STUDIES ON OPERATIONAL CALCULUS AND ITS NUMERICAL TREATMENTS FOR MATRICES OF FUNCTIONS AND THEIR APPLICATIONS TO ANALYSIS OF SURGES ON TRANSMISSION SYSTEMS

Satoshi ICHIKAWA

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1. Introduction

1.1 Purpose of this thesis

In physical or engineering applications, the operational calculus is used as a convenient method to solve a problem such that the value of the variable is equal to 0 in the domain $t<0$ and described by a differential equation with respect to t in $t\geq0$.

The operational calculus has its origin in the method of symbolic analysis by Heaviside. This method had too many intuitive elements and was doubted its strictness in the mathematical sence at that time. But after that time, it has been modified or extended by many reserchers and theoretical correctness of the method was assured by Carson's integral equation.

On the other hand, there was a method called the Laplace transform originated by Laplace on the basis of the function theory and later extended by Poincaré. At that time, this method had the main purpose to get the fundamental solutions of a linear differential equation, but after that time, correspondence between this method and the operational calculus was made by the integral transform of Bromwich.

Strictly speaking, the operational calculus of Heaviside and the Laplace transform are different fundamentally and the former agrees with the special form of the later. But in practical applications, they are different only in the expression and we can get the same results by either of them and the Laplace transform is more generally used today than the operational calculus of Heaviside.

In the Laplace transform, the problem with respect to the initial values was made clear, and for the intuitive element such as the unit impuls mathematical strictness was assured by the theory of distributions of Schwartz and the Laplace transform is established mathematically perfect.

Today transformation pairs of fairly many functions are given in a transformation table and the Laplace transform is used as the most powerful tool in physical or engineering applications.

However, in the Laplace transform, the class of the functions to be treated must have the property of transformable in infinite integral and if a given operational solution does not exist in a transformation table, we doubt whether the solution in the time domain corresponding to it exists or not, whether the solution is unique or not and if the unique solution exists it is difficult or sometimes impossible to get its analytical form. Thus we need an advanced mathematical theory as the degree of complexity of the problem increases, and cannot deny that the Laplace transform becomes far off from the spirit of the operational calculus intended implicitly by Heavisied that an operator is treated same as a number.

On the other hand, Miksinsky constructed the operational calculus algebraically without depending integral transformation. His idea lies on that the convolution quotient is defined as the operator and in his operational calculus a set of operators contains the sets of functions and numbers and for any operator, the positive real number which is the norm of continuous and bounded function in $t \geq 0$ is determined uniquely and operators can be treated without distinction between functions and numbers.

This operational calculus contains all the results of the Laplace transform including those given by the theory of distributions and the existence and uniqueness of the solution for a certain operational solution can be proved by the theory of the functional analysis, therefore, it is a more general method and gives an advanced approach to the spirit of the operational calculus intended by Heaviside.

In this way the correspondence between the operational solutions and positive real numbers is given, then we shall consider the relation between the operator s in the operational domain and time t in the time domain. The Laplace transform is the special case of the Fourier transform and there, the operator s is regarded as a complex number and the left half plane which is decided for an arbitrary positive real number a greater than the real part of the pole of the operational function by the inequality

$$\mathrm{Re}(s)<a$$

corresponds to the region $t\geq 0$ in the time domain.

By the descrete Fourier transform, the Laplace transform is approximated by the Fourier series and the value of the function in the time domain is related to a series of values of the operational function at a certain series of s in the operational domain. The above series is determined not uniquely, but once the parameters are determined, it is determined uniquely, and by the arithmetic operations in the s domain we can get a value of the function at any value of $t\geq 0$ in the time domain, or reversely by the arithmetic operations in the t domain a value of the operational function in the operational domain.

In this way, a numerical correspondence between the time t and the operator s is given and the method gives a remarkable advantage in numerical analysis. For example, if we do not find the time function in a transformation table for a given operational solution, first we assure the existence of the solution by the theory of the operational calculus and then get the numerical solution in the t domain by the above method and regard it as the solution, provided that it is not contradictory to the practical phenomenon. This method is one of the methods of numerical inversion of the Laplace transform and is the method which makes the most of the spirit of the operational calculus among them.

In the mean time, we shall often encounter the case to analyse a large system whose property is described by a set of differential equations. In this case, we can get the time function algebraically by solving the simultaneous linear equation given by applying the operational calculus, but recently to carry out the analysis systematically such a method is used that the analysis is performed by the theory of the matrix functions, for example, the state variable analysis developed from the control theory.

Then, if we extend the operational calculus to the matrix form and get the transformation pairs for various matrix functions, we can treat the simultaneous differential equation in a matrix form formally by the same method as the corresponding scalar equation. However, the results given by the above method is merely the formal solutions and we can get the practical solutions when every element of the matrix functions is determined analytically or numerically. Therefore, at the same time we must give a method to determine every element of the time solution for a given matric operational solution analytically or numerically.

As the scale of the system increases, the order of the equation increases and it becomes extremely difficult to get the analytical solution, but by the progress of the numerical treatment by the large scale digital computers, the numerical method is not so difficult and is used more widely today than the analytical method.

In this case, the procedure to get the matric operational solution for a given matric differential equation is fairly easy and if we apply the numerical inversion method of the Laplace transform, we can get the numerical solution in the time domain directly by extending the scalar operation to the corresponding vector or matrix operation without getting the formal solution, and to do so, the advantage in numerical analysis of the many variable systems increases remarkably.

Analysis of the wave propagation properties is one of the typical examples where the above method can be used effectively for the following reasons. There the operational calculus has been used as the tool of analysis for a long time and an operational solution for the phenomena is given easily. But if we consider the factors which give remarkable influences to the phenomena such as the skin effect and the absorption effect, the operational solution becomes very complicated and it is impossible to get the analytical solution and the sufficient numerical solutions in practical applications were not given because of imperfection of the numerical method. For such a case we can get the sufficient numerical solution by applying the numerical inversion method of the Laplace transform.

As was given in the outline above, this thesis has the purpose to establish the theoretical and numerical treatments of large scale systems by the operational calculus and is made out under the following policies.

(1) To extend ~~the~~ Miksinsky's operational calculus to a matrix form and to make clear the relation between the operational calculus and matrix functions.

(2) To make clear the relation between the operators and numbers and give a practical method of numerical inversion.

(3) To give a numerical method in analysing the wave propagation properties on transmission systems.

1.2 Construction of this thesis

This thesis consists of 9 chapters and some appendices as shown below. Chapter 1 is devoted to the introduction where the historical background and the purpose of this study are shown.

Chapters 2,3 and 4 are devoted to an extension of Miksinsky's operational calculus to a matrix form. In Chapter 2, we shall show the algebraic properties such as definition of the operator, its extension to a matrix form, definition of the matrix function and its application to the linear simultaneous differential equation with constant coefficients. In Chapter 3, we shall show some properties concerning the convergence, such as a series and a sequence of the matric operators and their application to the analyis of the periodic functions in the operational calculus. The properties of the operators in the topological space are shown in Appendix A and formulation of the state equation by the topological method in Appendix B. In Chapter 4, we shall show the analytical properties such as the derivatives, integrals and the differential equations of the matric operational functions and their application to the telegraphic equations. The properties of derivatives and integrals of the operational functions in the topological spaces are shown in Appendix C.

Chapter 5 is devoted to extend the operational calculus to a multi-dimensional space, for example to analyse the telegraphic equations. The properties of the differential equations of the operational functions derived from the partial differential equations by the operational calculus are shown in Appendix D.

Chapter 6 is devoted to show the relation among the operators, the numbers and the numerical treatment of the Laplace transform by the Fourier series technique, and give a practical numerical inversion method of the Laplace transform. An algorithm of the Fast Fourier Transform used there is shown in Appendix E.

Chapter 7 and 8 are devoted to show the applications of the results obtained to the analysis of the wave propagation properties in the transmission systems. In Chapter 7, we shall give the differential equations of the operational forms describing the wave propagation properties considering the skin effect and the absorption effect. The detailed reductions of the equations are shown in Appendix F. In Chapter 8, we shall study the attenuation and the distortion of the traveling waves on the transmission systems by applying the numerical inversion method given in Chapter 6 to the results obtained in Chapter 7.

Chapter 9 is devoted to the conclusion and there the results obtained in this thesis are summarized.

1.3 General literatures for this thesis

In writing out this thesis, the auther made an effort to treat every subject as strictly as possible in the mathematical point of view and referred to the following literatures frequently.

They are shown as follows by some items and by every chapter of this thesis where they reffered.

Operational calculus (Chapters 2,3,4,5,6 and 7)

(I-1) J.Miksinsky; "Operational calculus"

Translated by S.Matsuura,H.Matsumura and K.Kasahara Shokabo (1961)

Theories of matrix and linear space (Chapters 2,3,4,5,6,7 and 8)

(I-2) M.Fujiwara; "Gyoretsu oyobi Gyoretsushiki" Iwanamishoten (1961)

(I-3) Д.К.ФАДДЕЕВ и В.Н.ФАДДЕЕВА; "ВЫЧИСЛИТЕЛЬНЫЕ МЕТОДЫ ЛИНЕЙНОП АЛГЕБРЫ"

Translated by S.Furuya and T.Oguni Sangyotosho (1970)

Lebesgue integral (Chapters 2 and 6)

(I-4) S.Mizohata; "Lebesgue sekibun" Iwanamishoten (1966)

Theory of distributions (Chapters 5 and 6)

(I-5) L.Schwartz; "Méthodes mathématiques pour les sciences physiques"

Translated by K.Yoshida and J.Watanabe Iwanamishoten (1966)

Functional analysis (Chapters 2,3,4,5,6 and 7)

(I-6) П.А.Люстерника and А.Р.Яниопьского;"Functional analysis"

Translated by T.Sato and R.Iino Sogotosho (1968)

(I-7) T.Kato; "Isokaiseki" Kyoritsushuppan (1967)

Theories of Fourier analysis and partial differential equations (Chapters 5 and 6)

(I-8) S.Mizohata; "Henbibun Hoteishiki Ron" Iwanamishoten (1965)

Numerical treatment of Fourier analysis (Chapter 6)

(I-9) T.Yoshizawa; "Suchi kaiseki(Kisokogaku Koza)" Iwanamishoten (1968)

Theory of skin effect of ground (Chaptes 7 and 8)

(I-10) E.D.Sunde; "Earth conduction effects in transmission systems" Van Nostrand (1949)

Theory of traveling waves (Chapters 7 and 8)

(I-11) S.Hayashi; "Surges on transmission systems" Denkishoin (1955)

2. Algebraic properties of operators

2.1 Preface

The starting point of Miksinsky's operational calculus is the concept of the convolution integral of continuous functions in the interval $0 \leq t < \infty$. His idea lies in the fact that a set of continuous functions forms a quotient field with respect to two operations, addition and convolution and that an element of this field is termed an operator. In this way he introduced the operational calculus and by using it, ~~any~~ linear ordinary differential equation with constant coefficients can be solved only by the algebraic operations.

