

On a broad class of univalent functions

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Abstract

The purpose of the present paper is to give some univalence conditions for a broad class of analytic functions. Moreover, we consider some special cases as corollaries of the main results.

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1. Introduction

It is well known that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $D = \{z \mid |z| < 1\}$ and we suppose that $f(z)$ satisfies one of the following conditions

$$\operatorname{Re} f'(z) > 0 \quad \text{in } D \quad (1.1)$$

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in } D, \quad (1.2)$$

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D, \quad (1.3)$$

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0 \quad \text{in } D, \quad (1.4)$$

$$\operatorname{Re} \frac{zf'(z)}{\phi(f(z))} > 0 \quad \text{in } D, \quad (1.6)$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is analytic and satisfies the condition

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > 0 \quad \text{in } D$$

or $g(z)$ is starlike in D , $0 < \beta$ and ϕ is analytic on $f(D)$ with $\phi(0) = 0$ and $\operatorname{Re} \phi'(0) > 0$, then $f(z)$ is univalent in D and we call $f(z)$ when $f(z)$ satisfies the condition (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6) as a Noshiro-Warschawski function, a convex function, a starlike function, a close-to convex function, a Bazilevič function of type β and ϕ -like function, respectively.

It is the purpose of the present paper to introduce a broad class of analytic functions and to investigate some sufficient conditions for univalence of the class.

2. Main Results

Theorem 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in D and suppose that

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} > 0 \quad \text{in } D,$$

where $\varphi(f(z), z)$ is analytic in $(f(D), D)$ and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} > 0 \quad \text{in } (f(D), D)$$

for $z = re^{i\theta}$, $0 < r < 1$ and $0 \leq \theta < 2\pi$. Then $f(z)$ is univalent in D .

Proof. If there exists a point z_0 , $|z_0| < 1$ such that

$$f(z) \text{ is univalent for } |z| < |z_0|$$

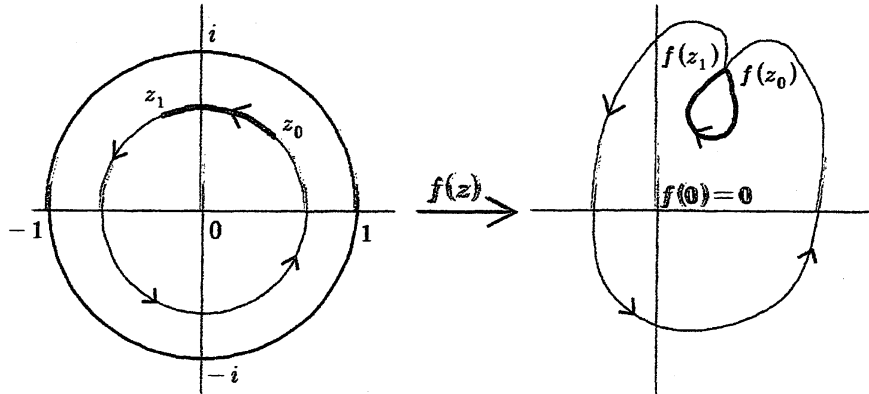
and

$$f(z) \text{ is not univalent for } |z| \leq |z_0|,$$

then there exists a point z_1 , $z_0 \neq z_1$, $|z_0| = |z_1|$, $z_0 = |z_0|e^{i\theta_0}$, $z_1 = |z_0|e^{i\theta_1}$ and $0 \leq \theta_0 < \theta_1 < 2\pi$ for which

$$f(z_0) = f(z_1),$$

as we see in the following figures.



