

A geometric constant induced by the Dunkl-Williams inequality

新潟大学大学院・自然科学研究科 水口 洋康 (Hiroyasu Mizuguchi)

Department of Mathematical Science,

Graduate School of Science and Technology, Niigata University

新潟大学・理学部 斎藤 吉助 (Kichi-Suke Saito)

Department of Mathematics, Faculty of Science, Niigata University

新潟大学大学院・自然科学研究科 田中 亮太郎 (Ryotaro Tanaka)

Department of Mathematical Science,

Graduate School of Science and Technology, Niigata University

1 Introduction

In this note, we mainly consider about the Dunkl-Williams constant. In particular, we describe some recent results obtained in [19].

Throughout this note, the term “normed linear space” always means a real normed linear space which has two or more dimension. For a normed linear space X , let B_X and S_X denote the unit ball and the unit sphere of X , respectively. In 1964, Dunkl and Williams [7] showed the following simple inequalities: Let X be a normed linear space. Then the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$, and if X is an inner product space, the stronger inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$. These inequalities are so called the Dunkl-Williams inequality. In the same paper, it was also proved that for any $\varepsilon > 0$ there exist $x, y \in (\mathbb{R}^2, \|\cdot\|_1)$ such that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| > (4 - \varepsilon) \frac{\|x-y\|}{\|x\| + \|y\|}.$$

This means that the constant 4 is the best possible choice for the Dunkl-Williams inequality in the space $(\mathbb{R}^2, \|\cdot\|_1)$. There are many result related to this inequality (cf. [1, 4, 5, 6, 16, 17, 20, 21, 22, 23, 24], and so on).

2 The Dunkl-Williams inequality

In this section, we list some results related to the Dunkl-Williams inequality. First, we see the original proof of the inequality.

Theorem 2.1 (The Dunkl-Williams inequality). *Let X be a normed linear space. Then, the inequality*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x - y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$, and if X is an inner product space, the stronger inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x - y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$.

Proof. Let x and y be two nonzero elements of X . Then we have

$$\begin{aligned} \|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| &\leq \|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|x\|} \right\| + \|x\| \left\| \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\| \\ &= \|x - y\| + |\|x\| - \|y\|| \\ &\leq 2\|x - y\|. \end{aligned}$$

Replacing x with y , we also have

$$\|y\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq 2\|x - y\|.$$

Therefore we obtain

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x - y\|}{\|x\| + \|y\|}.$$

Next, we assume that X is an inner product space. Then, for each nonzero elements $x, y \in X$, we obtain

$$\begin{aligned} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 &= 2 - 2\operatorname{Re} \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle \\ &= \frac{1}{\|x\|\|y\|} (2\|x\|\|y\| - 2\operatorname{Re}\langle x, y \rangle) \\ &= \frac{1}{\|x\|\|y\|} (\|x - y\|^2 - (\|x\| - \|y\|)^2). \end{aligned}$$

Hence we have

$$\begin{aligned} \|x - y\|^2 - \left(\frac{\|x\| + \|y\|}{2} \right)^2 \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \\ = \frac{(\|x\| - \|y\|)^2}{\|x\|\|y\|} ((\|x\| + \|y\|)^2 - \|x - y\|^2) \geq 0, \end{aligned}$$

and so the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x - y\|}{\|x\| + \|y\|}$$

holds. □

Dunkl and Williams asked in their paper [7] whether the second inequality in Theorem 2.1 characterizes inner product spaces. A bit later, Kirk and Smiley [14] solved this problem affirmatively. They used the following result of Lorch [15].

Lemma 2.2 (Lorch, 1948). *Let X be a normed linear space. Then, X is an inner product space if and only if $x, y \in X$ and $\|x\| = \|y\|$ implies $\|\alpha x + \alpha^{-1}y\| \geq \|x + y\|$ for all $\alpha > 0$.*

Now, we show the result of Kirk and Smiley.