A set of linear ordinary differential equations with constant coefficients is represented briefly by a matrix form and if the operational calculus is extended to a matrix form, it can be treated systematically by the theory of matrix functions in the same way as a scalar form.

Hence it is the purpose of this chapter to show the fundamental properties about the algebraic operations of matrix functions in the operational calculus.

First we shall show the definition of operators in terms of the abstract algebra formulated by Miksinsky and the definition of matrix functions. And then we shall extend this operational calculus to the matrix form and show some basic properties.

2.2 Definition of operators in terms of abstract algebra

2.2.1 Continuous functions and operators[1)] The terminologies, operators and continuous function in terms of the abstract algebra which are defined by Miksinsky are shown as follows.

In a set $A=\{a,b,c,...\}$ two operations, addition $a+b\in A$, and multiplication $ab\in A$ are defined for arbitrary elements. Then A is called a commutative ring if it has the following properties.

(i) $a+b=b+a$

(ii) $(a+c)+c=a+(b+c)$

(iii) For any pair of elements a,b, there exists a third element $x\in A$, satisfying the equation $a+x=b$.

(iv) $ab=ba$

(v) $(ab)c=a(bc)$

(vi) $a(b+c)=ab+ac$

A ring is said to have no divisors of zero if

(vii) $ab=0$ implies $a=0$ or $b=0$.

Any ring without divisors of zero can be extended to a quotient field which is a set of fractions with the following properties.

(1) $\frac{b}{a}=\frac{d}{c}$ if and only if $bc=ad$

(2) $a=\frac{ak}{k}$ for any $k\neq 0$

In the quotient field, addition and multiplication are defined as follows.

$$\frac{b}{a}+\frac{d}{c}=\frac{bc+ad}{ac}\ ,\ \frac{b}{a}\frac{d}{c}=\frac{bd}{ac}$$

The elements of the quotient field with the introduced addition and multiplication satisfy also axioms (i) - (vii). Moreover the following condition is satisfyed.

(viii) For any pair of elements a,b, provided $a\neq 0$, there exists a third element x, satisfying the equation $ax=b$.

Let us consider a set $C=\{a,b,c,...\}$ of complex functions of a real variable t, defined to be continuous in the interval $0\leq t<\infty$. This set is a commutative ring with respect to addition and multiplication defined as follows

$$a+b=\{a(t)\}+\{b(t)\}=\{a(t)+b(t)\} \qquad (2.2.1)$$

$$ab=\{a(t)\}\{b(t)\}=\{\int_0^t a(t-\tau)b(\tau)d\tau\} \qquad (2.2.2)$$

where

$\{a(t)\}$ is a function $a(t)$

$a(t)$ is a value of function $a(t)$ at $t=t$

a is an operator for function $a(t)$.

For these operations, the properties $a+b\in C$ and $ab\in C$ hold and all the axioms for the commutative ring (i) - (vi) are satisfied.

The operation (2.2.2) is a convolution integral and by Tichmarsh's theorem*

<< *If two functions $a\in C$ and $b\in C$ are not identically equal to zero, then their convolution is not identically equal to zero.*>>

the considered ring C has no divisors of zero. Therefore, it can be extended to a quotient field. The elements of this quatient field are called operators.

2.2.2 Locally integrable functions and operators In applying the operational calculus, it is not sufficient to restrict a set of functions whose elements belong to the class C. From the theoretical point of view, it is convenient to introduce the class L of locally integrable functions in the interval $0\le t<\infty$, which means a set of complex functions integrable in the sence of Lebesgue in any bounded interval $0<t<T$. For the convolution of these functions, the following theorems are given**.

<< *If $f\in L$ and $g\in L$, then $fg\in L$.*>>

<< *If the convolution fg of two functions $f\in L$ and $g\in L$ vanishes*

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* See I-1 Part 1 Chapter II

** See I-1 Part 6 Chapter II

almost everywhere in the interval $0 \le t < \infty$, then at least one of these functions vanishes almost everywhere in this interval.>>

From the standpoint of the functional analysis, two functions f_1 and f_2 of class L are said to be equivalent if they are equal almost everywhere in the interval to be considered because of the following relation

$$\{\int_0^t f_1(\tau)\,d\tau\}=\{\int_0^t f_2(\tau)\,d\tau\}.$$

Therefore, for a function $f \in L$, let us consider the convolution integral

$$lf=\{\int_0^t f(\tau)\,d\tau\} \qquad (2.2.3)$$

where $l=\{1\}$, then $lf \in C$ and because $l \neq 0$ it can be regarded that the function f and the operator lf/l are equivalent in the functional analysis. By using this relation, two operations, addition and multiplication for any pair of functions in a class L are defined as follows.

$$f+g=\frac{a+b}{l} \qquad (2.2.4)$$

$$fg=\frac{ab}{l^2} \qquad (2.2.5)$$

where $a=lf$ and $b=lg$. Because $a \in C, b \in C$ and $l \in C$, two rings L and C can be treated equivalently in the operational calculus and all the results obtained for the quotient field can be applied to the functions of class L.

2.3 Definition of matrix functions

In applying the operational calculus to a system with many variables, a matrix of the following form is introduced

$$\{F(z)\}=\begin{bmatrix} \{f_{11}(z)\} & \{f_{12}(z)\} & \cdots\cdots & \{f_{1n}(z)\} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \{f_{m1}(z)\} & \{f_{m2}(z)\} & \cdots\cdots & \{f_{mn}(z)\} \end{bmatrix} \qquad (2.3.1)$$

where every element $\{f_{ij}(z)\}$ is a complex function of a complex variable z.

Specially a matrix of function $\{f(A)\}$ arising from a scalar function $\{f(z)\}$ is treated frequently where A is a square numerical matrix and we call it matrix function $f(A)$. Such a matrix function is defined as follows [2].

(i) $\{f(z)\}=\{k\}$ then $\{f(A)\}=\{kI\}$

(ii) $\{f(z)\}=\{z\}$ then $\{f(A)\}=\{A\}$

(iii) $\{f(z)\}=\{g(z)+h(z)\}$ then $\{f(A)\}=\{g(A)+h(A)\}$

(iv) $\{f(z)\}=\{g(z)h(z)\}$ then $\{f(A)\}=\{g(A)h(A)\}$

(v) $\{f(z)\}=\{h[g(z)]\}$ then $\{f(A)\}=\{h[g(A)]\}$

For a polynomial with complex coefficients

$$\{f(z)\}=\{\sum_{i=0}^{n} a_i z^i\}$$

we mean that A^i is determined by ordinary matric multiplication, A^0 is replaced by unit matrix, $a_i A^i$ interpleted ordinary as scalar multiplication of A^i by a_i and the addition as matric addition, then we have a matrix polynomial

$$\{f(A)\}=\{\sum_{i=0}^{n} a_i A^i\}$$

satisfying the above properties (i) - (v).

Next we shall consider a matrix function $\{f(A)\}$ arising from an analytical function $\{f(z)\}$. Assume that a matrix A is transformed into the Jordan canonical form C by the following transformation.

$$\left.\begin{aligned} &C=P^{-1}AP \\ &C=C_1\dot{+}C_2\dot{+}\dots\dot{+}C_n \\ &C_i=\begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & 0 \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \lambda_i \end{bmatrix} \end{aligned}\right\} \qquad (2.3.2)$$

By Taylor's expansion, $\{f(z)\}$ can be given as the following infinite series in the neighborhood of $z=\lambda_i$

$$\{f(z)\}=\{\sum_{n=0}^{\infty}\frac{f^{(i)}(\lambda i)}{i!}(z-\lambda_i)^i\}$$

then we have

$$\{f(C_i)\}=\{\sum_{n=0}^{\infty}\frac{f^{(i)}(\lambda i)}{i!}(C_i-\lambda_i 1)^i\}.$$

Since $(C_i-\lambda_i 1)^i$ vanishes if $i\geq\nu_i$ where ν_i is the order of C_i, we have

$$\{f(C_i)\}=\begin{bmatrix} \{f(\lambda_i)\} & \{f'(\lambda_i)\} & \{\frac{f''(\lambda i)}{2!}\} & \cdots & \{\frac{f^{(\nu i-1)}(\lambda i)}{(\nu i-1)!}\} \\ 0 & \{f(\lambda i)\} & \{f'(\lambda i)\} & \cdots & \{\frac{f^{(\nu i-2)}(\lambda i)}{(\nu i-2)!}\} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \{f(\lambda_i)\} \end{bmatrix} \qquad (2.3.3)$$

and

$$\{f(C)\}=\{f(C_1)\dot{+}f(C_2)\dot{+}\cdots\cdots\dot{+}f(C_n)\} \qquad (2.3.4)$$

then we have

$$\{f(A)\}=\{f(PCP^{-1})\}=\{Pf(C)P^{-1}\}. \qquad (2.3.5)$$

This matrix function $\{f(A)\}$ satisfies the properties (i) - (v) if $\{f(z)\}$ is single valued.

For a many valued function, above formula can be expanded by introducing the Riemann plane.

2.4 Operational calculus for matrices of functions

2.4.1 Definition of matric operational calculus When matrices of functions $\{F(t)\}=[\{F_{ij}(t)\}]$ and $\{G(t)\}=[\{G_{ij}(t)\}]$ are given, where $F_{ij}\in C$ and $G_{ij}\in C$, as in the case of ordinary matrix algebra, we shall define sum $\{H(t)\}=[\{H_{ij}(t)\}]$ satisfying the following relation

$$\{H_{ij}(t)\}=\{F_{ij}(t)\}+\{G_{ij}(t)\}.$$

and denote it by

$$\{H(t)\}=\{F(t)\}+\{G(t)\}. \qquad (2.4.1)$$

Furthermore when $\{F(t)\}$ and $\{G(t)\}$ of the above class are comformable and not nilfactors, we shall define product $\{H(t)\}$ satisfying the following relation

$$\{H_{ij}(t)\}=\{\sum_k\int_0^t F_{ik}(t-\tau)G_{kj}(t)d\tau\}$$

and denote it

$$\{H(t)\}=\{F(t)\}\{G(t)\}. \qquad (2.4.2)$$

In a set of square matrices of functions, for any pair of its elements the sum and the product by the above definition belong to the same set and these operations satisfy the axioms (i) - (vi) in 2.2.1 except (iv). In other word, this set is a ring but non-commutative, therefore, a matric operator cannot be defined in the same way as in the scalar functions.

In general we shall consider a matrix of operators in the following form

$$Q=\begin{pmatrix} Q_{11} & Q_{12} & \cdots\cdots & Q_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ Q_{m1} & Q_{m2} & \cdots\cdots & Q_{mn} \end{pmatrix} \qquad (2.4.3)$$

where $Q_{ij}\in Q$ and Q is a quatient field of operators.

