

Fourier Transformation of 2D $O(N)$ Spin Model and Anderson Localization

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Abstract

We Fourier transform the 2D $O(N)$ spin model $N > 2$, and start with a representation of the correlation functions in terms of integrals by complex random fields. Since this integral is complicated, we use the idea of the Anderson localization to discard non-local terms which make the integrals difficult. Through this approximation, we obtain the correlation functions which decay exponentially fast for all $\beta > 0$ if $N \gg 3$.

1 Introduction: Result and Motivation

It is a longstanding problem to prove or disprove non-existence of phase transitions in 4 dimensional non-Abelian lattice gauge theories. In many points, this is similar to the same problem in the two-dimensional $O(N)$ symmetric spin models (Heisenberg or σ model) with $N \geq 3$.

In models such as $O(N)$ spin models and $SU(N)$ lattice gauge models [13, 17], the field variables form compact manifolds and the block spin transformations break the original

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structures. In some cases, this can be avoided by introducing an auxiliary field ψ [3] which can be regarded as complex random field. The ν dimensional $O(N)$ spin (Heisenberg) model at the inverse temperature $N\beta$ is defined by the Gibbs expectation values

$$\langle f \rangle \equiv \frac{1}{Z_\Lambda(\beta)} \int f(\phi) \exp[-H_\Lambda(\phi)] \prod_{i \in \Lambda} \delta(\phi_i^2 - N\beta) d\phi_i \quad (1.1)$$

Here Λ is an arbitrarily large square with the center at the origin, $\phi(x) = (\phi(x)^{(1)}, \dots, \phi(x)^{(N)})$ is the vector valued spin at $x \in \Lambda$ and Z_Λ is the partition function defined so that $\langle 1 \rangle = 1$. The Hamiltonian H_Λ is given by

$$H_\Lambda \equiv -\frac{1}{2} \sum_{|x-y|_1=1} \phi(x)\phi(y), \quad (1.2)$$

where $|x|_1 = \sum_{i=1}^\nu |x_i|$.

We substitute the identity $\delta(\phi^2 - N\beta) = \int \exp[-ia(\phi^2 - N\beta)] da/2\pi$ into eq.(1.1) with the condition that $\text{Im} a_i < -\nu$ [3], and set

$$\text{Im} a_i = -(\nu + \frac{m^2}{2}), \quad \text{Re} a_i = \frac{1}{\sqrt{N}} \psi_i \quad (1.3)$$

where $m > 0$ will be determined soon. Thus we have

$$\begin{aligned} Z_\Lambda &= c^{|\Lambda|} \int \dots \int \exp[-\frac{1}{2} \langle \phi, (m^2 - \Delta + \frac{2i}{\sqrt{N}} \psi) \phi \rangle + \sum_j i\sqrt{N}\beta\psi_j] \prod \frac{d\phi_j d\psi_j}{2\pi} \\ &= c^{|\Lambda|} \det(m^2 - \Delta)^{-N/2} \int \dots \int F(\psi) \prod \frac{d\psi_j}{2\pi} \end{aligned} \quad (1.4)$$

where c 's are constants being different on lines, $\Delta_{ij} = -2\nu\delta_{ij} + \delta_{|i-j|,1}$ is the lattice Laplacian,

$$F(\psi) = \det(1 + i\kappa G\psi)^{-N/2} \exp[i\sqrt{N}\beta \sum_j \psi_j], \quad (1.5)$$

and $\kappa = 2/\sqrt{N}$. Moreover $G = (m^2 - \Delta)^{-1}$ is the covariant matrix discussed later. In the same way, the two-point function is given by

$$\langle \phi_0 \phi_x \rangle = \frac{1}{\tilde{Z}} \int \dots \int \left(\frac{1}{m^2 - \Delta + i\kappa\psi} \right) (0, x) F(\psi) \prod \frac{d\psi_j}{2\pi} \quad (1.6)$$

namely by an average of the Green's function which includes complex fields $\psi(x)$, $x \in \mathbb{Z}^2$, where the constant $\tilde{Z}$ is chosen so that $\langle \phi_0^2 \rangle = N\beta$. We choose the mass parameter $m > 0$ so that $G(0) = \beta$, where

$$G(x) = \int \frac{e^{ipx}}{m^2 + 2 \sum (1 - \cos p_i)} \prod_{i=1}^\nu \frac{dp_i}{2\pi} \quad (1.7)$$

This is possible for any β if $\nu \leq 2$, and we easily find that

$$m^2 \sim 32e^{-4\pi\beta} \text{ for } \nu = 2 \quad (1.8)$$

as $\beta \rightarrow \infty$. Thus for $\nu = 2$, we can rewrite

$$F(\psi) = \det_3^{-N/2}(1 + i\kappa G\psi) \exp[-\langle \psi, G^{\circ 2}\psi \rangle], \quad (1.9)$$

$$\det_3(1 + A) \equiv \det[(1 + A)e^{-A+A^2/2}] \quad (1.10)$$

where $G^{\circ 2}(x, y) = G(x, y)^2$ so that $\text{Tr}(G\psi)^2 = \langle \psi, G^{\circ 2}\psi \rangle$. For any two matrices A and B of equal size, the Hadamard product [18] $A \circ B$ is defined by $(A \circ B)_{ij} = A_{ij}B_{ij}$ and we denote $G \circ G$ by $G^{\circ 2}$.

Decompose $\Lambda \subset Z^2$ into small blocks Δ_i , and define $G_\Lambda = \chi_\Lambda G \chi_\Lambda$:

$$\Lambda = \cup_{i=1}^n \Delta_i,$$

Then we use the Feshbach-Krein formula (blockwise diagonalizations of matrices), to decompose $\det(1 + i\kappa G_\Lambda \psi_\Lambda)$ into a product of $\det(1 + i\kappa G_{\Delta_i} \psi_{\Delta_i})$ as follows:

$$\begin{aligned} & \det^{-N/2}(1 + i\kappa G_\Lambda \psi_\Lambda) \\ &= \left[\prod_{i=1}^{n-1} \det^{-N/2}(1 + W(\Delta_i, \Lambda_i)) \right] \prod_{i=1}^n \det^{-N/2}(1 + i\kappa G_{\Delta_i} \psi_{\Delta_i}). \end{aligned} \quad (1.11)$$

where $\kappa = 2/\sqrt{N}$, $\Lambda_k = \cup_{i=k+1}^n \Delta_i$,

$$W(\Delta_i, \Lambda_i) = -(i\kappa)^2 \frac{1}{1 + i\kappa G_{\Delta_i} \psi_{\Delta_i}} G_{\Delta_i, \Lambda_i} \psi_{\Lambda_i} \frac{1}{1 + i\kappa G_{\Lambda_i} \psi_{\Lambda_i}} G_{\Lambda_i, \Delta_i} \psi_{\Delta_i} \quad (1.12)$$

$$= -(i\kappa)^2 \frac{1}{[G_{\Delta_i}]^{-1} + i\kappa \psi_{\Delta_i}} [G_{\Delta_i}]^{-1} G_{\Delta_i, \Lambda_i} \psi_{\Lambda_i} \frac{1}{[G_{\Lambda_i}]^{-1} + i\kappa \psi_{\Lambda_i}} [G_{\Lambda_i}]^{-1} G_{\Lambda_i, \Delta_i} \psi_{\Delta_i} \quad (1.13)$$

and $[G_\Delta]^{-1}$ is the Laplacian with free boundary condition and almost equal to the Laplacian restricted to the square Δ with no boundary. Thus $\inf \text{spec } [G_\Delta]^{-1} \sim 0$ and we can prove that $([G_{\Lambda_i}]^{-1} + i\kappa \psi_{\Lambda_i})^{-1}$ behaves like a massive Green's function which decreases fast since ψ behaves like a Gaussian random variable of zero mean and covariance $[G^{\circ 2}]^{-1}$.