Theorem 2.3 (Kirk-Smiley, 1964). *Let X be a normed linear space. Then, X is an inner product space if the inequality*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x - y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$.

Proof. Let x and y be nonzero elements of X such that $\|x\| = \|y\|$, and let $\alpha > 0$. Applying the inequality for αx and $\alpha^{-1}y$, we have

$$\begin{aligned} \|\alpha x + \alpha^{-1}y\| &\geq \frac{\|\alpha x\| + \|\alpha^{-1}y\|}{2} \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \\ &= \frac{\alpha + \alpha^{-1}}{2} \|x + y\| \\ &\geq \|x + y\|. \end{aligned}$$

Thus, X is an inner product space by Lemma 2.2. $\square$

As a consequence of Theorems 2.1 and 2.3, it turns out that a normed linear space X is an inner product space if and only if the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x - y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$. Thus, the best possible choice for the Dunkl-Williams inequality measures “how much” the space is close (or far) to be an inner product space. Motivated by this fact, Jiménez-Melado et al. [13] defined the Dunkl-Williams constant $DW(X)$ of a normed linear space X as the best constant for the Dunkl-Williams inequality, that is,

$$DW(X) = \sup \left\{ \frac{\|x\| + \|y\|}{\|x - y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| : x, y \in X \setminus \{0\}, x \neq y \right\}.$$

As was mentioned in Section 1, $DW((\mathbb{R}^2, \|\cdot\|_1)) = 4$, and Theorems 2.1 and 2.3 are restated as follows: Let X be a normed linear space. Then,

(i) $2 \leq DW(X) \leq 4$.

(ii) X is an inner product space if and only if $DW(X) = 2$.

Furthermore, it is known that $DW(X) = 4$ if and only if the space X is not uniformly non-square. Recall that a normed linear space X is said to be uniformly non-square if there exists $\delta > 0$ such that $x, y \in S_X$ and $\|x - y\| > 2(1 - \delta)$ implies $\|x + y\| \leq 2(1 - \delta)$. However, the Dunkl-Williams constant is very hard to calculate. In fact, except the case of $DW(X) = 2$ or 4, there have been probably no other example of the space X for which $DW(X)$ is determined precisely.

3 A calculation method for $DW(X)$

In [19], we constructed a new calculation method for the Dunkl-Williams constant. In this section, we describe the calculation method. As an application, we determine the precise value of $DW(\ell_2\text{-}\ell_\infty)$, where $\ell_2\text{-}\ell_\infty$ is the Day-James space defined as the space $\mathbb{R}^2$ endowed with the norm $\|\cdot\|_{2,\infty}$ given by

$$\|(a, b)\|_{2,\infty} = \begin{cases} \|(a, b)\|_2 & \text{if } ab \geq 0, \\ \|(a, b)\|_\infty & \text{if } ab \leq 0. \end{cases}$$

for all $(a, b) \in \mathbb{R}^2$.

When constructing a method, the notion of Birkhoff orthogonality plays an important role. We recall that for two elements x, y of a normed linear space X , x is said to be Birkhoff orthogonal to y , denoted by $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. Obviously, Birkhoff orthogonality is always homogeneous, that is, $x \perp_B y$ implies $\alpha x \perp_B \beta y$ for all $\alpha, \beta \in \mathbb{R}$. More details about Birkhoff orthogonality can be found in Birkhoff [3], Day [8, 9] and James [10, 11, 12].

To construct a calculation method, we introduce some notations. Suppose that X is a normed linear space. For each $x \in S_X$, let $V(x)$ be a subset of X defined by $V(x) = \{y \in X : x \perp_B y\}$. For each $x \in S_X$ and each $y \in V(x)$, we define $\Gamma(x, y)$ and $m(x, y)$ by

$$\Gamma(x, y) = \left\{ \frac{\lambda + \mu}{2} : \lambda \leq 0 \leq \mu, \|x + \lambda y\| = \|x + \mu y\| \right\}$$

and $m(x, y) = \sup\{\|x + \gamma y\| : \gamma \in \Gamma(x, y)\}$, respectively. Furthermore, let

$$M(x) = \sup\{m(x, y) : y \in V(x)\}.$$

Using these notions, we obtain a new calculation method for the Dunkl-Williams constant.