For the algebraic operations of this matrix of operators, all of them in the quotient field hold for every element. Let us call such a matrix of operators as a matric operator and define some special examples.

Matric integral operator: For a matrix function $\{1\}$ and a matrix of functions $\{F(t)\}$, we have

$$\{1\}\{F(t)\}=\{\int_0^t F(\tau)d\tau\}. \qquad (2.4.4)$$

Denote $\ell=\{1\}$ and call it the matric integral operator, then from the definition follow the following formulas.

$$\ell^n=\{\frac{t^{n-1}}{(n-1)!}1\} \qquad n=1,2,\ldots.. \qquad (2.4.5)$$

$$\left.\begin{array}{l}\ell^n F=\{\underbrace{\int_0^t dt\ldots\int_0^t}_{n} F(t)\,dt\}=\{\int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}F(\tau)\,d\tau\} \\ n=1,2,\ldots.\end{array}\right\} \tag{2.4.6}$$

Matric numerical operator: For a matrix function $\{1\}$ and a matrix of functions $\{A\}$ where A is a numerical matrix, we shall consider the matrix of functions $\{X\}$ satisfying the following relation

$$\{1\}\{X\}=\{A\}$$

We shall denote $\{X\}=\{A\}/\ell$, write it $[A]$ and call it a numerical operator. For any pair of numerical operator, we have

$$[A]+[B]=[A+B] \tag{2.4.7}$$

$$[A]\,[B]=[AB] \tag{2.4.8}$$

because by the definition yield

$$[A]+[B]=\frac{\{A\}}{\ell}+\frac{\{B\}}{\ell}=\frac{\{A+B\}}{\ell}=[A+B]$$

$$[A]\,[B]=\frac{\{A\}\{B\}}{\ell^2}=\frac{\ell\{AB\}}{\ell^2}=\frac{\{AB\}}{\ell}=[AB].$$

Owing to these formulas, the blackets [ ] can be ommited in the operational calculus and matric numerical operators can be treated same as ordinary numerical matrices.

Furthermore, the following relations hold

$$A\{B\}=[A]\{B\}=\{AB\} \tag{2.4.9}$$

$$\{1\}A=\{A\} \tag{2.4.10}$$

because

$$A\{B\}=\frac{\{A\}\{B\}}{\ell}=\frac{\ell\{AB\}}{\ell}=\{AB\}$$

$$\{1\}A=\frac{\ell\{A\}}{\ell}=\{A\}.$$

Let $A=1$ then by (2.4.9) becomes the formula

$$1\{B\}=\{B\} \tag{2.4.11}$$

and let $A=0$ then by (2.4.9) become the formulas

$$0\{B\}=0 \tag{2.4.12}$$

$$\{0\}=0 \tag{2.4.13}$$

and

$$0+\{B\}=\{B\} \tag{2.4.14}$$

because

$$0+\{B\}=\frac{\{0\}+\ell\{B\}}{\ell}=\frac{\ell\{B+0\}}{\ell}=\{B\}.$$

Matric differential operator: In the operational calculus the operator defined by the following relation

$$\ell s=s\,\ell=1 \tag{2.4.15}$$

plays an important role, then we shall denote it by

$$s=\frac{1}{\ell} \tag{2.4.16}$$

and call it the matric differential operator. For this operator the following theorems hold.

<< *Denote* $C^{m\times n}$ *a* $m\times n$ *dimensional space of functions of class* C *and suppose a matrix of functions* $\{X(t)\}$ *of* $C^{m\times n}$ *has a derivative matrix* $\{X'(t)\}$ *of class* $C^{m\times n}$*, then follows the formula*

$$s\{X(t)\}=\{X'(t)\}+X(0) \tag{2.4.17}$$

where $X(0)$ *is the value of* $\{X(t)\}$ *at* $t=0$. >>

Proof: From the relation

$$\{X(t)\}=\{\int_0^t X'(\tau)d\tau\}+\{X(0)\}$$

we have

$$\{X(t)\}=\ell\{X'(t)\}+\ell X(0)$$

and multiplying both sides by s, we have

$$s\{X(t)\}=\{X'(t)\}+X(0). \qquad \text{Q.E.D}$$

<< *Suppose a matrix of functions* $\{X(t)\}$ *of* $C^{m\times n}$ *has n-th derivative* $\{X^{(n)}(t)\}$ *of* $C^{m\times n}$*, then follows the formula*

$$s^n\{X(t)\}=\{X^{(n)}(t)\}+X^{(n-1)}(0)+sX^{(n-2)}(0)+..+s^{n-1}X(0) \tag{2.4.18}$$

where $s^n = \underbrace{ss..s}_{n}$ *and* $X^{(i-1)}(0)$ $(i=1,2,\ldots.n)$ *is the value of* $\{X^{(i-1)}(t)\}$ *at* $t=0$. >>

Proof: Multiplying both sides of (2.4.17) by s, we have

$$s^2\{X(t)\}=s\{X'(t)\}+sX(0)$$

and by (2.4.17)

$$s\{X'(t)\}=\{X''(t)\}+X'(0)$$

then we have

$$s^2\{X(t)\}=\{X''(t)\}+X'(0)+sX(0).$$

Repeating this procedure, we have (2.4.18). Q.E.D

So far discussions were limited to matrices of continuous functions, but they can be extended to matrices of locally integrable functions in the following way.

Let $L^{m\times n}$ a $m\times n$ dimensional space of functions of class L and for matrices of functions $\{F(t)\}$ and $\{G(t)\}$ of $L^{m\times n}$ give the definition

(1) $\ell F=\ell G$ then $F=G$

(2) $F+G=\frac{1}{\ell}(\ell F+\ell G)$

(3) $FG=\frac{1}{\ell^2}(\ell F\ell G)$

then matrices of locally integrable functions can be treated in the same way as matrices of continuous function in the operational calculus.

2.4.2 Operational forms of matric exponential functions Let us consider a matrix function $\{e^{At}\}$, then by (2.4.17) the following formula yields

$$s\{e^{At}\}=A\{e^{At}\}+1$$

and it becomes

$$(s-A)\{e^{At}\}=1.$$

For this relation, we shall denote $\{e^{At}\}$ as follows.

$$\{e^{At}\}=\frac{1}{s-A} \qquad (2.4.19)$$

By the definition of the convolution, the following general forms are given.

$$\left.\begin{array}{l}\frac{1}{(s-A)^n}=\{e^{At}\}\underbrace{\{e^{At}\}\cdots\{e^{At}\}}_{n}=\{\frac{t^{n-1}}{(n-1)!}e^{At}\} \\ \qquad n=1,2,\ldots\ldots \end{array}\right\} \quad (2.4.20)$$

Let us consider matrix functions $\{\cos At\}$ and $\{\sin At\}$, then by (2.4.19) they become

$$\{\cos At\}=\frac{1}{2}\{e^{iAt}+e^{-iAt}\}=\frac{1}{2}\left(\frac{1}{s-iA}+\frac{1}{s+iA}\right)$$

$$\{\sin At\}=\frac{1}{2i}\{e^{iAt}-e^{-iAt}\}=\frac{1}{2i}\left(\frac{1}{s-iA}-\frac{1}{s+iA}\right).$$

The matrices s and A are commutative, therefore, above formulas become as follows.

$$\{\cos At\}=\frac{1}{s^2+A^2} \qquad (2.4.21)$$

$$\{\sin At\}=\frac{A}{s^2+A^2} \qquad (2.4.22)$$

In the same way we have the following formula.

$$\{\cosh At\}=\frac{1}{s^2-A^2} \qquad (2.4.23)$$

$$\{\sinh At\}=\frac{A}{s^2-A^2} \qquad (2.4.24)$$

When the matrix A is non-singular we have

$$\frac{1}{(s^2+A^2)^2}=A^{-2}\{\sin At\}\{\sin At\}=\{\frac{1}{2}A^{-3}(\sin At-At\cos At)\} \qquad (2.4.25)$$

$$\frac{1}{(s^2+A^2)^3}=A^{-3}\{\sin At\}\{\sin At\}\{\sin At\}$$

$$=\{\frac{1}{8}A^{-5}(3\sin At-3At\cos At-A^2t^2\sin At)\}. \qquad (2.4.26)$$

By (2.4.20) we have

$$\frac{n!1}{(s-iA)^{n+1}}=\{t^n e^{iAt}\}$$

and comparing real and imaginary parts of the both sides, we have

$$\left.\begin{aligned}\{t^n\cos At\}&=\frac{n!\,s^{n+1}}{(s^2+A^2)^{n+1}}\sum_{m}(-1)^m\,{}_{n+1}C_{2m+1}\left(\frac{A}{s}\right)^{2m+1}\\ &\qquad 0\le 2m\le n\end{aligned}\right\}\quad(2.4.27)$$

$$\left.\begin{aligned}\{t^n\sin At\}&=\frac{n!\,s^{n+1}}{(s^2+A^2)^{n+1}}\sum_{m}(-1)^m\,{}_{n+1}C_{2m}\left(\frac{A}{s}\right)^{2m}\\ &\qquad 0\le 2m< n+1\end{aligned}\right\}.\quad(2.4.28)$$

For matrix functions with more than one coefficient matrix, the scalar forms can be extended to the matrix forms in the operational domain if all the coefficient matrices are commutative.

For example if numerical matrices A and B are commutative, the following equality holds

$$e^{A}e^{B}=e^{B}e^{A}=e^{(A+B)}$$

and the following relations hold.

$$\{e^{At}\cos Bt\}=\frac{s-A}{(s-A)^2+B^2}\qquad(2.4.29)$$

$$\{e^{At}\sin Bt\}=\frac{B}{(s-A)^2+B^2}\qquad(2.4.30)$$

$$\{e^{At}\cosh Bt\}=\frac{s-A}{(s-A)^2-B^2}\qquad(2.4.31)$$

$$\{e^{At}\sinh Bt\}=\frac{B}{(s-A)^2-B^2}\qquad(2.4.32)$$

2.5 Applications to linear simultaneous ordinary differential equations with constant coefficients

2.5.1 First order equations The properties of linear time-invariant systems are described by the following set of differential equations

$$\{x'(t)+Ax(t)\}=\{Bu(t)\}\qquad(2.5.1)$$

in the interval $0\le t<\infty$ where x is the state vector, u is the input vector, A and B are numerical matrices.

According to the results given in 2.4 the operational solution becomes

$$\{x(t)\}=\frac{1}{s+A}x(0)+\frac{1}{s+A}B\{u(t)\} \qquad (2.5.2)$$

and by (2.4.15) and the definition of convolution, the time solution becomes

$$\{x(t)\}=\{e^{-At}x(0)+\int_0^t e^{-A(t-\tau)}Bu(\tau)d\tau\} \qquad (2.5.3)$$

where $x(0)$ is the value of $x(t)$ at $t=0$.

