

Caratheodory-type equations in a Banach space

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We consider in a real Banach space E the initial value problem of an ordinary differential equation of Carathéodory type:

$$(1) \quad du/dt = f(t, u), \quad u(0) = a.$$

We assume the following conditions are satisfied.

(I) For each $x \in E$, the function $f(., x)$:

$I \rightarrow E$ is strongly measurable, where $I = [0, T]$ for a constant $T > 0$. For each $t \in I$, the function $f(t, .)$ on E_s with the strong topology into E_w with the weak topology is continuous.

(II) For each $\rho > 0$, there exists $\gamma_\rho \in L^1(I; \mathbb{R})$ such that

$$|f(t, x)| \leq \gamma_\rho(t)$$

whenever $t \in I$ and $|x| \leq \rho$.

In case E is finite-dimensional and f is continuous with respect to (t, x) , Okamura [3] showed the following necessary and sufficient condition for the uniqueness of solutions of the initial value problem (1) : If f is a continuous function on a domain $D \subset \mathbb{R}^{N+1}$ into $\mathbb{R}^N$, a necessary and sufficient condition for the uniqueness of solutions of $du/dt = f(t, u)$, existing to the right of the initial value (t_0, u_0) for each $(t_0, u_0) \in D$, is that there exists a $\Phi \in C^1(\hat{D}; \mathbb{R})$, $\hat{D} = \{(t, x_1, x_2); (t, x_i) \in D, i=1, 2\}$, such that

$$(2) \quad \Phi(t; x_1, x_2) \geq 0, \quad \Phi(t; x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$$

and

$$(3) \quad \frac{\partial \Phi}{\partial t}(t; x_1, x_2) + \sum_{j=1}^2 \langle f(t, x_j), \text{grad}_{x_j} \Phi(t; x_1, x_2) \rangle \leq 0$$

hold.

It may be natural that we introduce the Okamura's function Φ in our case. Actually Murakami [2] made use of the Okamura's function Φ in the case of Banach spaces to give a sufficient condition for the existence and uniqueness of solutions of the initial value problem (1) when f is continuous with respect to (t, x) . Also, Murakami [2] mentioned Carathéodory-type equations, but he did not enter into details. So we shall treat Carathéodory-type equations in detail to some extent.

Let E be a real Banach space. Let $\Phi(t; x, y)$ be a real-valued function defined on $I \times E \times E$. Suppose Φ satisfies the following conditions (i)-(iii):

$$(i) \quad \Phi(t; x, y) \geq 0 \text{ and } \Phi(t; x, x) = 0$$

for all $t \in I$ and all $x, y \in E$.

(ii) For any $t \in I$ and any $\rho > 0$, if two sequences $\{x_n\}$ and $\{y_n\}$ in $B_\rho \equiv \{z \in E; |z| \leq \rho\}$ satisfies $\Phi(t; x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$, then $|x_n - y_n| \rightarrow 0$ as $n \rightarrow \infty$.

(iii) For any $\rho > 0$, there exist a $\beta_\rho \in L^1(I; \mathbb{R})$ and a positive constant C_ρ such that

$$\begin{aligned} & |\Phi(t_1; x_1, y_1) - \Phi(t_2; x_2, y_2)| \\ & \leq \int_{t_1}^{t_2} \beta_\rho(s) ds + C_\rho (|x_1 - x_2| + |y_1 - y_2|) \end{aligned}$$

holds whenever $t_1, t_2 \in I$, $t_1 \leq t_2$, and $x_j, y_j \in B_\rho, j=1, 2$.

We define

$$\Phi'_{\pm}(t; x, y; a, b)$$

$$= \lim_{\xi \rightarrow \pm 0} \inf \{ \Phi(t+\xi; x+\xi a, y+\xi b) - \Phi(t; a, b) \} / \xi$$

for all $t \in I$ and all $x, y, a, b \in E$.

Example. Let $\beta \in L^1(I; \mathbb{R})$. If we set

$$(4) \quad \Phi(t; x, y) = |x - y|^2 \exp(-2 \int_0^t \beta(s) ds)$$

for all $x, y \in E$ and a.a. $t \in I$, then Φ satisfies (i)-(iii) and

$$(5) \quad \begin{aligned} \Phi'_{\pm}(t; x, y; a, b) \\ = 2 \{ \langle a-b, x-y \rangle_{\pm} - \beta(t) |x-y|^2 \} \exp(-2 \int_0^t \beta(s) ds) \end{aligned}$$

holds for all x, y, a and $b \in E$ and a.a. $t \in I$. Here $\langle \cdot, \cdot \rangle_{\pm}$ is defined by

$$\langle u, v \rangle_{\pm} = \pm \{ \sup \pm \langle u, v^* \rangle; v^* \in F(v) \}$$

for all $u, v \in E$, where F is the duality map of E into its dual E' .