Let us consider the measure localized on each block Δ :

$$d\mu_\Delta = \frac{1}{Z_\Delta} \det_3^{-N/2}(1 + i\kappa G_\Delta \psi_\Delta) \exp[-(\psi_\Delta, G_\Delta^{\circ 2} \psi_\Delta)] \prod_{x \in \Delta} d\psi(x) \quad (1.14)$$

where Z_Δ is chosen so that $\int d\mu_\Delta = 1$. Since the norm of G_Δ is of order $O(|\Delta|\beta) \gg 1$, one may think that it is still impossible to expand the determinant. However, this comes with the factor $\exp[-(\psi_\Delta, G_\Delta^{\circ 2} \psi_\Delta)]$, which makes the norm of $\frac{2i}{\sqrt{N}} G_\Delta \psi_\Delta$ small. To see if this is the case, we introduce new variables $\tilde{\psi}_\Delta(x)$ by

$$\psi_\Delta(x) = \frac{1}{\sqrt{2}} \sum_{y \in \Delta} \hat{G}_\Delta^{-1}(x, y) \tilde{\psi}(y), \quad \hat{G}_\Delta \equiv [G_\Delta^{\circ 2}]^{1/2} \quad (1.15)$$

so that $d\mu_\Delta$ is rewritten

$$d\mu_\Delta = \frac{1}{Z_\Delta} \det_3^{-N/2} (1 + i\kappa K_\Delta) \prod_{x \in \Delta} \exp[-\frac{1}{2} \tilde{\psi}(x)^2] \frac{d\tilde{\psi}(x)}{\sqrt{2\pi}}, \quad (1.16)$$

$$K_\Delta = \frac{1}{\sqrt{2}} G_\Delta^{1/2} (\hat{G}_\Delta^{-1} \tilde{\psi}) G_\Delta^{1/2} \quad (1.17)$$

Put

$$d\nu_\Delta^{(0)} = \prod \exp[-\frac{1}{2} \tilde{\psi}^2(x)] \frac{d\tilde{\psi}(x)}{\sqrt{2\pi}} \quad (1.18)$$

$$= \det^{-1/2} [G_\Delta^{\circ 2}] \exp[-\langle \psi_\Delta, G_\Delta^{\circ 2} \psi_\Delta \rangle] \prod_{x \in \Delta} \frac{d\psi(x)}{\sqrt{2\pi}} \quad (1.19)$$

and define

$$\|K\|_p = \left(\int \text{Tr}(K^* K)^{p/2} d\nu_\Delta \right)^{1/p} \quad (1.20)$$

Lemma 1 *It holds that*

$$\int \text{Tr} K_\Delta^2 d\nu_\Delta = \frac{1}{2} |\Delta|, \quad (1.21)$$

$$\|K_\Delta\|_p \leq (p-1) \|K_\Delta\|_2, \quad \text{for all } p \geq 2 \quad (1.22)$$

Proof. The first equation is immediate. See [16] for the second inequality. Q.E.D.

Thus we see that κK_{Δ_i} are a.e. bounded with respect to $d\nu_\Delta$, and converges to 0 as $N \rightarrow \infty$. To see to what extent K_Δ is diagonal, we estimate

$$\begin{aligned} \int \text{Tr} K_\Delta^4 d\nu_\Delta &= \sum_{x_i \in \Delta} \frac{1}{4} \prod_{i=1}^4 G_\Delta(x_i, x_{i+1}) \\ &\times [2[G^{\circ 2}]^{-1}(x_1, x_2)[G^{\circ 2}]^{-1}(x_3, x_4) + [G^{\circ 2}]^{-1}(x_1, x_3)[G^{\circ 2}]^{-1}(x_2, x_4)] \end{aligned}$$

where $x_5 = x_1$. As is proved in [8]

$$[G_\Delta^{\circ 2}]^{-1}(x, y) = \frac{1}{2\beta} G_\Delta^{-1} - \hat{B}_\Delta, \quad \hat{B}_\Delta(x, y) = O(\beta^{-2}) \quad (1.23)$$

The main contribution comes from the term containing $2[G^{\circ 2}]^{-1}(x_1, x_2) \dots$. To bound this, set $G_\Delta(x_i, x_{i+1}) = \beta - \delta G(x_i, x_{i+1})$. Then $\delta G(x, x) = 0$, $\delta G(x, x + e_\mu) = 0.25 - O(\beta m^2)$, $(-\Delta)_{xy} = 0$ unless $|x - y| \leq 1$, and we have

$$\begin{aligned} \int \text{Tr} K_\Delta^4 d\nu_\Delta &\geq \text{const.} \sum_{x_1 \in \Delta} \frac{1}{4\beta^2} \left\{ \beta^2 \sum_{x_4} \delta_{x_1, x_4} + \sum_{x_4} G^2(x_1, x_4) \right\} \\ &\geq \text{const.} (|\Delta| + |\Delta|^2) \end{aligned}$$

which means that K_Δ is approximately diagonal but off-diagonal parts are still considerably large. However, there is a reason to believe that W functions are of short range and small. In fact we know that

$$\left| \frac{1}{[G_{\Lambda_i}]^{-1} + i\kappa\psi_{\Lambda_i}}(x, y) \right| \leq \frac{1}{[G_{\Lambda_i}]^{-1} + c(N\beta)^{-1}}(x, y)$$

for almost all ψ . Then $(G_{\Lambda_i}^{-1} + m^2 + i\kappa\psi_{\Lambda_i})^{-1}(x, y)$ is negligible if $|x - y| > \sqrt{N\beta}$. Moreover it is shown in two dimension that

$$\int \frac{1}{[G_{\Lambda_i}]^{-1} + i\kappa\psi_{\Lambda_i}}(x, y) d\mu \leq \frac{1}{[G_{\Lambda_i}]^{-1} + m_{eff}^2}(x, y),$$

$$m_{eff}^2 = c \frac{\log(N\beta)}{N\beta}$$

if $d\mu(\psi)$ is Gaussian of mean zero and covariance $[G^{\circ 2}]^{-1}$. This logarithmic correction comes from the two-dimensionality. This implies that

$$\lim_{N\beta \rightarrow \infty} \frac{1}{N\beta} \sum_x \int \frac{1}{-\Delta + m^2 + i\kappa\psi}(0, x) d\mu = 0$$

Furthermore ψ in the numerators of W acts as a differential operators since

$$\psi = \frac{1}{\sqrt{2}} [G_\Delta^{\circ 2}]^{-1/2} \tilde{\psi} \sim \frac{1}{2\sqrt{\beta}} [G_\Delta]^{-1/2} \tilde{\psi}$$

Thus $W(\Delta_i, \Lambda_i)$ seems to be small as $N\beta \rightarrow \infty$.

We choose N larger than $|\Delta| = L^2$, i.e.,

$$N^{1/3-\epsilon} \geq |\Delta| = L^2 \quad (1.24)$$

This assumption is artificial and its role is to simplify the large field problem to bound the integrals in the region where $|\psi_x|$ are large. So more elaborate idea may remove this condition (it is natural to think that $N \geq 3$ is enough).

To imagine that the non-diagonal terms W are small, we perhaps choose L larger than some power of β , say $L > (\beta)^{1+\delta}$, $\delta > 0$, but we do not know how to determine it yet though it is now under investigation, see [9].

Assumption: We take N larger than $|\Delta| = L^2$ as above, and for sufficiently large Δ , non-local terms W are negligible in this case.

Once W is neglected and N is chosen larger than $|\Delta|$, we can prove the following result uniformly in β :

Main Theorem: Assume N is sufficiently large: $N^{1/3-\epsilon} \geq |\Delta|$. Neglect non-local terms $W(\Delta, \Lambda)$. Then the two point correlation function

$$\int \frac{1}{-\Delta + m^2 + i\kappa\psi}(x, y) \prod_{\Delta} d\mu_{\Delta} \quad (1.25)$$

decays exponentially fast for all $\beta \geq 0$.