2.5.2 Second order equations In the analysis of physical systems the following differential equation is given if the analysis is based on Lagrange's equation

$$\{x''(t)+A_1x'(t)+A_0x(t)\}=\{Bu(t)\} \qquad (2.5.4)$$

where A_0, A_1 and B are numerical matrices.

The operational solution becomes

$$\{x(t)\}=\frac{x'(0)+(s+A_1)x(0)+B\{u(t)\}}{s^2+A_1s+A_0} \qquad (2.5.5)$$

and for this operational solution, to get the corresponding time solution analytically as in the case of scalar function, matrices A_0 and A_1 must be commutative. But this requirement is satisfied scarcely, therefore, (2.5.4) is transformed into the first order equation of the following form by introducing a new variable $y=x'$

$$\{z'(t)+Az(t)\}=\{Cu(t)\} \qquad (2.5.6)$$

where

$$\{z(t)\}=\begin{Bmatrix}x(t)\\y(t)\end{Bmatrix} \quad A=\begin{pmatrix}0 & -1\\A_0 & A_1\end{pmatrix} \quad C=\begin{pmatrix}0\\B\end{pmatrix}$$

and is solved by the method of 2.5.1.

As an example that can be treated analytically without decreasing its order, let us consider a case where A_1 and u vanish in (2.5.4). Then the equation becomes

$$\{x''(t)+A_0x(t)\}=0 \qquad (2.5.7)$$

and the operational solution is given

$$\{x(t)\}=\frac{1}{s^2+C^2}x'(0)+\frac{1}{s^2+C^2}x(0) \qquad (2.5.8)$$

where $C^2=A_0$.

The time solution is given

$$\{x(t)\}=\{C^{-1}\sin Ctx'(0)+\cos Ctx(0)\} \qquad (2.5.9)$$

and from the definition of matrix functions

$$\{\cos Ct\}=\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}C^{2n}t^{2n}\}=\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}A_0^n t^{2n}\} \qquad (2.5.10)$$

$$\{C^{-1}\sin Ct\}=\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}C^{2n}t^{2n+1}\}=\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}A_0^n t^{2n+1}\}. \qquad (2.5.11)$$

2.6 Discontinuous functions in operational calculus

In practical applications, a function $\{f(t)\}$ considered in the interval $0\leq t<\infty$ satisfying the following two conditions is treated frequently.

(1) It has at most a finite number of points of discontinuity in every finite interval.

(2) The integral $\int_0^t|f(\tau)|d\tau$ has a finite value for every $t>0$.

Miksinsky defind a set of such functions as class K.

Let us denote K^{mxn} a mxn dimensional space of functions of K and consider a matrix of functions of K^{mxn}. Theoretically the operational calculus has been extended to functions of class L and because the class K is included in the class L , it is unnecessary to define any new algebraic operations in functions of class K.

For an example which gives a matrix of functions of K^{mxn}, we shall define non-integer power of a matric operator by

$$(s-A)^{-\lambda}=\{\frac{t^{\lambda-1}}{\Gamma(\lambda)}e^{At}\} \qquad \lambda>0 \qquad (2.6.1)$$

where $\Gamma(\lambda)$ is the Euler's gamma function, and for all real values of λ define

$$(s-A)^0=1 \quad (s-A)^{\lambda}=\frac{1}{(s-A)^{-\lambda}} \qquad \lambda>0 \qquad (2.6.2)$$

then we have the following formula for all values of λ and μ

$$(s-A)^{-\lambda}(s-A)^{-\mu}=(s-A)^{-\lambda-\mu} \qquad (2.6.3)$$

because

$$(s-A)^{-\lambda}(s-A)^{-\mu}=\{\frac{t^{\lambda-1}}{\Gamma(\lambda)}e^{At}\}\{\frac{t^{\mu-1}}{\Gamma(\mu)}e^{At}\}=\frac{1}{\Gamma(\lambda)\Gamma(\mu)}\{\int_0^t (t-\tau)^{\lambda-1}\tau^{\mu-1}e^{At}d\tau\}$$

and by substituting $t-\tau=t\sigma$

$$(s-A)^{-\lambda}(s-A)^{-\mu}=\frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_0^1 (1-\sigma)^{\mu-1}\sigma^{\lambda-1}d\sigma\{t^{\lambda+\mu-1}e^{At}\}$$

$$=\{\frac{\lambda+\mu-1}{\Gamma(\lambda+\mu)}e^{At}\}.$$

The operator $(s-A)^{-\lambda}$ represents

(1) a matrix function in the class $C^{m\times m}$ if $\lambda\geq 1$

(2) a matrix function in the class $K^{m\times m}$ discontinuous at $t=0$ if $0<\lambda<1$

(3) not a matrix function if $\lambda\leq 0$.

By this definition follows the relation

$$\frac{1}{\sqrt{s+A}}=\{\frac{t^{-1/2}}{\Gamma(1/2)}e^{-At}\}=\{\frac{1}{\sqrt{\pi t}}e^{-At}\} \qquad (2.6.4)$$

and by the definition of the convolution, follow the following relations if A is positive definite.

$$\frac{1}{\sqrt{s}(s-A)}=\{\frac{1}{\sqrt{\pi t}}\}\{e^{At}\}=\{\frac{e^{At}}{\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{\tau}}e^{-A\tau^2}d\tau\}=A^{-1/2}\{\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{t}}e^{-A\tau^2}d\tau\}$$

$$=\{A^{-1/2}e^{At}\mathrm{erf}(\sqrt{At})\} \qquad (2.6.5)$$

$$\frac{1}{s\sqrt{s+A}}=\{\int_0^t\frac{1}{\sqrt{\pi\tau}}e^{A\tau}d\tau\}=\{A^{-1/2}\mathrm{erf}(\sqrt{At})\} \qquad (2.6.6)$$

$$\frac{1}{(s+A)^{\nu+\beta}}=\{\frac{t^{\nu-1}}{(\nu-1)!}e^{-At}\}\{\frac{t^{\beta-1}}{\Gamma(\beta)}e^{-At}\}=\{\frac{e^{-At}}{(\nu-1)!\Gamma(\beta)}\int_0^t(t-\tau)^{\nu-1}\tau^{\beta-1}d\tau\}$$

$$=\{\frac{t^{\beta-1}(-t)^{\nu}e^{-At}}{\Gamma(\beta)\nu!}\Big/\binom{-\beta}{\nu}\}$$

$$\beta>0 \quad \nu \text{ natural number} \qquad (2.6.7)$$

Next we shall consider a case that in the following differential equation

$$\{x'(t)+Ax(t)\}=\{Bu(t)\}$$

an input function $\{u(t)\}$ is a function of K^m. Then the function $\{x(t)\}$ will be the solution if

(1) $\{x(t)\}$ is continuous

(2) $\{x(t)\}$ has the first derivative at all the points where $\{x(t)\}$ is continuous

(3) $\{x(t)\}$ satisfies the equation at all the points where $\{u(t)\}$ is continuous.

In such a case, if initial or boundary conditions are given at the points where $\{u(t)\}$ is continuous, then the equation can be treated in the same way as in the case where $\{u(t)\}$ is continuous.

2.7 Summary

In this chapter the algebraic operations of the matrix forms of functions and operators in the operational calculus were shown. Some operational forms for matrix functions which are derived from matric exponential functions were given, and the given results were applied to the analysis of linear simultaneous ordinary differential equations with constant coefficients. Almost every result given here remains a rearrangement of the known result, but it must be the foundation of the studies in successive chapters.

3. Sequences and series of operators

3.1 Preface

In applying the operational calculus to various problems, an analytical treatment of operators becomes necessary, for example in a case where we apply it to the analysis of Bessel functions, periodic functions or linear difference equations.

Here we shall extend Miksinsky's definition of the operational convergence of an operator in the topological space to a matrix form.

Hence it is the purpose of this chapter to show the analytical treatment of a matrix of functions and a matric operator specially related to the properties about their operational convergence.

First we shall show the fundamental properties of the sequences and series of matric operators, and then, as an example of their applications, present a method to analyse linear electric networks with periodic external forces by using the property of operational convergence of a certain infinite series of operators.

The fundamental properties of operators in the topological space will be show in Appendix A and formulation of a state equation of electric network by the topological method will be shown in Appendix B.

3.2 Sequences of matric operators

Let $Q^{l\times m}$ is a $l\times m$ dimensional space of operators, then a sequence of matric operator A_n of $Q^{l\times m}$ is termed to be convergent if every element $A_{n,ij}$ of A_n converges to A_{ij} operationally.

By the definition of operational convergence, every element $A_{n,ij}$ has a unique limit A_{ij} of the following form

$$\lim_{n\to\infty} A_{n,ij}=q_{ij}\lim_{n\to\infty}\frac{A_{n,ij}}{q_{ij}}=A_{ij} \qquad (3.2.1)$$

where $q_{ij}\in Q$ and $A_{n,ij}/q_{ij}\in C$ and because by this fact A_n has a unique limit A.

By the following theorem*

<< *Suppose* $\lim_{n\to\infty} a_n=a$ *and* $\lim_{n\to\infty} b_n=b$, *then* $\lim_{n\to\infty}(a_n\pm b_n)=a\pm b$ *and* $\lim_{n\to\infty} a_n b_n=ab$. *Furthermore suppose* $b\neq 0$ *and* $b_n\neq 0$ *for* $n=1,2,\ldots,$ *and the sequence of the quatients* a_n/b_n *has a limit, then* $\lim_{n\to\infty} a_n/b_n=a/b$. >>

if

$$\lim_{n\to\infty} A_n=A \quad \text{and} \quad \lim_{n\to\infty} B_n=B$$

then

$$\lim_{n\to\infty}(A_n\pm B_n)=A\pm B \qquad (3.2.2)$$

$$\lim_{n\to\infty} A_n B_n=AB \qquad (3.2.3)$$

and when every element of B_n, B and A_n/B_n satisfies the condition of the theorem for every element, then

$$\lim_{n\to\infty} A_n/B_n=A/B \qquad (3.2.4)$$

where A_n/B_n means left multiplication of B_n^{-1} to A_n.

3.3 Series of matric operators

3.3.1 Series of matric operators For an infinite seres of matric operator A_n of $Q^{l\times m}$ in the form $\sum_{n=0}^{\infty} A_n$, if the sequences of its partial sums

$$S_n=A_0+A_1+\ldots+A_n$$

converges to A, it is said to have the sum A and is written

$$\sum_{n=0}^{\infty} A_n=A_0+A_1+\ldots\ldots\ldots=A. \qquad (3.3.1)$$

For an example, we shall consider the following infinite series

$$1+h^{\lambda}B+h^{2\lambda}B^2+\ldots\ldots \qquad (3.3.2)$$

*See I-1 Part 2 Chapter I

where B is a square numerical matrix and multiplications in $B^2, B^3, \ldots$ mean ordinary matric multiplications and h^λ is a translation operator defined by

$$h^\lambda = s\begin{Bmatrix} 0: & 0 \le t < \lambda \\ 1: & 0 < \lambda < t \end{Bmatrix}$$

and causes for a matrix of functions $\{F(t)\}$

$$h^\lambda\{F(t)\} = \begin{Bmatrix} 0: & 0 \le t < \lambda \\ F(t-\lambda): & 0 < \lambda < t \end{Bmatrix}.$$

By multiplying the operator $1-h^\lambda B$, we have

$$(1-h^\lambda B)(1+h^\lambda B+h^{2\lambda}B^2+\ldots.)=1$$

and

$$1+h^\lambda B+h^{2\lambda}B^2+\ldots. = \frac{1}{1-h^\lambda B}. \qquad (3.3.3)$$

This series is important in the analysis of the periodic functions as will be shown later.

