144} \cdot \frac{\varepsilon_4(\varepsilon_2 + \varepsilon_3 + \varepsilon_4)}{(\varepsilon_3 + \varepsilon_4)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)} \\ &\quad + \sum_{j=1}^4 O(\varepsilon_j).\end{aligned}$$

4 Proof of the Main Theorem

In this section, we prove the main theorem. If $d = 1$, $\zeta_1(s_1)$ is the Riemann zeta function. Hence, the main theorem is clear. So we prove the theorem in the case $d > 1$.

First, we prove the meromorphic continuation of $\zeta_d(s_1, \dots, s_d)$. $\zeta_d(s_1, \dots, s_d)$ has an integral representation as the following,

$$\begin{aligned}\Gamma(s_1) \cdots \Gamma(s_d) \zeta_d(s_1, \dots, s_d) \\ = \int_0^1 \cdots \int_0^1 \int_0^\infty \prod_{j=1}^d x_j^{s_d(j)-d+j-2} \prod_{j=2}^d (1-x_j)^{s_{j-1}-1} \prod_{j=1}^d \frac{x_1 \cdots x_j}{e^{x_1 \cdots x_j} - 1} dx_1 \cdots dx_d. \quad (4.1)\end{aligned}$$

Dividing the integral into two parts and integrating by parts, (4.1) can be written

$$\begin{aligned} \zeta_d(s_1, \dots, s_d) &= \frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \sum_{k=0}^{n_1} \sum_{p_1 + \dots + p_d = k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} \frac{1}{s_d(1) - d + k} \times \\ &\quad \times \prod_{j=2}^d B(s_d(j) - d + j + p_d(j) - 1, s_{j-1}) \\ &\quad + \frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \int_0^1 x^{s_d(1) - d + n_1} F_\varphi(x, n_2, \dots, n_d) dx \\ &\quad + \frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \int_1^\infty \frac{x^{s_d(1) - d}}{e^x - 1} F_\psi(x, n_2, \dots, n_d) dx, \end{aligned} \quad (4.2)$$

where $n_1, \dots, n_d \in \mathbb{Z}_{\geq 0}$. (4.2) can be continued meromorphically to

$$\left\{ (s_1, \dots, s_d) \in \mathbb{C}^d \mid \begin{array}{l} \Re(s_d(j)) > d - j - n_j \ (j = 1, \dots, d), \\ \Re(s_{j-1}) > -n_j - 1 \ (j = 2, \dots, d) \end{array} \right\}.$$

We use (4.2) with $s_j = -m_j + \varepsilon_j$ ($j = 1, \dots, d$) and $n_1 = \dots = n_d = M + d$. By estimating the second term and the third terms of (4.2), we obtain

$$\frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \left(\int_0^1 x^{s_d(1) - d + n_1} F_\varphi(x) dx + \int_1^\infty \frac{x^{s_d(1) - d}}{e^x - 1} F_\psi(x) dx \right) = \sum_{j=1}^d O(\varepsilon_j).$$

Consider the first term of (4.2) by writing it as the following,

$$\sum_{k=0}^{M+d} = \sum_{k=0}^{M+d-1} + \sum_{k=M+d}, \quad (4.3)$$

and estimating the first term of (4.3), we obtain

$$\sum_{k=0}^{M+d-1} = \sum_{j=1}^d O(\varepsilon_j).$$

Next, we consider the second term of (4.3). First, we estimate the factors containing gamma functions and beta functions as the following,

$$\begin{aligned} &\frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \prod_{j=2}^d B(s_d(j) - d + j + p_d(j) - 1, s_{j-1}) \\ &= \frac{1}{\Gamma(s_d)} \prod_{j=2}^d \frac{\Gamma(s_d(j) - d + j + p_d(j) - 1)}{\Gamma(s_d(j-1) - d + j + p_d(j) - 1)} \\ &= \frac{1}{(\varepsilon_d)_{-m_d} \Gamma(\varepsilon_d(1))} \prod_{j=2}^d \frac{(\varepsilon_d(j))_{m_d(j) - d + j + p_d(j) - 1}}{(\varepsilon_d(j-1))_{m_d(j-1) - d + j + p_d(j) - 1}}. \end{aligned} \quad (4.4)$$

