

LU-Decomposition of a Matrix with Entries of Different Kinds

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February 1982

Abstract

Let $\underline{F} \supset \underline{K}$ be fields and consider a matrix A over $\underline{F}$ whose entries not belonging to $\underline{K}$ are algebraically independent transcendentals over $\underline{K}$. It is shown that if $\det A \in \underline{K}^*$ ($=\underline{K}-\{0\}$), the matrix A , with suitable permutations of its rows and columns, is decomposed into LU-factors with the entries of the U-factor belonging to $\underline{K}$.

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1. Introduction

Let $\underline{K}$ be a field and $\underline{F}$ ($\supset \underline{K}$) an extension field. For $S \subset \underline{F}$ we denote by $M(S)$ the set of matrices with entries belonging to S . Suppose an n by n matrix $A = (A_{ij}) \in M(\underline{F})$ is expressed as

$$A = Q + T, \quad (1)$$

where

- i) $Q \in M(\underline{K})$,
- ii) non-zero entries of T are algebraically independent transcendentals over $\underline{K}$.

In the following we shall denote by T^* the set of non-zero entries of T .

As is well known, A is invertible in the ring $\underline{K}[T^*]$ of polynomials in T^* over $\underline{K}$, i.e., $A^{-1} \in M(\underline{K}[T^*])$, iff $\det A \in \underline{K}^* (= \underline{K} - \{0\})$. Here we are interested in whether we can compute A^{-1} by means of pivot operations in $\underline{K}[T^*]$; moreover, how simple we can make the LU-factors of A by applying suitable permutations to its rows and columns.

By way of illustration, we will start with an example. Let $\underline{K} = \underline{Q}$ (the field of rational numbers) and set $\underline{F} = \underline{Q}(x, y, z)$, where $\{x, y, z\}$, as a collection, is assumed to be algebraically independent over $\underline{Q}$. Matrix

$$A = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & x & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ y & -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & z & 0 \end{pmatrix} \end{matrix},$$

is expressed as the sum of the following Q and T according to (1):

$$Q = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 \end{pmatrix}.$$

Note that $\det A = 2$ and hence A is invertible in $\underline{Q}[x,y,z]$. The matrix A is decomposed into LU-factors in $\underline{F}$ as

$$A = L U,$$

with

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -y & y-1 & y-1-2/x & 1 & 0 \\ -1 & 2 & 2+1/x & -(xz+1)/2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & x+1 & 1 & 1 \\ 0 & 0 & -x & -1 & 0 \\ 0 & 0 & 0 & -2/x & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is observed that some of the entries of L and U , especially some of the diagonals of U , do not belong to $\underline{K}[T^*]$.

However, after rearranging the rows and the columns of A as

$$\tilde{A} = \begin{matrix} & \begin{matrix} 1 & 5 & 3 & 4 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 4 \\ 2 \\ 5 \end{matrix} & \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ y & -1 & 1 & 0 & -1 \\ 1 & 0 & x & 1 & 0 \\ 1 & 0 & 0 & z & 1 \end{pmatrix} \end{matrix},$$

we obtain the LU-decomposition

$$\tilde{A} = \tilde{L} \tilde{U}$$

with

$$\tilde{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -y & y-1 & 1 & 0 & 0 \\ -1 & 1 & x/2 & 1 & 0 \\ -1 & 1 & 0 & z & 1 \end{pmatrix}, \quad \tilde{U} = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The LU-factors are much simpler in the sense that all the entries of $\tilde{U}$ are numbers in $\underline{K}=\underline{Q}$, i.e., $\tilde{U} \in M(\underline{K})$ and, consequently, the entries of $\tilde{L}$ are polynomials in x, y and z over $\underline{K}$ of degree at most 1.