Theorem 3.1 ([19]). *Let X be a normed linear space. Then,*

$$DW(X) = 2 \sup\{M(x) : x \in S_X\}.$$

If $\dim X = 2$, we have the following improvement of the preceding theorem.

Theorem 3.2 ([19]). *Let X be a normed linear space with $\dim X = 2$. Then,*

$$DW(X) = 2 \sup\{M(x) : x \in \text{ext}(B_X)\},$$

where $\text{ext}(B_X)$ denotes the set of all extreme points of B_X .

When we put this theory into practice, the following results are needed.

Proposition 3.3. *Let X be a normed linear space. Suppose that $x \in S_X$ and $y \in V(x)$. Then, the following hold:*

- (i) $0 \in V(x)$.
- (ii) $\alpha y \in V(x)$ for all $\alpha \in \mathbb{R}$.
- (iii) $m(x, 0) = 1 \leq m(x, y)$.

(iv) $m(x, \alpha y) = m(x, y)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$.

Proposition 3.4. *Let X, Y be normed linear spaces and let $x \in S_X$ and $y \in V(x)$. Suppose that T is an isometric isomorphism from X onto Y . Then, the following hold:*

(i) $m(Tx, Ty) = m(x, y)$.

(ii) $M(Tx) = M(x)$.

Proposition 3.5. *Let X be a normed linear space and let $x \in S_X$ and $y \in V(x) \setminus \{0\}$. Then, $\Gamma(x, y)$ is a bounded subset of $\mathbb{R}$. Furthermore, $m(x, y) = \max\{\|x + \alpha y\|, \|x + \beta y\|\}$, where $\alpha = \inf \Gamma(x, y)$ and $\beta = \sup \Gamma(x, y)$, respectively.*

Theorem 3.6. *Let X be a normed linear space and let $x \in S_X$ and $y \in V(x)$. Suppose that $\{x_n\}$ is a sequence in S_X which converges to x . If the sequence $\{y_n\}$ satisfies $y_n \in V(x_n)$ for each $n \in \mathbb{N}$ and converges to y , then*

$$m(x, y) \leq \liminf_{n \rightarrow \infty} m(x_n, y_n).$$

All of these results can be found in [19].

4 The Dunkl-Williams constant of the space $\ell_2\text{-}\ell_\infty$

Applying Theorem 3.2, we obtain the following example.

Theorem 4.1 ([19]). $DW(\ell_2\text{-}\ell_\infty) = 2\sqrt{2}$.

To prove Theorem 4.1, we need a lot of works. First, one can easily show that

$$\text{ext}(B_{\ell_2\text{-}\ell_\infty}) = \{(a, b) \in \mathbb{R}^2 : ab \geq 0, a^2 + b^2 = 1\} \cup \{(1, -1), (-1, 1)\}.$$

Now, let $M_0 = \sup\{M((a, b)) : 0 < b < a, a^2 + b^2 = 1\}$. Then, we have the following lemma by Theorems 3.2 and 3.6, and Proposition 3.4.

Lemma 4.2. $DW(\ell_2\text{-}\ell_\infty) = 2 \max\{M_0, M((1, -1))\}$.

We remark that $0 < b < a$ and $a^2 + b^2 = 1$ implies $b < 1/\sqrt{2} < a$. Next, to calculate $M(x)$, we find the set $V(x)$ for each x .

Lemma 4.3. *Suppose that $0 < b < a$ and $a^2 + b^2 = 1$. Then,*

$$V((a, b)) = \{\alpha(b, -a) \in \mathbb{R}^2 : \alpha \in \mathbb{R}\}.$$

Lemma 4.4. $V((1, -1)) = \{(a, b) \in \mathbb{R}^2 : ab \geq 0\}$.