Let

$$C_z = \{z \mid |z| = |z_0|, z = |z_0|e^{i\theta} \text{ and } \theta_0 \leq \theta \leq \theta_1\}.$$

Then from the hypothesis, we have

$$\begin{aligned} -\pi &= \int_{C_z} d \arg df(z) = \int_{C_z} d \arg \frac{df(z)}{dz} dz \\ &= \int_{C_z} d \arg \left(\frac{zf'(z)}{\varphi(f(z), z)} \right) + \int_{C_z} d \arg \left(\frac{dz}{z} \right) + \int_{C_z} d \arg \varphi(f(z), z) \\ &> -\pi + (\arg \varphi(f(z_1), z_1) - \arg \varphi(f(z_0), z_0)) \\ &= -\pi + (\arg \varphi(f(z_0), z_1) - \arg \varphi(f(z_0), z_0)) \\ &= -\pi + \int_{\theta_0}^{\theta_1} \frac{d \arg \varphi(f(z_0), |z_0|e^{i\theta})}{d\theta} d\theta \\ &> -\pi. \end{aligned}$$

This is contradiction and so, we completes the proof. □

Corollary 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in D and suppose that

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} > 0 \quad \text{in } D,$$

where

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} > 0 \quad \text{in } (f(D), D)$$

and $\varphi(f(z), z)$ satisfies one of the following conditions

$$\varphi(f(z), z) = z = re^{i\theta} \quad [5, 8], \quad (2.1)$$

$$\varphi(f(z), z) = zf'(z) = re^{i\theta} f'(e^{i\theta}) \quad [7], \quad (2.2)$$

$$\varphi(f(z), z) = f(z) = f(re^{i\theta}) \quad [4], \quad (2.3)$$

$$\varphi(f(z), z) = g(z) = g(re^{i\theta}) \quad [2, 3, 6], \quad (2.4)$$

$$\varphi(f(z), z) = \varphi(w, re^{i\theta}) = w^{1-\beta} g(z)^\beta = w^{1-\beta} g(re^{i\theta}) \quad [1], \quad (2.5)$$

where $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$, $0 < \beta$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is analytic and starlike in D or

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > 0 \quad \text{in } D.$$

Then $f(z)$ is univalent in D .

Proof. For the case (2.1), from hypothesis we have

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \operatorname{Re} \frac{zf'(z)}{z} = \operatorname{Re} f'(z) > 0 \quad \text{in } D$$

and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} = \frac{d\theta}{d\theta} = 1 > 0 \quad \text{in } D.$$

Applying Theorem 1, $f(z)$ is univalent in D .

For the case (2.2), we have

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \operatorname{Re} \frac{zf'(z)}{zf'(z)} = 1 > 0 \quad \text{in } D$$

and

$$\begin{aligned} \frac{d \arg \varphi(w, re^{i\theta})}{d\theta} &= \frac{d \arg zf'(z)}{d\theta} = \frac{d \arg(\frac{z}{dz})}{d\theta} + \frac{d \arg df(z)}{d\theta} \\ &= \frac{d \arg df(z)}{d\theta} = 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in } D. \end{aligned}$$

This shows that $f(z)$ is convex and univalent in D .

For the case (2.3), we have

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D$$

and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} = \frac{d \arg f(z)}{d\theta} = \operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D.$$

This shows that $f(z)$ is starlike in D .

For the case (2.4), we have

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \operatorname{Re} \frac{zf'(z)}{g(z)} > 0 \quad \text{in } D$$

and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} = \frac{d \arg g(z)}{d\theta} = \operatorname{Re} \frac{zg'(z)}{g(z)} > 0 \quad \text{in } D.$$

This shows that $f(z)$ is univalent in D and close-to-convex in D .

For the case (2.5), we have

$$\operatorname{Re} \frac{zf'(z)}{\varphi(f(z), z)} = \operatorname{Re} \frac{zf'(z)}{f(z)^{1-\beta}g(z)^\beta} > 0 \quad \text{in } D$$

and

$$\frac{d \arg \varphi(w, re^{i\theta})}{d\theta} = \frac{d \arg w^{1-\beta}g(re^{i\theta})^\beta}{d\theta} = \beta \frac{d \arg g(re^{i\theta})}{d\theta} = \beta \operatorname{Re} \frac{zg'(z)}{g(z)} > 0 \quad \text{in } D$$

where $0 < \beta$. This shows that $f(z)$ is univalent in D and $f(z)$ is Bazilevič function of type $0 < \beta$. $\square$

If $f(z)$ is a Bazilevič function of type β , then β must be a positive real number. But we can obtain the following theorem.