In this paper, we establish a theorem stating to the effect that this is always the case for any matrix A which admits the expression of (1) with $\det A \in \underline{K}^*$, i.e., that it is always possible to find a permutation of rows and that of columns, through which the matrix A can be brought to the form decomposable into LU-factors with a U -factor in $M(\underline{K})$. Furthermore, it is shown how to find suitable permutations. Some implications of the theorem are also discussed.

2. The Theorem

In this section we prove the following theorem.

Theorem. Let A be a matrix of form (1). If $\det A \in \underline{K}^*$, then there exist permutation matrices P_r, P_c and LU-factors $\tilde{L} = (\tilde{L}_{ij}), \tilde{U} = (\tilde{U}_{ij})$:

$$P_r' A P_c = \tilde{L} \tilde{U}$$

such that

- (i) $\tilde{L}_{ij}$ is a polynomial of degree at most 1 in non-zero entries T^* of T over $\underline{K}$ ($\tilde{L}_{ii}=1; \tilde{L}_{ij}=0$ for $i < j$)

and

- (ii) $\tilde{U} \in M(\underline{K}); \tilde{U}_{ii} \in \underline{K}^* (\tilde{U}_{ij}=0 \text{ for } i > j)$. $\square$

To prove the theorem, the following lemma is crucial, giving a necessary and sufficient condition for a matrix of form (1) to be invertible in $\underline{K}[T^*]$. We will say that a matrix is strictly lower triangular if it is a lower triangular matrix with zero diagonals.

Lemma 1. Let A be a matrix in (1). Then $\det A \in \underline{K}^*$ iff $\det Q \neq 0$ and $P_r'(TQ^{-1})P_r$ is strictly lower triangular for some permutation matrix P_r . $\square$

Proof: ["if" part] Suppose $P_r'(TQ^{-1})P_r$ is strictly lower triangular for some permutation matrix P_r . Then, since $\det Q \neq 0$ and $A = Q + T$, we have

$$\begin{aligned}\det A &= \det[(I + TQ^{-1})Q] \\ &= \det[I + P_r'(TQ^{-1})P_r] \det Q \\ &= \det Q \in \underline{K}^*.\end{aligned}$$

["only if" part] If $\det A \in \underline{K}^*$, then $\det Q = \det A \neq 0$, so that we may put $S = Q^{-1}$. Suppose, to the contrary, that $P_r'(TS)P_r$ is not strictly lower triangular for any permutation matrix P_r . Then TS has a cycle of non-zero entries, that is, there exist an integer $M \geq 1$ and a sequence of indices $i(m)$ and $j(m)$ ($m=1, \dots, M$) such that

$$T_{i(m-1), j(m)} \neq 0 \text{ and } S_{j(m), i(m)} \neq 0 \text{ for } m=1, \dots, M,$$

where $i(0)=i(M)$. Choose M to be the minimal of such integers. For notational simplicity, we write $T_{i(m-1), j(m)} = t_m$ and $S_{j(m), i(m)} = s_m$.

For $k=0, 1, \dots$, consider the expression of the $(j(1), i(1))$ entry of $S(TS)^{kM}$ in the form of the sum of products of T_{ij} 's and S_{ji} 's.

Corresponding to the above cycle, it contains a term

$$s_1(s_1s_2\dots s_M)^k(t_1\dots t_M)^k,$$

since no other similar terms of $(t_1\dots t_M)^k$ exist due to the minimality of M and since it cannot be cancelled out by non-similar terms by virtue of the algebraic independence of elements of T^* .

Next we formally expand A^{-1} as

$$\begin{aligned}
A^{-1} &= [(I+TQ^{-1})Q]^{-1} \\
&= S - STS + STSTS - \dots
\end{aligned}$$

Each entry of A^{-1} on the left-hand side is a polynomial in T^* over $\underline{K}$ since $\det A \in \underline{K}^*$. On the right-hand side, we first observe that each entry of the m -th term is a homogeneous polynomial in T^* of degree $m-1$. Hence, by algebraic independence of T^* , no cancellation occurs among distinct terms in this expansion.