To reduce the amount of computation, we make use of Proposition 3.3.

Lemma 4.5. *Suppose that $0 < b < a$ and $a^2 + b^2 = 1$. Then,*

$$M((a, b)) = m((a, b), (b, -a)).$$

Lemma 4.6. $M((1, -1)) = \sup\{m((1, -1), (a, b)) : 0 < b < a, a^2 + b^2 = 1\}$.

We need to determine the set $\Gamma(x, y)$ to calculate the value of $m(x, y)$.

Lemma 4.7. *Suppose that $0 < b < a$ and $a^2 + b^2 = 1$. Then,*

$$\Gamma((a, b), (b, -a)) = \begin{cases} [0, b/a] & \text{if } a \leq 2b, \\ [0, (a + b - \sqrt{2ab})/(a - b)] & \text{if } a > 2b. \end{cases}$$

Lemma 4.8. *Suppose that $0 < b < a$ and $a^2 + b^2 = 1$. Then,*

$$\Gamma((1, -1), (a, b)) = [b - a, 0].$$

Now we prove Theorem 4.1. Proposition 3.5 is used in this phase.

Proof of Theorem 4.1. Suppose that $0 < b < a$ and $a^2 + b^2 = 1$. First, we assume that $a \leq 2b$. Then, by Proposition 3.5 and Lemma 4.7, we have

$$\begin{aligned} M((a, b)) &= m((a, b), (b, -a)) \\ &= \max \left\{ \|(a, b)\|_{2,\infty}, \left\| (a, b) + \frac{b}{a}(b, -a) \right\|_{2,\infty} \right\} \\ &= \frac{1}{a}. \end{aligned}$$

On the other hand, if $0 < b < a$ and $a^2 + b^2 = 1$, then $a \leq 2b$ if and only if $a \leq 2/\sqrt{5}$. Hence we obtain

$$\begin{aligned} \{M(a, b) : 0 < b < a \leq 2b, a^2 + b^2 = 1\} &= \{1/a : 1/\sqrt{2} < a \leq 2/\sqrt{5}\} \\ &= [\sqrt{5}/2, \sqrt{2}). \end{aligned}$$

Next, we suppose that $a > 2b$. Then we have

$$0 < \frac{a + b - \sqrt{2ab}}{a - b} < 1.$$

Since the function $t \mapsto \|(a, b) + t(b, -a)\|$ is convex and increasing on $[0, \infty)$, we obtain

$$\begin{aligned} M((a, b)) &= m((a, b), (b, -a)) \\ &= \left\| (a, b) + \frac{a + b - \sqrt{2ab}}{a - b}(b, -a) \right\|_{2,\infty} \\ &\leq \|(a, b) + (b, -a)\|_{2,\infty} \\ &= \|(a + b, b - a)\|_{2,\infty} \\ &= a + b \leq \sqrt{2}(a^2 + b^2)^{1/2} = \sqrt{2} \end{aligned}$$

by Proposition 3.5 and Lemma 4.7. Thus, we have

$$M_0 = \sup\{M((a, b)) : 0 < b < a, a^2 + b^2 = 1\} = \sqrt{2}.$$

Finally, by Proposition 3.5 and Lemma 4.8, we obtain

$$\begin{aligned}
 m((1, -1), (a, b)) &= \|(1, -1) + (b - a)(a, b)\|_{2,\infty} \\
 &= \|(a^2 + b^2, -a^2 - b^2) + (ab - a^2, b^2 - ab)\|_{2,\infty} \\
 &= \|(ab + b^2, -a^2 - ab)\|_{2,\infty} \\
 &= (a + b)\|(b, -a)\|_{2,\infty} \\
 &= a(a + b) \leq \sqrt{2}.
 \end{aligned}$$

This implies that $M((1, -1)) \leq \sqrt{2} = M_0$.