Theorem 2. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in D and suppose that

$$\left| \arg \frac{zf'(z)}{f(z)^{1-\beta}g(z)^\beta} \right| < \frac{\pi}{2} \alpha \quad \text{in } D,$$

where $0 < \alpha < 1$, $\beta < 0$, $\alpha - 2\beta < 1$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is analytic and starlike in D . Then $f(z)$ is univalent in D .

Proof. If there exists a point z_0 , $|z_0| < 1$ such that

$$f(z) \text{ is univalent for } |z| < |z_0|$$

and

$$f(z) \text{ is not univalent for } |z| \leq |z_0|,$$

then there exists a point z_1 , $z_0 \neq z_1$, $|z_1| = |z_0|$, $z_0 = |z_0|e^{i\theta_0}$, $z_1 = |z_0|e^{i\theta_1}$ and $0 \leq \theta_0 < \theta_1 < 2\pi$ for which

$$f(z_0) = f(z_1).$$

Then the image picture under the mapping $w = f(z)$ for $|z| = |z_0|$ is the same as the picture of the proof of Theorem 1.

Let

$$C_z = \{z \mid |z| = |z_0|, z = |z_0|e^{i\theta} \text{ and } \theta_0 \leq \theta \leq \theta_1\},$$

$$C'_z = \{z \mid |z| = |z_0|\} - C_z,$$

and

$$\Gamma'_w = f(C'_z).$$

Then we have

$$\begin{aligned} 3\pi &= \int_{\Gamma'_w} d \arg dw = \int_{\Gamma'_w} d \arg df(z) \\ &= \int_{C'_z} d \arg \left(\frac{zf'(z)}{f(z)^{1-\beta}g(z)^\beta} \right) + \int_{C'_z} d \arg \left(\frac{dz}{z} \right) + \int_{C'_z} d \arg f(z)^{1-\beta} + \int_{C'_z} d \arg g(z)^\beta \\ &= \int_{C'_z} d \arg \left(\frac{zf'(z)}{f(z)^{1-\beta}g(z)^\beta} \right) + (1-\beta) \int_{C'_z} d \arg f(z) + \beta \int_{C'_z} d \arg g(z) \\ &< \alpha\pi + (1-\beta)2\pi = \pi(2+\alpha-2\beta) < 3\pi. \end{aligned}$$

This is a contradiction and so, we completes the proof. $\square$

References

- [1] I. E. Bazilevič, *On a case of integrability in quadratures of the Loewner-Kutarev equation*, Mat. Sb. 37 (1955), 471–476 (Russian).
- [2] A. W. Goodman, *Univalent Function II*, Mariner Publishing Com. Inc. Tampa Florida. 1983.
- [3] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. 1 (1952), 169–185.
- [4] R. Nevanlinna, *Über die konforme Abbildung Sterngebieten*, Oeversikt av Finska-Vetenskaps Societen Forhandlingar 63(A), No. 6 (1921), FM 48–403.
- [5] Noshiro, *On the theory of schlicht functions*, J. Fac. Sci. Hokkaido Univ. (1) 2 (1943–1935), 129–155.

- [6] S. Ozaki, *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku. Sect. A, 2 (1935), 167–188.
- [7] E. Strudy, *Konforme Abbildung Einfachzusammenhangender Bereiche*, B. C. Tueber, Leipzig and Berlin, 1913, FM 44–755.
- [8] S. Warshawski, *On the higher derivatives at the boundary in conformal mappings*, Trans. Amer. Math. Soc. 38 (1935), 310–340.