We are concerned with the following condition imposed on Φ and f :

$$(III) \quad \Phi'_{-}(t; x, y; f(t, x), f(t, y)) \leq 0$$

holds for a.a. $t \in I$ and all $x, y \in E$.

Example. If Φ is given by (4), then (III) is reduced to the following condition:

$$\langle f(t, x) - f(t, y), x - y \rangle_{-} \leq \beta(t) |x - y|^2$$

holds for a.a. $t \in I$ and all $x, y \in E$. See (5). Hence, if $\beta = 0$, (III) is equivalent to the condition:

(III') $-f(t, \cdot)$ is accretive for a.a. $t \in I$.

Now we shall prove a main result.

Theorem 1. Let E be an arbitrary real Banach space. Suppose f satisfies (I)-(III) with a Φ having properties (i)-(iii) and, moreover, for each $t \in I$, the function $f(t, \cdot) : E_s \rightarrow E_s$ is locally uniformly continuous. Then, for any $a \in E$, there exist an $r \in (0, T]$ and a unique $u \in C([0, r]; E)$ such that

$$(6) \quad u(t) = a + \int_0^t f(s, u(s)) ds$$

holds for all $t \in [0, r]$.

Proof. Let $a \in E$ and $\rho > 0$. By (II) there exists a function $\gamma \in L^1(I; \mathbb{R})$ such that

$$|f(t, x)| \leq \gamma(t)$$

whenever $t \in I$ and $|x - a| \leq \rho$. We choose an $r \in (0, T]$ satisfying

$\int_0^r \gamma(t) dt \leq \rho$. For each integer $n > 1/r$, we define

$$(7) \quad u_n(t) = \begin{cases} a & (t \leq 1/n) \\ a + \int_{\frac{1}{n}}^t f(s, u_n(s - n^{-1})) ds & (1/n \leq t \leq r). \end{cases}$$

Clearly we have $|u_n(t) - a| \leq \rho$ whenever $n > 1/r$ and $t \leq r$.

Let $m, n > 1/r$. By using (iii), we can verify easily that

$\Phi(t; u_m(t), u_n(t))$ is absolutely continuous on $0 \leq t \leq r$. Hence we have

$$(8) \quad \Phi(t; u_m(t), u_n(t)) = \int_0^t \frac{d}{ds} \Phi(s; u_m(s), u_n(s)) ds$$

for $t \in [0, r]$. Using (iii) again, we obtain

$$\frac{d}{ds} \Phi(s; u_m(s), u_n(s))$$

$$= \frac{d}{d\xi} \Phi(s+\xi; u_m(s) + \xi \cdot f(s, u_m(s-m^{-1})), u_n(s) + \xi \cdot f(s, u_n(s-n^{-1}))) \Big|_{\xi=0}$$

for almost all s if $du_k(s)/ds = f(s, u_k(s-k^{-1}))$ exists for $k=m, n$. Hence, by (III), we have

$$\begin{aligned} & \frac{d}{ds} \Phi(s; u_m(s), u_n(s)) \\ & \leq \Phi'_-(s; u_m(s), u_n(s); f(s, u_m(s-m^{-1})), f(s, u_n(s-n^{-1}))) \\ & \quad - \Phi'_-(s; u_m(s), u_n(s); f(s, u_m(s)), f(s, u_n(s))) \\ & \equiv A(s; m, n) \end{aligned}$$

for a.a. $s \in (\max(m^{-1}, n^{-1}), r)$. Then, by the assumption of the locally uniform continuity of $f(s, \cdot)$,

$$\lim_{m, n \rightarrow \infty} A(s; m, n) = 0$$

holds for almost all s . Then, by using (ii) and applying the Lebesgue-Fatou theorem to (8) and (7), it is verified that $u(t) = \lim_{n \rightarrow \infty} u_n(t)$ exists for all $t \in [0, r]$ and u satisfies (6).

The uniqueness of solutions is shown by a standard argument.

We can prove the following result in the same manner as we proved the preceding theorem, noticing that the duality map F is single-valued and locally uniformly continuous if the dual space E' of E is uniformly convex. See Kato[1], Lemma 1.2.

Theorem 2. Suppose E' is uniformly convex and f satisfies (I)-(III). Then the conclusion of Theorem 1 holds, if E is separable.

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References

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