Since we have

$$\begin{aligned} \frac{1}{(\varepsilon_d)_{-m_d} \Gamma(\varepsilon_d(1))} &= ((-1)^{m_d} m_d! + O(\varepsilon_d)) \left(\frac{\sin(\pi \varepsilon_d(1))}{\pi} \Gamma(1 - \varepsilon_d(1)) \right) \\ &= (-1)^{m_d} m_d! \varepsilon_d(1) + O(\varepsilon_d(1)^2) + O(\varepsilon_d(1) \varepsilon_d) \end{aligned}$$

and

$$\begin{aligned} &\prod_{j=2}^d \frac{(\varepsilon_d(j))_{m_d(j)-d+j+p_d(j)-1}}{(\varepsilon_d(j-1))_{m_d(j-1)-d+j+p_d(j)-1}} \\ &= \prod_{j=2}^d \left(h(-m_d(j) - d + j + p_d(j) - 1, -m_d(j-1) - d + j + p_d(j) - 1) \right. \\ &\quad \times \frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}} \Bigg) + \\ &\quad + \sum_{j=2}^d \left\{ O\left(\frac{\varepsilon_d(j)}{\varepsilon_d(j-1)} \varepsilon_d(j) \right) + O(\varepsilon_d(j-1)) + O(\varepsilon_d(j)) \right\}, \end{aligned}$$

we find (4.4) is

$$\begin{aligned} &\frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \prod_{j=2}^d B(s_d(j) - d + j + p_d(j) - 1, s_{j-1}) \\ &= (-1)^{m_d} m_d! \varepsilon_d(1) \times \\ &\quad \times \prod_{j=2}^d \left(h(-m_d(j) - d + j + p_d(j) - 1, -m_d(j-1) - d + j + p_d(j) - 1) \right. \\ &\quad \times \frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}} \Bigg) \\ &\quad + \sum_{j=1}^d O(\varepsilon_j \varepsilon_d(1)), \end{aligned} \tag{4.5}$$

where $h(m, n)$ is defined by

$$h(m, n) := \begin{cases} 0 & (m \geq 1 > n), \\ 1 & (\text{otherwise}). \end{cases}$$

Using (4.5), we can estimate the second term of (4.3) as the following,

$$\begin{aligned} &(-1)^{m_d} m_d! \sum_{p_1 + \cdots + p_d = d+M} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} \times \\ &\quad \times \prod_{j=2}^d h(-m_d(j) - d + j + p_d(j) - 1, -m_d(j-1) - d + j + p_d(j) - 1) \times \\ &\quad \times \frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}} + \sum_{j=1}^d O(\varepsilon_j). \end{aligned} \tag{4.6}$$

Finally, to remove the function h from (4.6), we restrict the summation. Then we obtain the main theorem.

References

- [1] S. Akiyama, S. Egami and Y. Tanigawa, *Analytic continuation of multiple zeta-functions and their values at non-positive integers*, Acta Arith, **98** (2001), 107–116.
- [2] S. Akiyama, Y. Tanigawa, *Multiple zeta values at non-positive integers*, Ramanujan J. **5** (2001), 327–351.
- [3] Y. Komori, *An integral representation of multiple Hurwitz-Lerch zeta functions and generalized multiple bernoulli numbers*, Quart. J. Math. (Oxford) (2009), 1–60.
- [4] K. Matsumoto, *On analytic continuation of various multiple zeta-functions*, Number Theory for the Millenium (Urbana, 2000), Vol. II, M. A. Bennett et. al. (eds.), A. K. Peters, Natick, MA, 2002, pp. 417–440.
- [5] K. Matsumoto, *The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions I*, J. Number Theory **101** (2003), 223–243.
- [6] Y. Sasaki, *Multiple zeta values for coordinatewise limits at non-positive integers*, Acta Arith. **136** (2009), 299–317.
- [7] J. Zhao, *Analytic continuation of multiple zeta functions*, Proc. Amer. Math. Soc. **128** (2000), 1275–1283.

Graduate School of Mathematics
 Nagoya University
 Chikusa-ku, Nagoya 464-8602, Japan
 E-mail: m11022v@math.nagoya-u.ac.jp