It follows in particular that the $(j(1), i(1))$ entry of the right-hand side contains a term of arbitrarily high degree, since the non-zero term $(t_1 \dots t_M)^k$ of degree kM , stemming from $S(TS)^{kM}$ as above, cannot be cancelled out for $k=0,1,\dots$. This is a contradiction. $\square$

We make use of the following well-known lemma, the proof of which is omitted.

Lemma 2. If $\det Q \neq 0$, then for any permutation matrix P_r , there exists a permutation matrix P_c and LU-factors $M, \tilde{U}$ such that

$$P_r' Q P_c = M \tilde{U},$$

where M is a lower triangular matrix with unit diagonals in $M(\underline{K})$ and $\tilde{U}$ a nonsingular upper triangular matrix in $M(\underline{K})$. $\square$

With Lemmas 1 and 2, the Theorem is easy to establish as shown below.

Proof of Theorem: Let P_r and P_c be permutation matrices as in Lemmas 1 and 2, respectively. Then from Lemma 2 we obtain

$$\begin{aligned}
\tilde{A} &= P_r' A P_c \\
&= P_r' (Q+T) P_c \\
&= (I+P_r'(TQ^{-1})P_r) (P_r' Q P_c) \\
&= (I+P_r'(TQ^{-1})P_r) M \tilde{U}
\end{aligned}$$

$$= \tilde{L} \tilde{U},$$

where

$$\tilde{L} = (I + P_r'(TQ^{-1})P_r) M.$$

Since both factors of $\tilde{L}$ are lower triangular matrices with unit diagonals,

$\tilde{L}$ is also a lower triangular matrix with unit diagonals and therefore

$\tilde{A} = \tilde{L} \tilde{U}$ is actually the LU-decomposition of $\tilde{A}$. Obviously $\tilde{U}$ belongs to $M(\underline{K})$

and, consequently, the entries of $\tilde{L} = \tilde{A} \tilde{U}^{-1}$ are polynomials in

T^* of degree at most 1. $\square$

Remark 1. In parallel with the Theorem, it is likewise possible to find permutations through which A can be brought to a form decomposable into LU-factors in such a way that the L -, instead of U -, factor belongs to $M(\underline{K})$.

Remark 2. Consider a matrix A in $M(\underline{F})$. Then it can be written as

$$A = Q + T,$$

where $Q \in M(\underline{K})$ and $T \in M(\underline{F} \setminus \underline{K})$. In general, the non-zero entries T^* of T are not algebraically independent over $\underline{K}$ and the LU-decomposition of the above-mentioned kind does not necessarily exist even if $\det A \in \underline{K}^*$, as is the case with

$$A = \begin{pmatrix} x & 1+x \\ 1-x & -x \end{pmatrix},$$

where $\underline{K} = \underline{Q}$ and $\underline{F} = \underline{Q}(x)$.

However, it may happen that the matrix $A_0 = Q + T_0$, where T_0 is obtained from T by replacing its non-zero entries by algebraically independent transcendentals, satisfies the condition $\det A_0 \in \underline{K}^*$. Then the Theorem can be applied to A_0 , which, in turn, implies that A itself can be decomposed, with suitable permutations, into the LU-factors with a U -factor belonging

to $M(\underline{K})$.

3. Discussions

When given a matrix A of form (1) satisfying the condition $\det A \in \underline{K}^*$, we can find the suitable permutations P_r and P_c on the basis of Lemmas 1 and 2. P_r can be determined by the zero/non-zero pattern of TQ^{-1} and P_c by pivoting operations on the matrix Q . Thus both permutations can be found with $O(n^3)$ arithmetic operations in $\underline{K}$.

Lemma 1 gives an efficient way, with $O(n^3)$ arithmetic operations in $\underline{K}$, for testing whether a matrix A of form (1) satisfies the condition $\det A \in \underline{K}^*$.

