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Recent results on the Index Theorem

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1 Review on elliptic pairs [4]

Let X be a complex manifold of dimension n and $\pi : T^*X \to X$ its cotangent bundle. We shall make use of the sheaves $\omega_X \simeq \mathbb{C}_X [2n]$, the dualizing complex, $\mathcal{O}_X$ the structural sheaf, $\mathcal{D}_X$ the sheaf of linear holomorphic partial differential operators on X, $\mathcal{E}_X$ the sheaf on T^*X of microdifferential operators ([3]).

If F is an $\mathbb{R}$-constructible sheaf, its microlocal Euler class $\mu eu(F)$ is a Lagrangian cycle supported by $SS(F)$, the micro-support of F (see [2]):

$$\mu eu(F) \in H^0_{SS(F)}(T^*X; \pi^{-1}\omega_X)$$

If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, its microlocal Euler class is a cohomology class supported by $char(\mathcal{M})$, the characteristic variety of $\mathcal{M}$

$$\mu eu(\mathcal{M}) \in H^0_{char(\mathcal{M})}(T^*X; \pi^{-1}\omega_X)$$

The pair $(\mathcal{M}, F)$ is elliptic if

$$char(\mathcal{M}) \cap SS(F) \subset T^*_X X$$

Under this ellipticity hypothesis, the microlocal convolution product$*_\mu$:

$$H^0_{char(\mathcal{M})}(T^*X; \pi^{-1}\omega_X) \otimes H^0_{SS(F)}(T^*X; \pi^{-1}\omega_X) \to H^0_{char(\mathcal{M})+SS(F)}(T^*X; \pi^{-1}\omega_X)$$

is well defined, and the microlocal Euler class of the elliptic pair $(\mathcal{M}, F)$ is obtained as:

$$\mu eu(\mathcal{M}, F) = \mu eu(\mathcal{M}) *_\mu \mu eu(F)$$

One denote by $eu(\mathcal{M}, F)$ its restriction to the zero-section. Hence, if $supp(\mathcal{M}) \cap supp(F)$ is compact, we get:

$$eu(\mathcal{M}, F) \in H^0_c(X; \omega_X)$$

One of the main results of [4] asserts that the complex $\mathrm{RHom}_{\mathcal{D}}(\mathcal{M} \otimes \mathrm{F}, \mathcal{O}_\mathrm{X})$ has finite dimensional cohomology and its Euler-Poincaré index $\chi(X, \mathcal{M}, F)$ may be calculated by the formula:

$$\chi(X, \mathcal{M}, F) = \int_X eu(\mathcal{M}, F)$$

The remaining problem is to calculate the microlocal Euler class of $\mathcal{M}$.

Assume $\mathcal{M}$ is endowed with a good filtration Then $gr(\mathcal{M})$ is a coherent module over the sheaf $\mathcal{O}_{T^*X}$ and its Chern character is well defined. Let $Td_X(TX)$ denote the Todd class. The following conjecture is made in [4]:

$$\mu eu(shm) = [ch(gr(\mathcal{M})) \cup \pi^{-1} Td_X(TX)]^{2n}.$$

2 Chern character of $\mathcal{E}$-modules [1]

Let A be a unitary algebra over a field k of characteristic zero. It is a well known result that the Chern caracter of a finitely generated projective A-module may be calculated using negative cyclic homology. This construction has been generalized by Bressler-Nest-Tsygan to perfect sheaves of modules over sheaves of rings of k_X algebras. (There is also a recent and elegant construction due to B. Keller.) Hence, if $\mathcal{M}$ is a coherent $\mathcal{E}_X$-module, its Chern character $ch(\mathcal{M})$ belongs to $\bigoplus_{j=1}^{n} H^{2j}_{char(\mathcal{M})}(T^*X, \mathbb{C}_{T^*X})$.

Theorem 2.1 *[1] Assume $\mathcal{M}$ is endowed with a good filtration. Then:*

$$\begin{aligned} ch(\mathcal{M}) &= ch(gr(\mathcal{M})) \cup \pi^{-1} Td_X(TX) \\ ch(\mathcal{M})^{2n} &= \mu eu(\mathcal{M}) \end{aligned}$$

Of course, this result in particular implies the conjecture of [4],

In fact, the theorem of (loc.cit.) is more general and is stated in the framework of "quantized deformation algebras".

3 Comments

It is interesting to notice that the Todd class appears when passing from $\mathcal{D}$-modules to $\mathcal{O}$-modules, and that the index theorem, when formulated in terms of $\mathcal{D}$-modules (and Chern character of such modules) do not use Todd classes.

References

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[3] M. Sato, T. Kawai and M. Kashiwara *Hyperfunctions and pseudodifferential equations* in LN in Math 287, Springer-Verlag pp. 265-529 (1973).

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