Thus, by Lemma 4.2, we have

$$DW(\ell_2\text{-}\ell_\infty) = 2 \max\{M_0, M((1, -1))\} = 2M_0 = 2\sqrt{2}. \quad \square$$

References

- [1] A. M. Al-Rashed, Norm inequalities and characterization of inner product spaces, J. Math. Anal. Appl., 176(1993), 587-593.
- [2] M. Baronti and P. L. Papini, Up and down along rays, Riv. Mat. Univ. Parma, 2*(1999), 171-189.
- [3] G. Birkoff, Orthogonality in linear metric spaces, Duke Math. J., 1(1935), 169-172.
- [4] F. Dadipour, M. Fujii and M. S. Moslehian, Dunkl-Williams inequality for operators associated with p -angular distance, Nihonkai Math. J., 21(2010), 11-20.
- [5] F. Dadipour and M. S. Moslehian, An approach to operator Dunkl-Williams inequalities, Publ. Math. Debrecen, 79(2011), 109-118.
- [6] F. Dadipour and M. S. Moslehian, A characterization of inner product spaces related to the p -angular distance, J. Math. Anal. Appl., 371(2010), 667-681.
- [7] C. F. Dunkl and K. S. Williams, A simple norm inequality, Amer. Math. Monthly, 71(1964), 53-54.
- [8] M. M. Day, Polygons circumscribed about closed convex curves, Trans. Amer. Math. Soc., 62(1947), 315-319.
- [9] M. M. Day, Some characterizations of inner product spaces, Trans. Amer. Math. Soc., 62(1947), 320-337.
- [10] R. C. James, Orthogonality in normed linear spaces, Duke Math. J., 12(1945), 291-302.
- [11] R. C. James, Inner products in normed linear spaces, Bull. Amer. Math. Soc., 53(1947), 559-566.
- [12] R. C. James, Orthogonality and linear functionals in normed linear spaces, Trans. Amer. Math. Soc., 61(1947), 265-292.

- [13] A. Jiménez-Melado, E. Llorens-Fuster and E. M. Mazcunan-Navarro, The Dunkl-Williams constant, convexity, smoothness and normal structure, *J. Math. Anal. Appl.*, 342(2008), 298–310.
- [14] W. A. Kirk and M. F. Smiley, Another characterization of inner product spaces, *Amer. Math. Monthly*, 71(1964), 890–891.
- [15] E. R. Lorch, On certain implications which characterize Hilbert space, *Ann. of Math.* (2), 49 (1948), 523–532.
- [16] L. Maligranda, Simple norm inequalities, *Amer. Math. Monthly*, 113(2006), 256–260.
- [17] J. L. Massera and J. J. Schäffer, Linear differential equations and functional analysis I, *Ann. of Math.*, 67(1958), 517–573.
- [18] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer-Verlag, New York, 1998.
- [19] H. Mizuguchi, K. -S. Saito and R. Tanaka, On the calculation of the Dunkl-Williams constant of normed linear spaces, to appear in *Cent. Eur. J. Math.*
- [20] M. S. Moslehian and F. Dadipour, Characterization of equality in a generalized Dunkl-Williams inequality, *J. Math. Anal. Appl.*, 384(2011), 204–210.
- [21] M. S. Moslehian, F. Dadipour, R. Rajić and A. Marić, A glimpse at the Dunkl-Williams inequality, *Banach J. Math. Anal.*, 5(2011), 138–151.
- [22] J. E. Pečarić and R. Rajić, Inequalities of the Dunkl-Williams type for absolute value operators, *J. Math. Inequal.*, 4(2010), 1–10.
- [23] K. -S. Saito and M. Tominaga, A Dunkl-Williams type inequality for absolute value operators, *Linear Algebra Appl.*, 432(2010), 3258–3264.
- [24] K. -S. Saito and M. Tominaga, A Dunkl-Williams inequality and the generalized operator version, *International Series of Numerical Mathematics*, 161(2012), 137–148, Birkhauser.