2 Averaged Green's Function by the measure $d\mu_0$

Let us come back to the present case where Δ_i are boxes of equal size $L \times L$ ($L \geq 2$) such that $\cup_i \Delta_i = Z^2$ and $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$. Let us estimate

$$G^{(ave)}(x, y) \equiv \int G^{(\psi)}(x, y) d\mu(\psi) \quad (2.1)$$

where

$$G^{(\psi)}(x, y) \equiv \left(\frac{1}{G^{-1} + i\kappa\psi} \right) (x, y), \quad (2.2a)$$

$$d\mu(\psi) \equiv \prod \frac{1}{Z_\Delta} \det_3^{-N/2} (1 + i\kappa G_\Delta \psi_\Delta) d\nu_\Delta, \quad (2.2b)$$

$$d\nu_\Delta = \frac{1}{\det^{1/2}(G_\Delta^{\circ 2})} \exp[-(\psi_\Delta, G_\Delta^{\circ 2} \psi_\Delta)] \prod_{x \in \Delta} \frac{d\psi(x)}{\sqrt{2\pi}}. \quad (2.2c)$$

and $\kappa = 2/\sqrt{N}$. Expanding $G^{(\psi)}$ by random walk, we have

$$G^{(\psi)}(x, y) = \sum_{\omega: x \rightarrow y} \prod_{\zeta \in \omega} \frac{1}{(4 + m^2 + i\kappa\psi_\zeta)^{n_\zeta}} \quad (2.3)$$

where $n_\zeta \in N$ is the visiting number of ω at $\zeta \in Z^2$. We set

$$d\nu = \prod_{\Delta \subset Z^2} d\nu_\Delta \quad (2.4)$$

We first prove our assertion for the Gaussian case:

Theorem 2 *The following bound holds:*

$$\int G^{(\psi)}(x, y) d\nu \leq \frac{1}{-\Delta + m_{eff}^2} (x, y) \quad (2.5)$$

where

$$m_{eff}^2 = m^2 + \frac{c}{N\beta} \quad (2.6)$$

with a constant $c > 1$.

Proof. Let Δ be the square of size $L \times L$ centered at the origin, and let $n_x \in \{0, 1, 2, \dots\}$, $x \in \Delta$. We estimate

$$D_\Delta(\{n\}) \equiv \int \prod_{x \in \Delta} \frac{1}{(4 + m^2 + i\kappa\psi(x))^{n_x}} d\nu_\Delta(\psi) \quad (2.7)$$

For large $\sum_{x \in \Delta} n_x$ such that

$$\sum_{x \in \Delta} n_x \geq \beta(\#\{x \in \Delta; n_x \neq 0\})^2 \quad (2.8)$$

the bound follows by the complex translation estimate by putting $\psi_x \rightarrow \psi_x - ih_x$, where

$$h_x = \frac{c_x}{\beta\sqrt{N}}, \quad c_x = \begin{cases} c > 0 & \text{if } n_x \geq 1 \\ 0 & \text{if } n_x = 0 \end{cases} \quad (2.9)$$

In fact we have :

$$\begin{aligned} D_\Delta(\{n\}) &\leq \frac{e^{\langle h, G_\Delta^{\circ 2} h \rangle}}{\prod_{x \in \Delta} (4 + m^2 + \kappa h_x)^{n_x}} \leq \frac{e^{\beta^2 (\sum_{x \in \Delta} h_x)^2}}{(4 + m^2 + c(\beta N)^{-1})^{\sum n_x}} \\ &\leq \left(\frac{1}{4 + m^2 + c'(\beta N)^{-1}} \right)^{\sum n_x} \end{aligned} \quad (2.10)$$

with a constant $0 < c' < c$.

For small $\{n_x; x \in \Delta\}$, we start with the new expression of $D_\Delta(\{n\})$:

$$\begin{aligned} &\prod_{x \in \Delta} \frac{1}{(n_x - 1)!} \int_0^\infty \prod s_x^{n_x - 1} \exp[-(4 + m^2) \sum s_x - \frac{\kappa^2}{4} \langle s_\Delta, [G_\Delta^{\circ 2}]^{-1} s_\Delta \rangle] \prod ds_x \\ &= \prod_{x \in \Delta} T^{-n(x)} \int \exp[-\frac{1}{NT^2} \langle s_\Delta, [G_\Delta^{\circ 2}]^{-1} s_\Delta \rangle] \prod_x d\nu_{n_x}(s_x) \end{aligned} \quad (2.11)$$

where $T = 4 + m^2$ and

$$d\nu_n(s) = \frac{s^{n-1} e^{-s}}{(n-1)!} ds \quad (2.12)$$

Since $\int d\nu_n(s) = 1$ and $n \log s - s$ takes its maximum at $s = n$, we set $s_x = n_x + \sqrt{n_x} \tilde{s}_x$ ($x \in \Delta$) and note that

$$d\nu_n(s) = \exp[-\frac{1}{2} \tilde{s}^2] \frac{e^{\delta_n(\tilde{s})}}{\langle e^{\delta_n(\tilde{s})} \rangle} \frac{d\tilde{s}}{\sqrt{2\pi}}, \quad (2.13)$$

$$\delta_n(\tilde{s}) = -\sqrt{n} \tilde{s} + (n-1) \log(1 + \frac{\tilde{s}}{\sqrt{n}}) + \frac{1}{2} \tilde{s}^2 \quad (2.14)$$

$$\begin{aligned} &= -\log(1 + \frac{\tilde{s}}{\sqrt{n}}) + O\left(\frac{\tilde{s}^3}{\sqrt{n}}\right), \\ \langle e^{\delta_n(\tilde{s})} \rangle &= \int_{-\sqrt{n}}^\infty e^{\delta_n(\tilde{s})} e^{-\tilde{s}^2/2} \frac{d\tilde{s}}{\sqrt{2\pi}} \\ &= 1 + O(1/n) > 1 \end{aligned} \quad (2.15)$$

Put

$$\alpha^2 = \frac{1}{N\beta} \quad (2.16)$$

Then if $\alpha^2 n(x) < 1$ and $N^{-1} < n$, $[G_\Delta^{\circ 2}]^{-1} n$ is small, the integral (2.11) is carried out by perturbative calculations.

For large $\alpha^2 n(x) \geq 1$ or for non-smooth n such that $N^{-1} < n, [G_\Delta^{\circ 2}]^{-1} n \gg 1$ we use a priori bound. See [8]. Q.E.D.

In the case where $d\mu$ is Gaussian, we can obtain $G^{(ave)}$ in a closed form. See [8] where $m_{eff}^2 \sim \log(N\beta)/N\beta$ is obtained.

Remark 1 We note that this is similar to the pinch singularity encountered in the study of the Anderson localization [5], where

$$\int G(E + i\varepsilon, v)(x, y) dP(v)$$

has a convergent random walk expansion, and

$$\int |G(E + i\varepsilon, v)(x, y)|^2 dP(v)$$

does not have.