The problem dealt with in the present paper has arisen when the author was investigating the following problem of large-scale system analysis.

Let R and C be the set of row and column numbers, respectively, and $A(I, J)$ denote the submatrix of A corresponding to $I (\subset R)$ and $J (\subset C)$. For a matrix A of form (1), it is known [1] (cf. also the concept of 2-block rank in [2]) that we can find, by an efficient algorithm, two subsets $I \subset R$ and $J \subset C$ such that

$$\text{rank } A = \text{rank } A(I, J) + \text{rank } A(R \setminus I, C \setminus J),$$

$$\text{rank } A(I, J) = \text{rank } Q(I, J)$$

and

$$\text{rank } A(R \setminus I, C \setminus J) = \text{rank } T(R \setminus I, C \setminus J),$$

where the rank is considered over $\underline{F}$. If we take I and J to be the minimal of such subsets, we have $|I| = |J|$ and

$$\det A(I, J) \in \underline{K}^*,$$

The submatrix $A(I, J)$ above meets the condition of the Theorem. This implies that a matrix A of form (1) with $\det A \neq 0$ can be decomposed, after

suitable permutations P_r and P_c , into LU-factors as

$$P_r^t A P_c = \tilde{L} \tilde{U}$$

with a lower triangular matrix

$$\tilde{L} = \begin{pmatrix} \tilde{L}_{11} & 0 \\ \tilde{L}_{21} & \tilde{L}_{22} \end{pmatrix}$$

with unit diagonals and a nonsingular upper triangular matrix

$$\tilde{U} = \begin{pmatrix} \tilde{U}_{11} & \tilde{U}_{12} \\ 0 & \tilde{U}_{22} \end{pmatrix}$$

such that

- i) the entries of $\tilde{L}_{11}$ and $\tilde{L}_{21}$ are polynomials in T^* over $\underline{K}$ of degree at most 1.
- ii) $\tilde{U}_{11} \in M(\underline{K})$ and the diagonal entries of $\tilde{U}_{22}$ are algebraically independent over $\underline{K}$.

This procedure is applied to the iterative solution of a system of linear/non-linear equations $f(x)=0$ in real unknown variables x , as follows.

Let us suppose that a sequence of approximate solutions are computed by means of the Newton method, which would involve the solution of $J(x) \Delta x = f(x)$ for Δx through the LU-decomposition of $J(x)$, where $J(x)$ is the Jacobian matrix.

Since the non-constant derivatives of $f(x)$ may vary in value at each iteration, we regard them as being algebraically independent, or in other words, denoting the non-linear part of $J(x)$ by $T(x)$, we express $J(x)$ in the form (1):

$$J(x) = Q + T(x)$$

with $\underline{K}=\underline{Q}$ or $\underline{K}=\underline{R}$ (the field of real numbers). Furthermore we assume that $\det J(x) \in \underline{Q}^*$ or $\underline{R}^*$.

As the Theorem guarantees, we can obtain the LU-decomposition of $J(x)$:

$$J(x) = L(x) U$$

with

$$\begin{aligned} L(x) &= (I + T(x) Q^{-1}) M \\ &= M + T(x) U^{-1}, \end{aligned}$$

where $Q = M U$, as above, and the permutation matrices are suppressed for simplicity. Since M and U do not depend on x , they can be computed before the iteration process starts. At each iteration step, only the L -factor $L(x)$ of $J(x)$ is to be computed. Note that U^{-1} on the right-hand side of $L(x)$ does not cost much since U is triangular. As pointed out in Remark 1 in the previous section, we may alternatively adopt the LU-decomposition $J(x) = L U(x)$ with the L -factor being independent of x .

Acknowledgement.

The author expresses his gratitude to Professor Masao Iri of the University of Tokyo for his penetrating suggestions. The author is supported by The Sakkokai Foundation of Japan.

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