3.3.2 Power series of matric operators In applying the operational calculus, the power series of the following form is important

$$\Phi(F) = A_0 + A_1 F + A_2 F^2 + \ldots\ldots. \qquad (3.3.4)$$

where $A_0, A_1, A_2, \ldots.$ are numerical matrices and F is a matrix of functions of $C^{m\times m}$ and multiplications in $F^2, F^3, \ldots$ mean the convolutions.

For such a power series, the following theorem holds.

<< *Suppose the radius of convergence to the series*

$$\Phi(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + \ldots\ldots \quad \lambda \text{ complex} \qquad (3.3.5)$$

is positive, then the series (3.3.4) *is operationally convergent.* >>

Proof: In every domain $|\lambda| \le \lambda_0$, $0 \le t \le t_0$, let us consider the following series

$$\Psi(F\lambda) = A_1 F\lambda + A_2 F^2\lambda^2 + \ldots.. \qquad (3.3.6)$$

Denote $\max\limits_{i,j}|F_{ij}(t)| = M$, then

$$|F(t)| \le MH$$

$$|F^2(t)|\leq M^2m\frac{t}{1!}H\leq M^2m\frac{t_0}{1!}H$$

$$|F^3(t)|\leq M^3m^2\frac{t^2}{2!}H\leq M^3m^2\frac{t_0^2}{2!}H$$

and generally

$$|F^n(t)|\leq M^nm^{n-1}\frac{t^{n-1}}{(n-1)!}H\leq M^nm^{n-1}\frac{t_0^{n-1}}{(n-1)!}H$$

where the matric absolute means the matrix whose elements are the absolutes of the original matrix, the matric inequality means the inequalities for the corresponding pairs of elements, m is the order of F and H is the matrix whose elements are all unity.

Denote the radius of convergence of (3.3.4) by ρ, then

$$|A_nF^n(t)\lambda^n|\leq|\lambda_0^nM^nm^{n-1}\frac{t^{n-1}}{(n-1)!}A_nH|=|(\frac{2\lambda_0}{\rho})^nM^nm^{n-1}\frac{t_0^{n-1}}{(n-1)!}(\frac{\rho}{2})^nA_nH|$$

$$\leq K|A_n|(\frac{\rho}{2})^nH$$

where the following inequality is used

$$(\frac{2\lambda_0}{\rho})^nM^nm^{n-1}\frac{t_0^{n-1}}{(n-1)!}\leq K$$

which is valid because of the fact that the series $\sum_{n=0}^{\infty}x^n/n!$ is convergent, and consequently the series

$$K|A_1|\frac{\rho}{2}+K|A_2|(\frac{\rho}{2})^2+\ldots\ldots\ldots\ldots$$

is convergent by (3.3.5). Therefore the series (3.3.6) is uniformly convergent in the considered domain. Add A_0 to it and substitute $\lambda F=F$, then the theorem is proved. Q.E.D

By the binomial theorem for an arbitrary real exponent β,

$$(1+\lambda)^{\beta}=\sum_{\nu=0}^{\infty}\binom{\beta}{\nu}\lambda^{\nu}\quad |\lambda|<1$$

and the above theorem, we can define

$$(1+F)^{\beta}=\sum_{\nu=0}^{\infty}\binom{\beta}{\nu}F^{\nu}\tag{3.3.7}$$

where F^{ν} means the convolution and the follwing relations are given by this definition.

$$(1+F)^{\beta_1}(1+F)^{\beta_2}=(1+F)^{\beta_1+\beta_2} \qquad (3.3.8)$$

$$(1+F)^{-\beta}=\frac{1}{(1+F)^{\beta}} \qquad (3.3.9)$$

By (3.3.7) we have

$$\frac{1}{\sqrt{s^2+A^2}}=\{J_0(At)\} \qquad (3.3.10)$$

because

$$\frac{1}{\sqrt{s^2+A^2}}=\ell(1+\ell^2A^2)^{-1/2}=\sum_{\nu=0}^{\infty}\binom{-1/2}{\nu}\ell^{2\nu+1}A^{2\nu}=\{\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(2\nu!)}{2^{2\nu}(\nu!)^2}A^{2\nu}t^{2\nu}\}$$

$$=\{J_0(At)\}$$

and by replacing $A=iA$ we have

$$\frac{1}{\sqrt{s^2-A^2}}=\{J_0(iAt)\}. \qquad (3.3.11)$$

By the following series*

$$(\sqrt{1+\lambda}-1)^n=n\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{(n+2\nu-1)!}{2^{n+2\nu}\nu!(n+\nu)!}\lambda^{n+\nu} \qquad |\lambda|<1$$

we have

$$(\sqrt{1+F}-1)^n=n\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{(n+2\nu-1)!}{2^{n+2\nu}\nu!(n+\nu)!}F^{n+\nu}. \qquad (3.3.12)$$

By this formula we have

$$(\sqrt{s^2+A^2}-s)^n=s^n(\sqrt{1+\ell^2A^2}-1)^n=n\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{(n+2\nu-1)!}{2^{n+2\nu}\nu!(n+\nu)!}\ell^{n+2\nu}A^{2(n+\nu)}$$

$$=\{n\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{A^{2(n+\nu)}t^{n+2\nu-1}}{2^{n+2\nu}\nu!(n+\nu)!}\}=\{\frac{nA^n}{t}J_n(At)\}$$

$$n=1,2,\ldots \qquad (3.3.13)$$

and if A is non-singular we have

$$\frac{(\sqrt{s^2+A^2}-s)^n}{\sqrt{s^2+A^2}}=\{\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{A^{2(n+\nu)}t^{n+2\nu}}{2^{2n+\nu}\nu!(n+\nu)!}\}=\{A^nJ_n(At)\} \qquad n=1,2,\ldots (3.3.14)$$

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*See I-1 Part 2 Chapter IV

3.4 Analysis of electric networks with periodic external forces

3.4.1 Separation of periodic solutions I Here we shall consider an electric network such as periodically interrupted network of the second genus[3] and analyse it by the operational calculus.

The properties of a linear lamped time-invariant network with energy storage elements are described by the following state and output equations

$$\{x'(t)\}=\{Ax(t)+Bu(t)\} \qquad (3.4.1)$$

$$\{y(t)\}=\{Cx(t)+Du(t)\} \qquad (3.4.2)$$

where x means a set of necessary and sufficient variables to determine the state of the network and called the state vector, y means the output vector, u means the input vector and matrices A,B,C, and D are constant numerical matrices whose elements are determined by the value of the elements contained in the given network and by their connections.

It is not so intuitive to formulate the state equation for the given network compared to the classical method such as the node equation or the loop equation, but it is convenient to use the state equation in the theoretical considerations.

To formulate the state equation, a systematic method based on the topology of the network has recently been made.[4),5),6),7)] A procedure to formulate the state equation by this method is summarised briefly in Appendix B.

For the given state equation we shall consider a case where input is consisted in periodic functions of period $2T$ such that

$$\{u(t)\}=\{u(t+2T)\}$$

and try to separate the periodic and transient parts from the state vector by the operational method[8)].

The input vector can be written in the following form

$$\{u(t)\}=\frac{\{u_0(t)\}}{1-h^{2T}} \quad \text{and} \quad \{u_0(t)\}=\begin{cases} u(t): & 0\le t<2T \\ 0: & 2T<t<\infty \end{cases} \tag{3.4.3}$$

and $\{u(t)\}$ is generally a discontinuous function considered in 2.6.

By the above notation, the operational solution of the state vector can be written

$$\{x(t)\}=\frac{1}{s-A}x(0)+\frac{1}{s-A}B\frac{\{u_0(t)\}}{1-h^{2T}}. \tag{3.4.4}$$

Here we shall separate the solution to the following form

$$\{x(t)\}=\{x_0(t)+x_t(t)+x_s(t)\} \tag{3.4.5}$$

where x_0 is the term caused by the initial value $x(0)$, x_t is the term related to the input function and x_s is the periodic term with the same period as the input function, then we have

$$\left.\begin{array}{c}\{x_0(t)\}=\frac{1}{s-A}x(0)=\{e^{At}x(0)\}=\{e(t)x(0)\} \\ 0\le t<\infty\end{array}\right\} \tag{3.4.6}$$

where we use the notation $e^{At}=e(t)$, and have

$$\{x_s(t)+x_t(t)\}=\frac{\{Xs(t)\}}{1-h^{2T}}+\{x_t(t)\}=\frac{1}{s-A}B\frac{\{u_0(t)\}}{1-h^{2T}} \tag{3.4.7}$$

where Xs is the fundamental periodic part in the interval $0\le t<2T$.

From (3.4.7) we have

$$\{Xs(t)\}=\frac{1}{s-A}B\{u_0(t)\}-(1-h^{2T})\{x_t(t)\} \tag{3.4.8}$$

and because $Xs=0$ in $2T<t<\infty$, we have

$$\left.\begin{array}{c}\frac{1}{s-A}B\{u_0(t)\}=(1-h^{2T})\{x_t(t)\} \\ 2T<t<\infty\end{array}\right\}. \tag{3.4.9}$$

By this relation the function $\{x_t(t)\}$ satisfies the following differential equation in the interval $2T<t<\infty$

$$\{x'(t)\}=\{Ax(t)\}$$

then we have

$$\{x_t(t)\}=\{e^{At}C\}=\{e(t)C\} \tag{3.4.10}$$

where C is an arbitrary numerical matrix.

Substitute (3.4.10) into the right side of (3.4.9), then

$$(1-h^{2T})\{x_t(t)\}=\{e(t)C\}-\{e(t-2T)C\}=\{-e_0^{-1}(1-e_0)e(t)C\} \quad (3.4.11)$$

where $e_0=e(2T)=e^{2AT}$.