3 Averaged Green's Function by the measure $d\mu(\psi)$

It remains to discuss the effects of the determinants $\det_3^{-N/2}(1 + \dots)$. Set

$$S_\Delta = \{\psi_x; x \in \Delta, \text{Tr} K_\Delta^2 < N^{1-2\varepsilon}\}, \quad (3.1)$$

$$K_\Delta = G_\Delta^{1/2} \psi_\Delta G_\Delta^{1/2} \quad (3.2)$$

Since

$$\exp[-\text{Tr} K_\Delta^2] \leq \left| \det_2^{-N/2}(1 + i\kappa K_\Delta) \right| \leq \left(1 + \frac{4}{N} \text{Tr} K_\Delta^2 \right)^{-N/4} \quad (3.3)$$

and $\text{Tr} K_\Delta^2 = \sum \tilde{\psi}_x^2/2$, we have

$$\int \exp[-\text{Tr} K_\Delta^2] \prod_x \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} = \int \exp[-\sum_x \frac{1}{2} \tilde{\psi}(x)^2] \prod_x \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} = 1 \quad (3.4)$$

and $\int (1 + \frac{2}{N} \sum \tilde{\psi}^2(x))^{-N/4} \prod_{x \in \Delta} d\tilde{\psi}_x$ is convergent for $2|\Delta| < N$. Even so, it is obvious that $|\det^{-N/2}(1 + i\kappa G_\Delta \psi)|$ is integrable if and only if $N > 2$ since

$$\det^{-N/2}(1 + i\kappa G_\Delta \psi) = \det^{-N/2}(G_\Delta) \det^{-N/2}(G_\Delta^{-1} + i\kappa \psi) \quad (3.5)$$

$$\sim \det^{-N/2}(G_\Delta) \prod_{x \in \Delta} \left(\frac{1}{4 + m^2 + i\kappa \psi(x)} \right)^{N/2} \quad (3.6)$$

holds for ψ such that $|\psi/\sqrt{N}| > O(1)$

3.1 Small Fields, Large Fields and Complex Displacements

Let us estimate

$$D_{\Delta}(n) = \frac{1}{Z_{\Delta}} \int \frac{e^{i\sqrt{N}\beta \sum_{x \in \Delta} \psi_x}}{[\prod_{x \in \Delta} (4 + m^2 + i\kappa\psi_x)^{n_x}] \det^{N/2}(1 + i\kappa G_{\Delta} \psi_{\Delta})} \prod_{x \in \Delta} d\psi_x, \quad (3.7)$$

$$Z_{\Delta} = \int \frac{e^{i\sqrt{N}\beta \sum_{x \in \Delta} \psi_x}}{\det^{N/2}(1 + i\kappa G_{\Delta} \psi_{\Delta})} \prod_{x \in \Delta} d\psi_x \quad (3.8)$$

by putting $\psi_x \rightarrow \psi_x - ih_x$, $\psi_x \in R$. Then

$$\begin{aligned} D_{\Delta}(n) &= \frac{1}{Z_{\Delta}} \int \frac{e^{\sqrt{N}\beta \sum_{x \in \Delta} (i\psi_x + h_x)}}{[\prod_{x \in \Delta} (4 + m^2 + \kappa(i\psi_x + h_x))^{n_x}] \det^{N/2}(1 + \kappa G_{\Delta}^{1/2}(i\psi_{\Delta} + h_{\Delta}) G_{\Delta}^{1/2})} \prod_{x \in \Delta} d\psi_x \\ &= \frac{1}{Z_{\Delta}} \int \frac{\exp[-\langle \psi - ih, G_{\Delta}^{\circ 2}(\psi - ih) \rangle]}{[\prod_{x \in \Delta} (4 + m^2 + \kappa(i\psi_x + h_x))^{n_x}] \det_3^{N/2}(1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta}))} \prod_{x \in \Delta} d\psi_x \end{aligned}$$

where

$$K_{\Delta}(\psi_{\Delta}) \equiv G_{\Delta}^{1/2} \psi_{\Delta} G_{\Delta}^{1/2}, \quad K_{\Delta}(h_{\Delta}) \equiv G_{\Delta}^{1/2} h_{\Delta} G_{\Delta}^{1/2} \quad (3.9)$$

and $K_{\Delta}(h_{\Delta}) \geq 0$ since $h_x \geq 0$. We again put $h_x = c_x/(\sqrt{N}\beta)$ and then

$$\kappa K_{\Delta}(h_{\Delta}) \leq \frac{c|\Delta|}{N}, \quad c = O(1) > 0. \quad (3.10)$$

We repeat the previous arguments by using $(n-1)!x^{-n} = \int_0^{\infty} s^{n-1} e^{-sx} ds$. Define $I_n^{(k)} = \{s; k\sqrt{n} < |s - n| < (k+1)\sqrt{n}, s \geq 0\}$, $k = 0, 1, 2, \dots$, and let $\chi_x^{(k)}(s_x)$ be the characteristic function of the interval $I_{n_x}^{(k)}$. Then

$$\begin{aligned} D_{\Delta}(n) &= \frac{1}{Z_{\Delta}} \int_0^{\infty} \prod_{x \in \Delta} \left(\sum_k \chi_x^{(k)}(s_x) \right) \frac{s_x^{n(x)-1} ds_x}{(n(x)-1)!} \int_{-\infty}^{\infty} \prod_{x \in \Delta} d\psi_x \\ &\quad \times \exp\left[-\sum_x (4 + m^2 + \kappa(i\psi(x) + h(x))s(x))\right] \frac{e^{-\langle (\psi_{\Delta} - ih_{\Delta}), G^{\circ 2}(\psi_{\Delta} - ih_{\Delta}) \rangle}}{\det_3^{N/2}(1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta}))} \quad (3.11) \\ &= \frac{1}{Z_{\Delta}^{(0)}} \frac{1}{\prod_x T_x^{n_x}} \int_0^{\infty} \prod_{x \in \Delta} d\nu_{n_x}(s_x) \int_{-\infty}^{\infty} \prod_{x \in \Delta} \left(\sum_k \chi_x^{(k)}(s_x) \right) d\psi_x \\ &\quad \times \exp\left[-\langle (\psi - ih + i\zeta), G_{\Delta}^{\circ 2}(\psi - ih + i\zeta) \rangle - \frac{1}{N} \langle \frac{1}{T}s, [G_{\Delta}^{\circ 2}]^{-1} \frac{1}{T}s \rangle + \kappa \langle h, \frac{s}{T} \rangle\right] \\ &\quad \times \det_3^{-N/2}(1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta})) \quad (3.12) \end{aligned}$$

where $T_x = 4 + m^2 + \kappa h_x$, $(s/T)_x = s_x/T_x$

$$d\nu_n(s) = \frac{1}{(n-1)!} e^{-s} s^{n-1} ds, \quad \zeta_x = \frac{\kappa}{2} ([G_\Delta^{\circ 2}]^{-1} \frac{1}{T} s)(x) \quad (3.13)$$

and

$$\begin{aligned} Z_\Delta^{(0)} &= \int \exp[-\langle (\psi - ih), G_\Delta^{\circ 2}(\psi - ih) \rangle] \\ &\quad \times \det_3^{-N/2} (1 + i\kappa K_\Delta(\psi_\Delta) + \kappa K_\Delta(h_\Delta)) \prod_{x \in \Delta} d\psi_x \end{aligned} \quad (3.14)$$

We then change the contour of ψ_x by replacing $\psi_x + i\zeta_x$ by ψ (namely we put $\psi_x \rightarrow \psi - i\zeta_x$). The contours depend on $\{s_x; x \in \Delta\}$. This yields

$$\begin{aligned} D_\Delta(n) &= \frac{1}{Z_\Delta^{(0)}} \frac{1}{\prod_x T_x^{n_x}} \int_0^\infty \prod_{x \in \Delta} d\nu_{n_x}(s_x) \int_{-\infty}^\infty \prod_{x \in \Delta} \left(\sum_k \chi_x^{(k)}(s_x) \right) d\psi_x \\ &\quad \times \exp \left[-\langle (\psi - ih), G_\Delta^{\circ 2}(\psi - ih) \rangle - \frac{1}{N} \langle \frac{1}{T} s, [G_\Delta^{\circ 2}]^{-1} \frac{1}{T} s \rangle + \kappa \langle h, \frac{s}{T} \rangle \right] \\ &\quad \times \det_3^{-N/2} (1 + i\kappa K_\Delta(\psi_\Delta) + \kappa K_\Delta(h_\Delta) + \kappa K_\Delta(\zeta_\Delta)) \\ &= \frac{1}{\prod_x T_x^{n_x}} \int_0^\infty \prod_{x \in \Delta} d\nu_{n_x}(s_x) \exp \left[-\frac{1}{N} \langle \frac{1}{T} s, [G_\Delta^{\circ 2}]^{-1} \frac{1}{T} s \rangle + \kappa \langle h, \frac{s}{T} \rangle \right] \\ &\quad \times \frac{1}{Z_\Delta^{(0)}} \int_{-\infty}^\infty \prod_{x \in \Delta} \left(\sum_k \chi_x^{(k)}(s_x) \right) d\psi_x \det_3^{-N/2} (1 + i\kappa K_\Delta(\psi_\Delta)) \exp [-\langle \psi, G_\Delta^{\circ 2} \psi \rangle] \\ &\quad \times \det_3^{-N/2} (1 + \kappa J_\Delta(\zeta_\Delta)) \times \exp[R_3] \end{aligned} \quad (3.15)$$

where

$$\begin{aligned} K_\Delta(\zeta_\Delta)(x, y) &= \frac{1}{\sqrt{N}} \sum_\xi G_\Delta^{1/2}(x, \xi) \left([G_\Delta^{\circ 2}]^{-1} \frac{s}{T} \right)(\xi) G_\Delta^{1/2}(\xi, y), \\ J_\Delta(\zeta_\Delta) &= \frac{1}{\sqrt{1 + i\kappa K_\Delta(\psi_\Delta)}} K(\zeta_\Delta) \frac{1}{\sqrt{1 + i\kappa K_\Delta(\psi_\Delta)}} \end{aligned}$$

and

$$R_3 = \frac{N}{2} \text{Tr} \left[\left(\frac{1}{1 + i\kappa K_\Delta(\psi_\Delta)} - 1 + i\kappa K_\Delta(\psi_\Delta) \right) \kappa K_\Delta(\zeta_\Delta) + \kappa^2 (J_\Delta^2 - K_\Delta(\zeta_\Delta)^2) \right] \quad (3.16)$$

3.2 $K_\Delta(\psi_\Delta)$, $K_\Delta(\zeta_\Delta)$ and R_3

Let

$$G_\Delta = \sum_{i=0}^{|\Delta|-1} e_i P_i, \quad G_\Delta^{\circ 2} = \sum_{i=0}^{|\Delta|-1} \hat{e}_i \hat{P}_i \quad (3.17)$$

be the spectral resolutions of the positive matrices G_Δ and $G_\Delta^{\circ 2}$ respectively, where $e_0 \geq e_1 \geq \dots \geq e_{|\Delta|-1}$, $\hat{e}_0 \geq \hat{e}_1 \geq \dots \geq \hat{e}_{|\Delta|-1}$, $P_i P_j = \delta_{i,j} P_i$ and so on. Then

$$G_\Delta^{1/2} = \sum_{i=0}^{|\Delta|-1} \sqrt{e_i} P_i, \quad [G_\Delta^{\circ 2}]^{-1} = \sum_{i=0}^{|\Delta|-1} \frac{1}{\hat{e}_i} \hat{P}_i \quad (3.18)$$

It is convenient to introduce the abbreviation for the Green's function with the largest eigenvalue part extracted:

$$G_\Delta^{(0)} = \sum_{k \neq 0} e_k P_k = G_\Delta - e_0^{-1} P_0$$

We let $\{u_i\}_{i=0}^{|\Delta|-1}$ and $\{\hat{u}_i\}_{i=0}^{|\Delta|-1}$ be the normalized eigenvectors such that

$$G_\Delta u_i = e_i u_i, \quad G_\Delta^{\circ 2} \hat{u}_i = \hat{e}_i \hat{u}_i \quad (3.19)$$

Then

$$P_i = |u_i\rangle\langle u_i|, \quad \hat{P}_i = |\hat{u}_i\rangle\langle \hat{u}_i| \quad (3.20)$$

and for small Δ , we have

$$e_0 = |\Delta|\beta - O(1), \quad e_i = O(1) > 0 \quad (3.21)$$

$$\hat{e}_0 = |\Delta|\beta^2 - O(\beta), \quad \hat{e}_i = 2\beta e_i + O(1) \quad (3.22)$$

($i \neq 0$) and

$$P_0 \sim \hat{P}_0 \sim \frac{1}{|\Delta|} |U\rangle\langle U| = \frac{1}{|\Delta|} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}, \quad (3.23)$$

where $U = {}^t(1, 1, \dots, 1) \sim \sqrt{|\Delta|} u_0$. Moreover we can symbolically write

$$P_i \sim \hat{P}_i \sim \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (3.24)$$

namely P_i ($i \neq 0$) is a matrix which represents a lattice differentiation since $\langle u_i, u_0 \rangle = 0$. Note that $e_i \leq O(\log |\Delta|)$, $e_0 = \beta|\Delta| - O(|\Delta| \log |\Delta|)$ and

$$(P_0 \zeta P_0)_{x,y} = \sum_{\xi} \frac{1}{|\Delta|^2} \zeta_{\xi} = \left(\frac{1}{|\Delta|} \sum \zeta_{\xi} \right) P_0, \quad P_0(\hat{P}_i \zeta) P_0 = O(\beta^{-1}) \quad (3.25)$$

We insert $\psi = \hat{G}^{-1} \tilde{\psi} / \sqrt{2}$ into K_{Δ} and use $\hat{e}_i = 2\beta e_i + O(1)$ ($i \neq 0$), $P_i = \hat{P}_i + O(\beta^{-1})$ and $\sum_{i \neq 0} P_i = 1 - P_0$ to find that

$$\begin{aligned} K_{\Delta} &= \frac{\sum \tilde{\psi}(x)}{\sqrt{2|\Delta|}} P_0 + \frac{\sqrt{|\Delta|}}{2} \left(\sum_{i \neq 0} P_0(P_i \tilde{\psi}) + \sum_{i \neq 0} (\tilde{\psi} P_i) P_0 \right) + O(\beta^{-1}) \\ &= \frac{(\sum \tilde{\psi}(x))}{\sqrt{2|\Delta|}} (1 - \sqrt{2}) P_0 + \frac{\sqrt{|\Delta|}}{2} (P_0 \tilde{\psi} + \tilde{\psi} P_0) + O(\beta^{-1}) \\ K_{\Delta}^2 &= \left[\frac{1}{4} X + (1 - \sqrt{2}) Y^2 \right] P_0 + \frac{\sqrt{2} - 1}{4} \sqrt{|\Delta|} Y (P_0 \tilde{\psi} + \tilde{\psi} P_0) \\ &\quad + \frac{|\Delta|}{4} \tilde{\psi} P_0 \tilde{\psi} + O(\beta^{-1}) \end{aligned}$$

where

$$X = \sum_{x \in \Delta} \tilde{\psi}_x^2, \quad Y = \frac{1}{\sqrt{|\Delta|}} \sum_{x \in \Delta} \tilde{\psi}_x \quad (3.26)$$

Note that $\text{Tr} K_{\Delta}^2 = \sum \tilde{\psi}_x^2 / 2$ as expected. Just in the same way, we have

$$\begin{aligned} K_{\Delta}(\zeta) &= G_{\Delta}^{1/2} \zeta G_{\Delta}^{1/2} = G_{\Delta}^{1/2} \left(\frac{1}{\sqrt{N}} \sum_k \frac{1}{\hat{e}_k} \hat{P}_k \frac{s}{T} \right) G_{\Delta}^{1/2} \\ &= \left(\sum_x \zeta_x \right) \beta P_0 + \frac{1}{2} \left(\frac{|\Delta|}{\beta N} \right)^{1/2} \left[([G_{\Delta}^{(0)}]^{-1/2} \frac{s}{T}) P_0 + P_0 ([G_{\Delta}^{(0)}]^{-1/2} \frac{s}{T}) \right] + [G_{\Delta}^{(0)}]^{1/2} \zeta [G_{\Delta}^{(0)}]^{1/2} \end{aligned}$$