For the left side of (3.4.9), we have

$$\frac{1}{s-A}B\{u_0(t)\}=\{\int_0^t e(t-\tau)Bu_0(\tau)d\tau\}=\{e(t-2T)\int_0^{2T} e(2T-\tau)Bu_0(\tau)d\tau\}$$

$$=\{e_0^{-1}e(t)h_0\} \quad (3.4.12)$$

where

$$h_0=\int_0^{2T} e(2T-\tau)Bu_0(\tau)d\tau.$$

The matrix C is decided by (3.4.11) and (3.4.12) and finally we have

$$\left.\begin{aligned}&\{x_t(t)\}=\{-(1-e_0)^{-1}e(t)h_0\}\\ &\qquad 0\le t<\infty\end{aligned}\right\}. \quad (3.4.13)$$

For the fundamental periodic part, we have

$$\{Xs(t)\}=\frac{1}{s-A}B\{u_0(t)\}-\{x_t(t)\}=\{\int_0^t e(t-\tau)Bu_0(\tau)d\tau\}+\{(1-e_0)^{-1}e(t)h_0\}$$

$$=\{h(t)+(1-e_0)^{-1}e(t)h_0\} \qquad 0<t<2T \qquad (3.4.14)$$

where

$$h(t)=\int_0^t e(t-\tau)Bu_0(\tau)d\tau$$

and for the periodic solution, we have

$$\left.\begin{aligned}&\{x_s(t)\}=\{Xs(t-2nT)\}\\ &\qquad 2nT<t<2(n+1)T \quad n=0,1,2,\ldots\end{aligned}\right\}. \quad (3.4.15)$$

By using theses results we can separate the transient and periodic parts from the state vector.

3.4.2 Separation of periodic solutions II For the periodic functions of period $2T$, we shall often encounter a case where

$$\{u(t)\}=\{-u(t+T)\}.$$

This function can be treated by the above method, but it can be treated more conveniently than the above method in the following way.

The function $\{u(t)\}$ can be written in the following form.

$$\{u(t)\}=\frac{\{u_0(t)\}}{1+h^T} \qquad \{u_0(t)\}=\left\{\begin{matrix} u(t): & 0\leq t<T \\ 0 \quad : & T<t \end{matrix}\right\} \quad (3.4.16)$$

Denote the solution of the state equation by

$$\left.\begin{aligned} &\{x(t)\}=\{x_0(t)+x_t(t)+x_s(t)\} \\ &\{x_s(t)\}=\frac{\{X_s(t)\}}{1+h^T} \end{aligned}\right\} \quad (3.4.17)$$

then follow the results by the same procedure stated in the previous case.

$$\left.\begin{aligned} &\{x_0(t)\}=\{e(t)x(0)\} \\ &\qquad 0\leq t<\infty \end{aligned}\right\} \quad (3.4.18)$$

$$\left.\begin{aligned} &\{x_t(t)\}=\{(1+e_0)^{-1}e(t)h_0\} \\ &\qquad e_0=e^{AT} \quad h_0=\int_0^t e(t-\tau)Bu_0(\tau)d\tau \\ &\qquad 0\leq t<\infty \end{aligned}\right\} \quad (3.4.19)$$

$$\left.\begin{aligned} &\{X_s(t)\}=\{h(t)-(1+e_0)^{-1}e(t)h_0\} \\ &\qquad h(t)=\int_0^t e(t-\tau)Bu_0(\tau)d\tau \\ &\qquad 0\leq t<T \end{aligned}\right\} \quad (3.4.20)$$

$$\left.\begin{aligned} &\{x_s(t)\}=\{(-1)^n X_s(t-nT)\} \\ &\qquad nT<t<(n+1)T \end{aligned}\right\} \quad (3.4.21)$$

As is prove~~n~~d easily, every solution given here coincides perfectry with that given by the previous method in the considered interval.

3.4.3 Numerical examples Here we shall consider responce of an amplifire shown in Fig 3-4-1.

The equivalent circuit of this amplifi~~re~~er is shown in Fig 3-4-2 and a state equation of this circuit is given as follows.

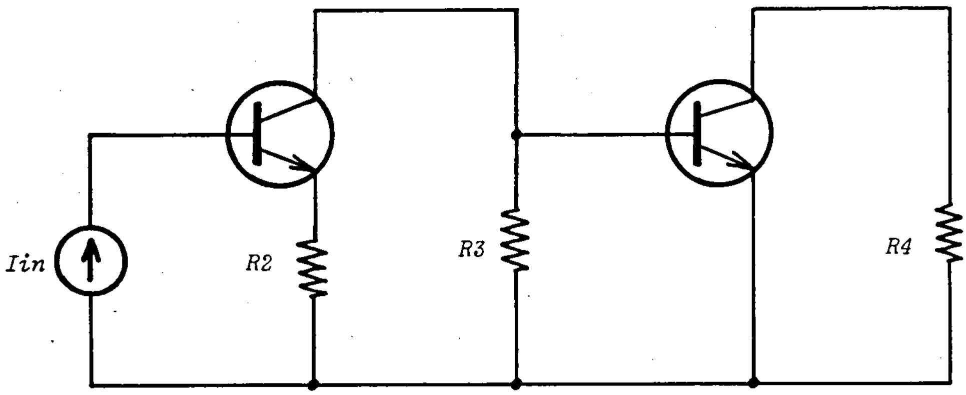

Fig 3-4-1 Circuit diagram of an amplifi~~re~~er

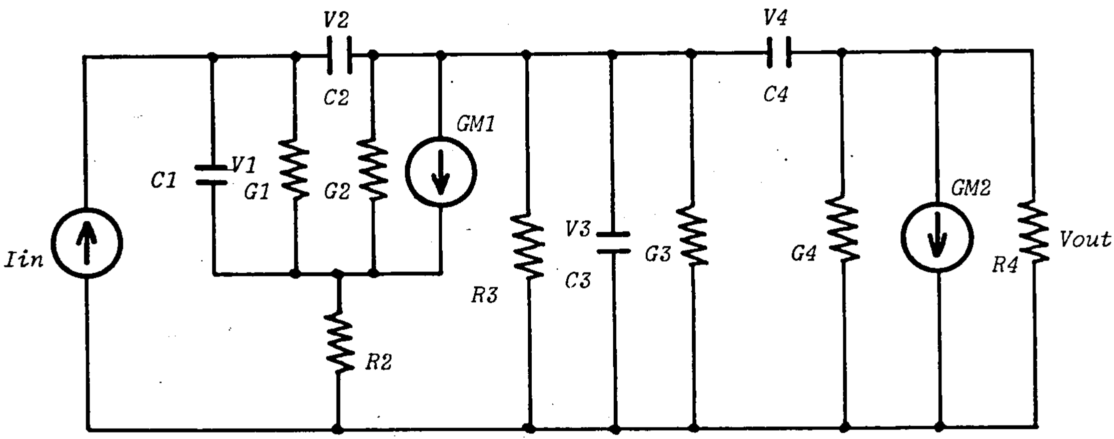

Fig 3-4-2 Equivalent circuit of the amplifi~~re~~er

$$\frac{d}{dt}\left\{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}\right\} = A\left\{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}\right\} + B\{Iin\} \qquad (3.4.22)$$

where

$$A=\left[\begin{array}{lll} -(G_1+G_2+Gm_1+1/R_2)/C_1 & (G_2+1/R_2)/C_1 & 1/C_2R_2 \\ (G_1+Gm_1+1/R_2)/C_2 & -(G_2+1/R_2)/C_2 & -1/C_2R_2 \\ 1/C_3R_2 & -1/C_3R_2 & -(G_3+G_4+Gm_2+1/R_2+1/R_3+1/R_4)/C_3 \\ 0 & 0 & (G_4+Gm_2+1/R_4)/C_4 \end{array}\right.$$

$$\begin{matrix} 0 \\ 0 \\ (G_4+1/R_4)/C_3 \\ -(G_4+1/R_4)/C_4 \end{matrix}\right] \qquad B=\begin{bmatrix} 0 \\ 1/C_2 \\ 1/C_3 \\ 0 \end{bmatrix}.$$

Suppose that Iin is given by

$$\{Iin\}=\left\{\begin{matrix} I: & 0\leq t<T \\ -I: & T<t<2T \end{matrix}\right\} \tag{3.4.23}$$

then the first method gives

$$\{x_0(t)\}=\{e^{At}x(0)\} \qquad 0\leq t<\infty \tag{3.4.24}$$

$$\{x_t(t)\}=\{-A^{-1}(1+e^{AT})^{-1}(1-e^{AT})e^{At}BI \qquad 0\leq t<\infty \tag{3.4.25}$$

$$\{Xs(t)\}=\left\{\begin{matrix} A^{-1}(e^{At}-1)BI+A^{-1}(1+e^{AT})^{-1}(1-e^{AT})e^{At}BI \\ :\ 0\leq t<T \\ A^{-1}[(1-2e^{-AT})e^{At}+1]BI+A^{-1}(1+e^{AT}) \\ (1-e^{AT})e^{At}BI\ :\ T<t<2T \end{matrix}\right\} \tag{3.4.26}$$

and the second method gives the same solution x_0 and x_t as these by the first method and

$$\{Xs(t)\}=\{A^{-1}(e^{At}-1)BI+A^{-1}(1+e^{AT})^{-1}(1-e^{AT})e^{At}BI\}$$

$$0\leq t<T. \tag{3.4.27}$$

For (3.4.26) we can prove easily that

$$\{Xs(t)\}=-\{Xs(t+T)\} \qquad 0\leq t<T \tag{3.4.28}$$

therefore, these two methods give the same solutions.

Numerical solution of

$$Vout/Iin=(V_3-V_4)/Iin$$

is shown in Fig 3-4-3

where

$C_1=C_2=C_3=C_4=10^{-4}\mu F$

$R_2=1k\Omega \qquad R_3=50k\Omega \qquad R_4=20k\Omega$

$G_1=G_3=2.8 mmho \quad G_2=G_4=0.1 mmho$

$Gm_1=Gm_2=25 mmho.$

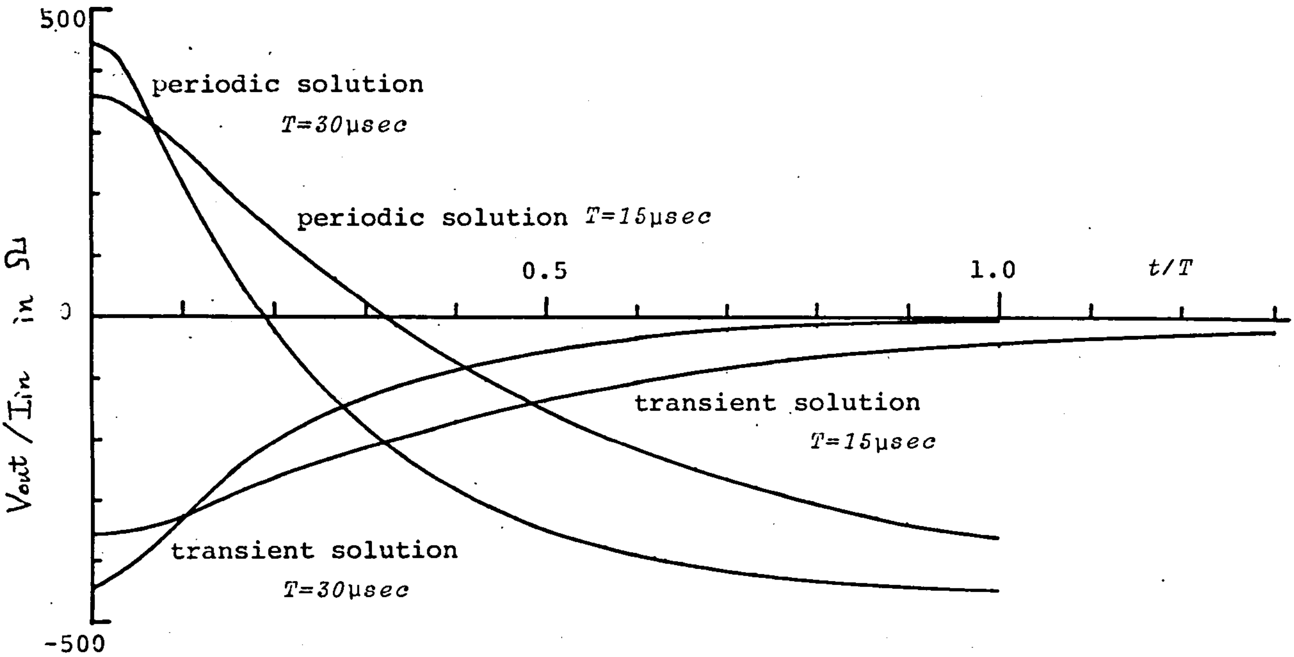

Fig 3-4-3 Numerical solution of *Vout/Iin*

3.5 Summary

In this chapter, we showed the analytical properties of matric operators with respect to the operational convergence and gave the fundamental properties of the sequences and the series of matric operators.