and

$$\begin{aligned} K_{\Delta}(\zeta)^2 &= \left[\beta^2 (\sum \zeta)^2 + \frac{1}{4\beta N} \langle \frac{s}{T}, G_{\Delta}^{-1} \frac{s}{T} \rangle + \left(\frac{\beta}{N|\Delta|} \right)^{1/2} (\sum \zeta) (\sum [G_{\Delta}^{(0)}]^{-1/2} \frac{s}{T}) \right] P_0 \\ &\quad + \left[\left(\frac{\beta|\Delta|}{N} \right)^{1/2} (\sum \zeta) + \frac{1}{4\beta N} (\sum [G_{\Delta}^{(0)}]^{-1/2} \frac{s}{T}) \right] \left(P_0 ([G_{\Delta}^{(0)}]^{-1/2} \frac{s}{T}) + ([G_{\Delta}^{(0)}]^{-1/2} \frac{s}{T}) P_0 \right) \\ &\quad + \frac{\sqrt{|\Delta|}}{2\sqrt{\beta N}} \left(P_0 \left(\frac{s}{T} \circ \zeta \right) [G_{\Delta}^{(0)}]^{1/2} + [G_{\Delta}^{(0)}]^{1/2} \left(\frac{s}{T} \circ \zeta \right) P_0 \right) \\ &\quad + \frac{|\Delta|}{4N\beta} ([G_{\Delta}^{(0)}]^{-1/2} \frac{s}{T}) P_0 ([G_{\Delta}^{(0)}]^{-1/2} \frac{s}{T}) + [G_{\Delta}^{(0)}]^{1/2} \zeta [G_{\Delta}^{(0)}] \zeta [G_{\Delta}^{(0)}]^{1/2} \end{aligned}$$

Here $\zeta_x = N^{-1/2}(G_\Delta^{\circ 2})^{-1}(s/T)(x)$, $(x \circ y)_k \equiv x_k y_k$ for two vectors x and y , $e_0 = \beta|\Delta| - O(|\Delta| \log |\Delta|)$ and we have used $P_0 P_i = 0$ ($i \neq 0$) and

$$P_0(G_\Delta^{-1/2} \frac{s}{T})[G_\Delta^{(0)}]^{1/2} \zeta [G_\Delta^{(0)}]^{1/2} = P_0(\frac{s}{T} \circ \zeta)[G_\Delta^{(0)}]^{1/2}$$

We can obtain similar expressions for $K(\psi)^n$ etc., and R_3 is represented by these functions of ψ and ζ . We decompose our set $\{s_x; s_x \geq 0, x \in \Delta\}$ into 2 regions:

- (1) small s region
- (2) large s region

and each region is also decomposed into large ψ region and small ψ region, where the small ψ field $\mathcal{S}_\Delta(\psi)$ means the set of ψ such that

$$\mathcal{S}_\Delta(\psi) = \{\psi_x = \frac{1}{\sqrt{2}}(\hat{G}_\Delta^{-1}\tilde{\psi})(x), \sum_{x \in \Delta} \tilde{\psi}_x^2 \leq N^{1-2\epsilon}\}$$

and small s field $\mathcal{S}_\Delta(s)$ means the set of s_x such that

$$\begin{aligned} \mathcal{S}_\Delta(s) = & \left\{ s_x = n(x) + \sqrt{n(x)}\tilde{s}(x) \geq 0, \frac{1}{N\beta} \sum_{n.n.} \left(\frac{s_x}{T_x} - \frac{s_y}{T_y} \right)^2 \leq O(1) \right. \\ & \left. \frac{1}{N^2\beta^2} \sum_{x \in \partial\Delta} \frac{s_x^2}{T_x^2} \leq O(1) \right\} \end{aligned} \quad (3.27)$$

3.3 Small field Region of s_x

For small smooth $\{s_x\}$, we see that $\det_3^{-N/2}(1 - \kappa J_\Delta(\psi))$ yields a convergent small factor uniformly in ψ_x . Put

$$\det_3^{-N/2}(1 + \kappa J_\Delta(\psi)) = \exp[\mathcal{E}_3]$$

Then

$$\begin{aligned} |\mathcal{E}_3| &= \left| \frac{4}{3\sqrt{N}} \text{Tr} J_\Delta^3 + \dots \right| = o(1) \text{Tr} K(\zeta_\Delta)^2 \\ &= o(1) \frac{1}{N} < \frac{s}{T}, [G_\Delta^{\circ 2}]^{-1} \frac{s}{T} > \end{aligned}$$

Contrary to the above, we must be careful about R_3 which depend on ψ sensitively.

3.3.1 small ψ region

We first assume ψ are small. Let us begin our calculation

$$\begin{aligned} I &= \frac{1}{Z_\Delta^0} \int \exp[\mathcal{E}_3 + R_3] \det_3^{-N/2}(1 + i\kappa K_\Delta) \exp[-<\psi, [G_\Delta^{\circ 2}]\psi>] \prod \frac{d\psi_x}{\sqrt{2\pi}} \\ &= \frac{1}{\bar{Z}_\Delta^0} \int \exp[\mathcal{E}_3 + R_3] \det_3^{-N/2}(1 + i\kappa K_\Delta) d\nu_\Delta \end{aligned} \quad (3.28)$$

$$d\nu_\Delta \equiv \exp[-\frac{1}{2} \sum \tilde{\psi}_x^2] \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} \quad (3.29)$$

by decomposing $\{\tilde{\psi}_x \in R; x \in \Delta\}$ into small field region

$$\mathcal{S}_\Delta = \left\{ \sum_x \tilde{\psi}_x^2 < |\Delta| N^\varepsilon \right\}, \quad \varepsilon \in (0, 1) \quad (3.30)$$

and its compliment $\mathcal{S}^c$, where the normalization constants $Z_\Delta^{(0)}$ and $\tilde{Z}_\Delta^{(0)}$ are defined in the obvious way. Thus we evaluate

$$I = I_S + I_{S^c} \quad (3.31)$$

where

$$I_S = \frac{1}{\tilde{Z}_\Delta^0} \int_S (1 + \mathcal{E}_3 + R_3 + O(R_3^2)) \det_3^{-N/2} (1 + i\kappa K_\Delta) d\nu_\Delta \quad (3.32)$$

$$I_{S^c} = \frac{1}{\tilde{Z}_\Delta^0} \int_{S^c} \det_1^{-N/2} (1 + i\kappa K_\Delta (\psi - ih + i\zeta)) e^{i\sqrt{N}\beta \sum_x (\psi_x - ih_x + i\zeta_x)} \\ \times \exp \left[-\frac{2}{N} \left\langle \frac{s}{T}, [G_\Delta^2]^{-1} \frac{s}{T} \right\rangle + 2\kappa \left\langle h, \frac{s}{T} \right\rangle + i\kappa \left\langle \psi, \frac{s}{T} \right\rangle \right] \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} \quad (3.33)$$

We first calculate the small field contribution I_S given by

$$I_S = \frac{\langle \chi_S \mathcal{D}_\Delta \rangle}{\langle \mathcal{D}_\Delta \rangle} \left\{ 1 + \langle \chi_S \mathcal{E}_3 \rangle + \langle \chi_S R_3 \rangle + \langle \chi_S O(R_3^2) \rangle \right. \\ \left. + \frac{\langle \chi_S \mathcal{D}_\Delta; \chi_S \mathcal{E}_3 \rangle}{\langle \chi_S \mathcal{D}_\Delta \rangle} \frac{\langle \chi_S \mathcal{D}_\Delta; \chi_S R_3 \rangle}{\langle \chi_S \mathcal{D}_\Delta \rangle} + \frac{\langle \chi_S \mathcal{D}_\Delta; \chi_S O(R_3^2) \rangle}{\langle \chi_S \mathcal{D}_\Delta \rangle} \right\} \quad (3.34)$$

where

$$\mathcal{D}_\Delta \equiv \det_3^{-N/2} (1 + i\kappa K_\Delta), \quad \langle A \rangle = \int A d\nu_\Delta$$

and

$$\langle A; B \rangle = \int AB d\nu - \left(\int A d\nu \right) \left(\int B d\nu \right)$$

We calculate $\langle \mathcal{D} \rangle$ and $\langle \chi_S \mathcal{D} \rangle$ first. We assumed that

$$\frac{|\Delta| - 2}{2} \leq N^{1/3-2\varepsilon}, \quad 0 < \varepsilon \ll 1 \quad (3.35)$$