Some operational forms for matrix functions derived from the power series such as some sorts of Bessel functions were given.

As an example of their applications, we gave the method of separating the periodic and the transient parts from the state vector of the electric network with periodic external forces by the operational method and gave some numerical examples about it.

4. Differential and integral properties of operators

4.1 Preface

In applying the operational calculus to the analysis of linear partial differential equations with constant coefficients, differentiation and integration of operators in a certain domain are necessary. For these problems, Miksinsky introduced an operational function and defined its derivative and integral. Here we shall extend his results to the matrix form.

Hence it is the purpose of this chapter to show the analytical properties of matrix of functions and matric operators with respect to the properties of differentiation and integration.

First we shall show the definition of a matric operational function, its derivative and integral and the fundamental properties of the differential equation of the matric operational function.

And then, as an example of their applications, we shall show the method to analyse the telegraphic equations by the operational calculus.

The fundamental properties of the operational derivatives and integrals in the topological space will be shown in Appendix C.

4.2 Derivatives and integrals of matric operational functions

For a given matrix of functions of two variables $\{F(\lambda,t)\}$ defined for $t\geq 0$ and for some values of λ, let us write it as

$$F(\lambda)=\{F(\lambda,t)\} \tag{4.2.1}$$

and call $F(\lambda)$ a matric operational function.

It is said continuous in a certain interval $\lambda\in I$ if each of its elements can be written as

$$f_{ij}(\lambda)=q_{ij}\{f_{1,ij}(\lambda,t)\} \qquad (4.2.2)$$

where q_{ij} is a certain non-zero operator and $f_{1,ij}$ is a function of two variables continuous in the domain $D(\lambda\in I, 0\leq t<\infty)$. In this case it is said to be differentiable in the interval I if every function $f_{1,ij}$ has the partial derivative $\{\frac{\partial}{\partial\lambda}f_{1,ij}(\lambda,t)\}$.

Then the derivative of $F(\lambda)$ is denoted by

$$\frac{d}{d\lambda}F(\lambda)=F'(\lambda)$$

where each of its elements is given by

$$q_{ij}\{\frac{\partial}{\partial\lambda}f_{1,ij}(\lambda,t)\}$$

in the domain D.

Furthermore, $F(\lambda)$ is said to have the n-th derivative $F^{(n)}(\lambda)$ if each of its elements can be written as

$$f_{ij}(\lambda)=q^{(n)}_{n,ij}\{f_{n,ij}(\lambda,t)\} \qquad (4.2.3)$$

where $q^{(n)}_{n,ij}$ is a certain non-zero operator and $f_{n,ij}$ has the n-th partial derivative $\{\frac{\partial^n}{\partial\lambda^n}f_{n,ij}(\lambda,t)\}$ in D.

For the derivatives of matric operational functions the following properties hold.

(1) $F(\lambda)=C$ then $F'(\lambda)=0$

(2) $[F(\lambda)\pm G(\lambda)]'=F'(\lambda)\pm G'(\lambda)$

(3) $[F(\lambda)G(\lambda)]'=F'(\lambda)G(\lambda)+F(\lambda)G'(\lambda)$

(4) $[CF(\lambda)]'=CF'(\lambda)$

For the integration, $F(\lambda)$ is said to be integrable in the interval $\alpha\leq\lambda\leq\beta$ if each of its elements is integrable in this interval, and the following properties hold. The definition of integration of the operational function is shown in Appendix C.

(5) $\int_\alpha^\alpha F(\lambda)d\lambda=0$

(6) $\int_\alpha^\beta F(\lambda)d\lambda=-\int_\beta^\alpha F(\lambda)d\lambda$

$$(7)\quad \int_{\alpha}^{\beta} F(\lambda)\,d\lambda=\int_{\alpha}^{\gamma} F(\lambda)\,d\lambda+\int_{\gamma}^{\beta} F(\lambda)\,d\lambda$$

$$(8)\quad \int_{\alpha}^{\beta} CF(\lambda)\,d\lambda=C\int_{\alpha}^{\beta} F(\lambda)\,d\lambda$$

$$(9)\quad \int_{\alpha}^{\beta} [F(\lambda)\pm G(\lambda)]\,d\lambda=\int_{\alpha}^{\beta} F(\lambda)\,d\lambda\pm\int_{\alpha}^{\beta} G(\lambda)\,d\lambda$$

$$(10)\quad \int_{\alpha}^{\beta} F'(\lambda)G(\lambda)\,d\lambda=F(\beta)G(\beta)-F(\alpha)G(\alpha)-\int_{\alpha}^{\beta} F(\lambda)G'(\lambda)\,d\lambda$$

$$(11)\quad \int_{\alpha}^{\beta} F[\phi(\lambda)]\phi'(\lambda)\,d\lambda=\int_{\phi(\alpha)}^{\phi(\beta)} F(\lambda)\,d\lambda$$

4.3 Differential equations of operational functions

Here we shall show the differential equations of the matric operational functions which will appear in analysing the partial differential equations by the operational calculus.

First we shall show the following fact called the Cauchy's problem which is concerned with the study of linear differential equations in a Banach space[9)].

<< *Denote by* A *a linear constant operator which is defined in the domain* $D(A)$ *to be dense at a Banach space* E*, and consider the following differential equation.*

$$\frac{dx}{dt}=Ax \qquad (4.3.1)$$

The function $x(t)$ *is said the solution of this equation in the interval* $0\le t\le T$ *if it satisfies the following conditions.*

(1) $x(t)\in D(A)$ *for all* t *in* $0\le t\le T$.

(2) There exists $x'(t)$ *for all* t *in* $0\le t\le T$.

(3) $x(t)$ *satisfies the equation for all* t *in* $0\le t\le T$.

If an initial value $x(0)\in D(A)$ *is determined,* (4.3.1) *has the unique solution in* $0\le t<\infty$ *which depends continuously on the initial value.* >>

For a differential equation of any matric operational functions, the following theorem holds.

<< *For given matric operators* W, K *whose elements are defined in a Banach space and a real number* λ_0, *there exists the unique matric operational function* $x(\lambda)$ *satisfying the equation for any real number* λ

$$x'(\lambda)=Wx(\lambda) \tag{4.3.2}$$

and the condition

$$x(\lambda_0)=K \tag{4.3.3}$$

if each of its elements lies on the same space as W *and* K. >>

Proof: By the assumption, W is a constant operator in the Banach space, therefore, $x(\lambda)$ is determined uniquely as the solutions of Cauchy's problem. Q.E.D

For an arbitrary matric operator W the generalized matric exponential function

$$x(\lambda)=e^{\lambda W} \tag{4.3.4}$$

is defined by the solution of the following differential equation

$$\left.\begin{aligned} &x'(\lambda)=Wx(\lambda)\\ &x(0)=1 \end{aligned}\right\} \tag{4.3.5}$$

and is determined uniquely by the above theorem if it exists.

As an example of such a matric exponential function, let us consider the following function defined by

$$h^{\lambda A}=Q^{-1}\mathrm{diag}[h^{\lambda a_1},h^{\lambda a_2},\ldots\ldots,h^{\lambda a_n}]Q \tag{4.3.6}$$

where A is a constant numerical matrix of the following form

$$A=Q^{-1}\mathrm{diag}[a_1,a_2,\ldots\ldots,a_n]Q$$

and $h^{\lambda a_1},h^{\lambda a_2},\ldots$ are translation operators shown in 3.3.1.

Let us assume $a_i>0$, then by the definition

$$h^{\lambda a_i}=s\left\{\begin{matrix} 0: & 0\le t<\lambda a_i\\ 1: & \lambda a_i<t \end{matrix}\right\}$$

and by using the relation $s=s^2l=s^3l^2$

it becomes*

$$h^{\lambda a_i}=s^2\{h_1(\lambda a_i,t)\}=s^3\{h_2(\lambda a_i,t)\}$$

where

$$\{h_1(\lambda a_i,t)\}=\left\{\begin{matrix}0: & 0\le t<\lambda a_i \\ (t-\lambda a_i): & \lambda a_i<t\end{matrix}\right\}$$

$$\{h_2(\lambda a_i,t)\}=\left\{\begin{matrix}0: & 0\le t<\lambda a_i \\ \frac{1}{2}(t-\lambda a_i)^2: & \lambda a_i<t\end{matrix}\right\}.$$

By extending above results to the matrix form $h^{\lambda A}$ can be written

$$h^{\lambda A}=s^2\{h_1(\lambda A,t)\}=s^3\{h_2(\lambda A,t)\} \qquad (4.3.7)$$

where

$$\{h_1(\lambda A,t)\}=Q^{-1}\mathrm{diag}[\{h_1(\lambda a_1,t)\},\{h_1(\lambda a_2,t)\},\ldots,\{h_1(\lambda a_n,t)\}]Q$$

$$\{h_2(\lambda A,t)\}=Q^{-1}\mathrm{diag}[\{h_2(\lambda a_1,t)\},\{h_2(\lambda a_2,t)\},\ldots,\{h_2(\lambda a_n,t)\}]Q$$

then by the definition of the differentiation $h^{\lambda A}$ satisfies the following differential equation

$$\left.\begin{matrix}(h^{\lambda A})'=s^2\{\frac{\partial}{\partial\lambda}h_2(\lambda A,t)\}=s^3\{-Ah_1(\lambda A,t)\}=-sAh^{\lambda A} \\ (h^{\lambda A})_{\lambda=0}=1\end{matrix}\right\} \qquad (4.3.8)$$

and becomes as follows.

$$h^{\lambda A}=e^{-\lambda As} \qquad (4.3.9)$$

As another example let us consider the matric operational function $e^{-\lambda A\sqrt{s}}$ where $\lambda\ge 0$ and A is positive definite.