Then

$$\langle \chi_S \mathcal{D} \rangle = \int_{\chi_S} \det_3^{-N/2} (1 + i\kappa K_\Delta) d\nu_\Delta = 1 - O(N^{-1/3}) \quad (3.36)$$

To bound $\langle \chi_{S^c} \mathcal{D} \rangle$, we use the bounds (3.3), and set $r^2 = 2\text{Tr} K_\Delta^2 = \sum \tilde{\psi}_x^2$. Then for $R^2 > \rho_0 = (|\Delta| - 2)/2$, we have that

$$\frac{(|\Delta| - 2)!!}{(2\pi)^{|\Delta|/2}} \int_{r>R} \left(\frac{1}{1 + \frac{2}{N} r^2} \right)^{N/4} r^{|\Delta|-1} dr \leq O(\exp[-N^{1/3}])$$

This means that

$$\frac{\langle \chi_S \mathcal{D}_\Delta \rangle}{\langle \mathcal{D} \rangle} = \frac{\langle \chi_S \mathcal{D}_\Delta \rangle}{\langle \chi_S \mathcal{D} \rangle + \langle \chi_{S^c} \mathcal{D} \rangle} = 1 - O(\exp[-cN^{1/3}]) \quad (3.37)$$

Estimates are straightforward and we see that the most significant contribution is from $\text{Tr} K_\Delta^2(\psi) K_\Delta(\zeta)$ in R_3 and we have:

$$\begin{aligned} \langle \chi_S R_3 \rangle &= -\frac{|\Delta|}{\sqrt{N}} \left(c_1 \beta \left(\sum_x \zeta_x \right) + \frac{c_2}{\sqrt{\beta|\Delta|}} \left(\sum [G_\Delta^{(0)}]^{-1/2} \frac{s}{T} \right) \right. \\ &\quad \left. - \frac{\sqrt{|\Delta|}}{\sqrt{2\beta N}} \left(\sum [G_\Delta^{(0)}]^{-1/2} \frac{s}{T} \right) - \frac{1}{\sqrt{N}} \text{Tr} G_\Delta^{(0)} \zeta + (\text{smaller terms}) \right) \end{aligned} \quad (3.38)$$

where $c_i = 1 + O(|\Delta|^{-1})$ ($i = 1, 2$) are positive constants. Moreover we have (see [8]) :

$$\sum_{x \in \Delta} \zeta_x = \frac{1}{\sqrt{N}} \sum_{x \in \Delta} \left([G_\Delta^{\circ 2}]^{-1} \frac{s}{T} \right) (x) = \frac{1}{\sqrt{N}} \left(\sum_{x \in \partial \Delta} \frac{1}{\beta} \delta_{\partial \Delta}(x) \frac{s_x}{T_x} + O(\beta^{-3}) \right) \quad (3.39)$$

$$\delta_{\partial \Delta}(x) = O\left(\frac{1}{\beta \sqrt{\Delta}}\right) \geq 0 \quad (3.40)$$

and

$$\begin{aligned} \text{Tr} G_\Delta^{(0)} \zeta &= \beta \sum \zeta_x - \text{Tr} e_0 P_0 \zeta \\ &= O\left(\frac{\log |\Delta|}{|\Delta|}\right) \left(\sum \zeta_x \right) - \left(\beta - \frac{\sigma_0}{|\Delta|} \right) \left(\sum \zeta_x \right) + \frac{1}{2\beta \sqrt{N} |\Delta|} \sum \frac{s_x}{T_x} \end{aligned}$$

Then the largest contribution comes from $\langle \chi_S R_3 \rangle$ and is negative, and other contributions can be made less than $\frac{1}{N\beta} \sum s_x/T_x$

3.3.2 large ψ region

For $\{\tilde{\psi}\} \notin \mathcal{S}_\Delta$, we start with

$$\begin{aligned} I_{S^c} &= \frac{1}{\tilde{Z}_\Delta^0} \int_{S^c} \det^{-N/2} (1 + i\kappa K_\Delta(\psi - ih + i\zeta)) e^{i\sqrt{N}\beta \sum_x (\psi_x - ih_x + i\zeta_x)} \\ &\quad \times \exp \left[-\frac{2}{N} \left\langle \frac{s}{T}, [G_\Delta^{\circ 2}]^{-1} \frac{s}{T} \right\rangle + 2\kappa \left\langle h, \frac{s}{T} \right\rangle + i\kappa \left\langle \psi, \frac{s}{T} \right\rangle \right] \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} \\ &= \frac{1}{\tilde{Z}_\Delta^0} \int_{S^c} \det^{-N/2} (1 + i\kappa K_\Delta(\psi)) e^{i\sqrt{N}\beta \sum_x \psi_x} \end{aligned} \quad (3.41)$$

$$\begin{aligned} &\times \det^{-N/2} (1 + \kappa J_\Delta(h - \zeta)) e^{\sqrt{N}\beta \sum_x (h_x - \zeta_x)} \\ &\times \exp \left[-\frac{2}{N} \left\langle \frac{s}{T}, [G_\Delta^{\circ 2}]^{-1} \frac{s}{T} \right\rangle + 2\kappa \left\langle h, \frac{s}{T} \right\rangle + i\kappa \left\langle \psi, \frac{s}{T} \right\rangle \right] \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} \end{aligned} \quad (3.42)$$

where

$$J_{\Delta}(h - \zeta) = \frac{1}{\sqrt{1 + i\kappa K_{\Delta}(\psi)}} K_{\Delta}(h - \zeta) \frac{1}{\sqrt{1 + i\kappa K_{\Delta}(\psi)}} \quad (3.43)$$

and then

$$\begin{aligned} & \det^{-N/2}(1 + \kappa J_{\Delta}(h - \zeta)) e^{\sqrt{N}\beta \sum_x (h_x - \zeta_x)} \\ & \times \exp \left[-\frac{2}{N} \left\langle \frac{s}{T}, [G_{\Delta}^{\circ 2}]^{-1} \frac{s}{T} \right\rangle + 2\kappa \left\langle h, \frac{s}{T} \right\rangle + i\kappa \left\langle \psi, \frac{s}{T} \right\rangle \right] \\ & = \det_3^{-N/2}(1 + \kappa J_{\Delta}(h - \zeta)) \\ & \times \exp \left[-\frac{1}{N} \left\langle \frac{s}{T}, [G_{\Delta}^{\circ 2}]^{-1} \frac{s}{T} \right\rangle + \kappa \left\langle h, \frac{s}{T} \right\rangle + i\kappa \left\langle \psi, \frac{s}{T} \right\rangle + \left\langle h, [G_{\Delta}^{\circ 2}]h \right\rangle \right] \\ & \times \exp \left[\text{Tr} \left(\frac{2iK_{\Delta}(\psi)}{1 + i\kappa K_{\Delta}(\psi)} \right) K_{\Delta}(h - \zeta) + \text{Tr}(J_{\Delta}^2(h - \zeta) - K_{\Delta}^2(h - \zeta)) \right] \end{aligned}$$

and

$$\begin{aligned} \text{Re Tr } J_{\Delta}^2(h - \zeta) & \leq \text{Tr } K_{\Delta}^2(h - \zeta) \\ & = \frac{1}{N} \left\langle \frac{s}{T}, [G_{\Delta}^{\circ 2}]^{-1} \frac{s}{T} \right\rangle - \kappa \left\langle h, \frac{s}{T} \right\rangle + \left\langle h, [G_{\Delta}^{\circ 2}]h \right\rangle \end{aligned}$$

Then putting $\mathcal{S}_{\Delta}^c = \cup_{k=1}^{\infty} S_k$ where

$$S_k = \{ \{ \tilde{\psi}_x \}; kN^{1-2\epsilon} \leq \sum \tilde{\psi}^2 \leq (k+1)N^{1-2\epsilon} \}$$

we estimate the integral on each shell of $\mathcal{S}^c$:

$$\int_{S_k} |\det^{-N/2}(1 + i\kappa K_{\Delta}(\psi))| \prod \frac{d\tilde{\psi}}{\sqrt{2\pi}} \leq \frac{(|\Delta| - 2)!! (kN^{1-2\epsilon})^{(|\Delta|-1)/2}}{(2\pi)^{|\Delta|/2} (1 + 2kN^{-2\epsilon})^{N/4}}$$

3.3.3 integration over small-smooth s_x

It remains to integrate over $\{s_x = n_x + \sqrt{n_x} \tilde{s}_x\}$ such that $0 < s_x < N\beta$ and $|s_x - s_{x \pm \mu_i}| < \sqrt{N}\beta$. Since the contribution from I_{S^c} is negligible, we can apply the previous methods of analysis: we set

$$\begin{aligned} d\nu_n(s) &= \frac{1}{(n-1)!} e^{-n - \sqrt{n}\tilde{s}} (n + \sqrt{n}\tilde{s})^{n-1} \sqrt{n} d\tilde{s} \\ &= \frac{e^{-n} n^n}{(n-1)! \sqrt{n}} \exp[-\sqrt{n}\tilde{s} + (n-1) \log(1 + \tilde{s}/\sqrt{n})] d\tilde{s} \\ &= \exp[-\frac{1}{2}\tilde{s}^2 + O(\tilde{s}/\sqrt{n})] \frac{d\tilde{s}}{\sqrt{\pi}} \end{aligned}$$

We note that $K_{\Delta}(x, y) \sim \frac{1}{\sqrt{2|\Delta|}} (\tilde{\psi}(x) + \tilde{\psi}(y))$ is not of short range, though $K_{\Delta}(x, y) = O(|\Delta|^{-1/2})$. This long range nature of the interaction is expected compensated by the Anderson localization like phenomena.

3.4 Large field Region of s_x

For $\{s_x; s_x > N\beta, \exists x \in \Delta\}$ or for $\{s_x; |s_x - s_{x'}| > \sqrt{N\beta}, \exists x \in \Delta, \exists x' \in \Delta, |x - x'| = 1\}$, we need a priori bound to estimate the two-point function. Continuing the argument in the previous section (3.1), we start from

$$D_\Delta(n) \Big|_{C_0} = \frac{1}{Z_\Delta} \int \frac{\exp[-\langle \psi - ih, G_\Delta^{\circ 2}(\psi - ih) \rangle]}{[\prod_{x \in \Delta} (4 + m^2 + \kappa(i\psi_x + h_x))^{n_x}] \det_3^{N/2}(1 + i\kappa K_\Delta(\psi_\Delta) + \kappa K_\Delta(h_\Delta))} \prod_{x \in \Delta} d\psi_x \quad (3.44)$$

$$= \frac{1}{Z_\Delta^{(0)}} \frac{1}{\prod_x T_x^{n_x}} \int_0^\infty \prod_{x \in \Delta} d\nu_{n_x}(s_x) \int_{-\infty}^\infty \prod_{x \in \Delta} \left(\sum_k \chi_x^{(k)}(s_x) \right) d\psi_x \\ \times \exp \left[-\langle (\psi - ih + i\zeta), G_\Delta^{\circ 2}(\psi - ih + i\zeta) \rangle - \frac{1}{N} \langle \frac{1}{T}s, [G_\Delta^{\circ 2}]^{-1} \frac{1}{T}s \rangle + \kappa \langle h, \frac{s}{T} \rangle \right] \\ \times \det_3^{-N/2}(1 + i\kappa K_\Delta(\psi_\Delta) + \kappa K_\Delta(h_\Delta)) \quad (3.45)$$

We choose $h_x = c_x/(\beta\sqrt{N})$. Then

$$\langle h, G_\Delta^{\circ 2} h \rangle \leq \frac{(\sum c_x)^2}{N} \leq \frac{|\Delta|^2}{N} \quad (3.46)$$

and we see

$$\left| \prod \frac{1}{(4 + m^2 + i\kappa(\psi_x - ih_x))^{n_x}} \right| \leq \left(\frac{1}{4 + m^2 + \frac{c_x}{\beta N}} \right)^{\sum n_x} \quad (3.47)$$

Then if $\sum n(x)$ is so large that $\sum n(x)h(x)/\sqrt{N} > |\Delta|^2/N$, namely if $\sum n_x > \beta|\Delta|^2$, we easily see that the following a priori bound holds

$$D_\Delta(n) \leq \left(\frac{1}{4 + m_{eff}^2} \right)^{\sum_{x \in \Delta} n(x)} \quad (3.48)$$

$$m_{eff}^2 = m^2 + \alpha^2, \quad \alpha^2 \equiv \frac{c}{N\beta} \quad (3.49)$$

Therefore in the following discussion, we assume that $\sum_{x \in \Delta} n_x \leq \beta|\Delta|^2$ and $\{s_x = n_x + \sqrt{n_x} \tilde{n}_x, x \in \Delta\}$ satisfy

- (1) $s_x \geq N\beta, \exists x \in \Delta$, or
- (2) $|s_x - s_{x'}| > \sqrt{N\beta}, \exists x \in \Delta, \exists x' \in \Delta, |x - x'| = 1$

If (1) occurs, then the factor

$$d_n(s) = \frac{s^n}{(n-1)!} e^{-s} ds$$

restricted on this region yields a small coefficient less than

$$\exp[-\frac{1}{2}N\beta] \leq \exp[-\frac{\sum n_x}{N\beta}]$$

If (1) does not take place and (2) happens, then we can implement the complex deformation $\psi_x \rightarrow \psi_x + i\tau\zeta_x$, where $\zeta = (N)^{-1/2}[G_\Delta^{\circ 2}]^{-1}(s/T)$ and $0 < \tau \leq 1$, and we see that the following factor arises from the complex deformation:

$$\begin{aligned} \exp[-\frac{1-(1-\tau)^2}{N} < \frac{s}{T}, [G_\Delta^{\circ 2}]^{-1} \frac{s}{T} >] &\leq \exp[-\frac{1-(1-\tau)^2}{2N\beta} < \frac{s}{T}, (-\Delta) \frac{s}{T} >] \\ &= \exp[-\frac{1-(1-\tau)^2}{2NT\beta} \sum_{nn} (s_x - s_{x'})^2] \end{aligned} \quad (3.50)$$

On the other hand, since

$$\|\kappa K_\Delta(\tau\zeta)\|_2^2 = \frac{4\tau^2}{N^2} < \frac{s}{T}, [G_\Delta^{\circ 2}]^{-1} \frac{s}{T} > \quad (3.51)$$

we have the bound

$$\left| \det_3^{-N/2} (1 + \kappa K_\Delta(\tau\zeta)) \right| \leq \exp \left[O \left(\frac{1}{\sqrt{N}} \right) \|K_\Delta(\tau\zeta)\|_2^2 \right] \quad (3.52)$$

which is close to 1 and has no effects on the bound (3.50) if N is large.

4 Conclusions and Discussions

We have shown that if the non-local factor

$$\prod_i \det^{-N/2} (1 + W(\Delta_i, \Lambda_i))$$

are discarded, then the resultant system exhibits exponential clustering for all β if N is large enough:

$$< s_0 s_x > \sim \int \frac{1}{-\Delta + m^2 + i\kappa\psi} (0, x) \prod d\mu_\Delta(\psi_\Delta), \quad (4.1)$$

$$\leq \exp[-m_{eff}|x|] \quad (4.2)$$

where $m_{eff}^2 = m^2 + c(N\beta)^{-1}$ and

$$d\mu_\Delta(\psi_\Delta) = \det_3^{-N/2} (1 + i\kappa K_\Delta) d\nu_\Delta \quad (4.3)$$

is the complex measure localized to each block Δ of size $L \times L$ in Z^2 . The assumption $N \gg 1$ is to simplify the large field problem and could be removed by additional efforts. The smallness of $W(\Delta, \Lambda)$ is due to the Anderson localization type arguments which remains to be justified [8, 9].

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