Suppose the function

$$\left.\begin{matrix}F(\lambda)=\{F(\lambda,t)\}=\{\frac{\lambda A}{2\sqrt{\pi t}}\exp(-\frac{\lambda^2A^2}{4t})\} \\ 0<\lambda<\infty\end{matrix}\right\} \qquad (4.3.10)$$

then the following relation holds.

$$\ell^{1/2}F(\lambda)=\{\frac{1}{\sqrt{\pi t}}\}F(\lambda)=\{\frac{\lambda A}{2}\int_0^t(t-\tau)^{-1/2}\tau^{-3/2}\exp(-\frac{\lambda^2A^2}{4\tau})d\tau\}.$$

Substitute $\lambda^2/4\tau=\lambda^2/4t+\sigma^2$, then it becomes

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*See I-1 Part 3 Chapter II

$$\ell^{1/2}F(\lambda)=\{\frac{1}{\sqrt{\pi t}}\exp(-\frac{\lambda^2A^2}{4t})\}\qquad(4.3.11)$$

and

$$\ell^{3/2}F(\lambda)=\{\int_0^t\frac{1}{\sqrt{\pi\tau}}\exp(-\frac{\lambda^2A^2}{4\tau})d\tau\}.\qquad(4.3.12)$$

Differentiate this function, then it becomes

$$\{\frac{\partial}{\partial\lambda}\int_0^t\frac{1}{\sqrt{\pi\tau}}\exp(-\frac{\lambda^2A^2}{4\tau})d\tau\}=-\{\int_0^t\frac{\lambda A^2}{2\sqrt{\pi\tau^3}}\exp(-\frac{\lambda^2A^2}{4\tau})d\tau\}$$

and

$$F'(\lambda)=-A\sqrt{s}F(\lambda).\qquad(4.3.13)$$

From (4.3.12) it can be seen that

$$F(0)=s^{3/2}\{\int_0^t\frac{1}{\sqrt{\pi\tau}}1d\tau\}=s^{3/2}\ell\ell^{1/2}=1\qquad(4.3.14)$$

therefore, we have

$$e^{-\lambda A\sqrt{s}}=\{\frac{\lambda A}{2\sqrt{\pi t^3}}\exp(-\frac{\lambda^2A^2}{4t})\}.\qquad(4.3.15)$$

From (4.3.11) we have

$$\frac{1}{\sqrt{s}}e^{-\lambda A\sqrt{s}}=\{\frac{1}{\sqrt{\pi t}}\exp(-\frac{\lambda^2A^2}{4t})\}\qquad(4.3.16)$$

and from (4.3.15) we have

$$\frac{1}{s}e^{-\lambda A\sqrt{s}}=\{\int_0^t\frac{\lambda A}{2\sqrt{\pi\tau^3}}\exp(-\frac{\lambda^2A^2}{4\tau})d\tau\}.\qquad(4.3.17)$$

Substitute $\sigma=1/2\sqrt{\tau}$, then (4.3.17) becomes

$$\frac{1}{s}e^{-\lambda A\sqrt{s}}=\{\frac{2}{\sqrt{\pi}}\int_{1/2\sqrt{t}}^{\infty}\lambda A\exp(-\lambda^2A^2\sigma^2)d\sigma\}=\{\mathrm{cerf}(\frac{\lambda A}{2\sqrt{t}})\}.\qquad(4.3.18)$$

For the derivative of a power series, the following theorem holds.

<< *If a numerical power series*

$$\Phi(\lambda)=A_0+A_1\lambda+A_2\lambda^2+......\qquad(4.3.19)$$

has a positive radius of convergence, then the function

$$\Phi(\lambda F)=A_0+A_1\lambda F+A_2\lambda^2F^2+......\qquad(4.3.20)$$

has a derivative in the form of a power series

$$\Psi(\lambda F)=A_1F+2A_2\lambda F^2+.....\qquad(4.3.21)$$

which is convergent for any complex number λ, *where* F *is a matrix of* *functions of $C^{m\times m}$.* >>

Proof: By the theorem in 3.3.2, (4.3.20) is operationally convergent. The series (4.3.19) is differentiable with respect to λ term by term and the derivative series has the same radius of convergence as (4.3.19), therefore, the series (4.3.21) is operationally convergent. By the relation

$$A_1F+2A_2\lambda F^2+.....=\frac{\partial}{\partial\lambda}(A_0+A_1\lambda F+A_2\lambda^2F^2+....)$$

we have

$$\Phi'(\lambda F)=\Psi(\lambda F) \tag{4.3.22}$$

Q.E.D

In particular, if

$$\Phi(\lambda F)=1+\frac{\lambda F}{1!}+\frac{\lambda^2F^2}{21}+....... \tag{4.3.23}$$

then

$$\Phi'(\lambda F)=F+\frac{\lambda F^2}{1!}+\frac{\lambda^2F^3}{2!}+.....=F\Phi(\lambda F). \tag{4.3.24}$$

Since $\Phi(0)=1$, by the definition of the exponential function we have

$$e^{\lambda F}=1+\frac{\lambda F}{1!}+\frac{\lambda^2F^2}{2!}+..... \tag{4.3.25}$$

Let $F=-A/s$ and A positive definite, then F is a matrix function of $C^{m\times m}$, therefore, for $\lambda\geq 0$

$$e^{-\lambda A/s}=\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^n\ell^nA^n}{n!}=s\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^n\ell^{n+1}A^n}{n!}$$
$$=s\{\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^nA^nt^n}{(n!)^2}\}=s\{J_0(2\sqrt{\lambda At})\}.$$

Hence we have

$$\frac{1}{s}e^{-\lambda A/s}=\{J_0(2\sqrt{\lambda At})\} \tag{4.3.26}$$

and

$$\frac{1}{s^2}e^{-\lambda A/s}=\{\int_0^t J_0(2\sqrt{\lambda A\tau})d\tau\}=\{\sqrt{(\lambda A)^{-1}}tJ_1(2\sqrt{\lambda At})\}. \tag{4.3.27}$$

In the same way, we have

$$\frac{1}{\sqrt{s}}e^{-\lambda A/s}=\{\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{\lambda A t}\} \tag{4.3.28}$$

$$\frac{1}{\sqrt{s}}e^{\lambda A/s}=\{\frac{1}{\sqrt{\pi t}}\cosh 2\sqrt{\lambda A t}\}. \tag{4.3.29}$$

Let $F=s-\sqrt{s^2+A^2}$, then by the relation

$$F=s-s(1+\ell^2 A^2)^{1/2}=-\sum_{\nu=1}^{\infty}\binom{1/2}{\nu}\ell^{2\nu-1}A^{2\nu}=\{-\sum_{\nu=1}^{\infty}\binom{1/2}{\nu}\frac{t^{2(\nu-1)}A^{2\nu}}{(2\nu-2)!}\}$$

F is the matrix function of $C^{m\times m}$, therefore for $\lambda\geq 0$

$$\begin{aligned}\exp\lambda(s-\sqrt{s^2+A^2})&=\sum_{\mu=0}^{\infty}\frac{1}{\mu!}(-\sum_{\nu=1}^{\infty}\binom{1/2}{\nu}\ell^{2\nu-1}A^{2\nu})^{\mu}\\&=1-\{\lambda\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{A^{2+2\nu}(t^2+2\lambda t)^{\nu}}{2^{1+2\nu}\nu!(\nu+1)!}\}\\&=1-\{\frac{\lambda A}{\sqrt{t^2+2\lambda t}}J_1(A\sqrt{t^2+2\lambda t})\}.\end{aligned} \tag{4.3.30}$$

In the same way, we have

$$\frac{\exp\lambda(s-\sqrt{s^2+A^2})}{\sqrt{s^2+A^2}}=\{\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{A^{2\nu}(t^2+2\lambda t)^{\nu}}{2^{2\nu}(\nu!)^2}\}=\{J_0(A\sqrt{t^2+2\lambda t})\}. \tag{4.3.31}$$

Let $A=iA$, then

$$\exp\lambda(s-\sqrt{s^2-A^2})=1-\{\frac{i\lambda A}{\sqrt{t^2+2\lambda t}}J_1(iA\sqrt{t^2+2\lambda t})\} \tag{4.3.32}$$

$$\frac{\exp\lambda(s-\sqrt{s^2-A^2})}{\sqrt{s^2-A^2}}=\{J_0(iA\sqrt{t^2+2\lambda t})\}. \tag{4.3.33}$$

Multiplying by $e^{-\lambda s}$ to (4.3.30) - (4.3.33), we have

$$\exp(-\lambda\sqrt{s^2+A^2})=e^{-\lambda s}-\left\{\begin{array}{ll}0: & 0\leq t<\lambda\\ \frac{\lambda A}{\sqrt{t^2-\lambda^2}}J_1(A\sqrt{t^2-\lambda^2}): & 0\leq\lambda<t\end{array}\right\} \tag{4.3.34}$$

$$\frac{\exp(-\lambda\sqrt{s^2+A^2})}{\sqrt{s^2+A^2}}=\left\{\begin{array}{ll}0: & 0\leq t\\ J_0(A\sqrt{t^2-\lambda^2}): & 0\leq\lambda<t\end{array}\right\} \tag{4.3.35}$$

$$\exp(-\lambda\sqrt{s^2-A^2})=e^{-\lambda s}-\left\{\begin{array}{ll}0: & 0\leq t<\lambda\\ \frac{i\lambda A}{\sqrt{t^2-\lambda^2}}J_1(iA\sqrt{t^2-\lambda^2}): & 0\leq\lambda<t\end{array}\right\} \tag{4.3.36}$$

$$\frac{\exp(-\lambda\sqrt{s^2-A^2})}{\sqrt{s^2-A^2}} = \begin{cases} 0: & 0 \le t < \lambda \\ J_0(iA\sqrt{t^2-\lambda^2}): & 0 \le \lambda < t \end{cases}. \qquad (4.3.37)$$

For a linear differential equation in a Banach space, the following fact is shown[9].

<< *Denote by* B *a linear constant operator which is defined in the domain* D(B) *to be dense at a Banach space* E, *and consider the following differential equation.*

$$\frac{d^2\mathrm{x}}{dt^2} = \mathrm{B}^2\mathrm{x} \qquad (4.3.38)$$

The function x(t) *is said the solution of this equation in the interval* $0 \le t \le T$ *if it satisfies the following conditions.*

(1) $\mathrm{x}(t) \in D(\mathrm{B}^2)$ *for all* t *in* $0 \le t \le T$.

(2) There exist $\mathrm{x}'(t)$ *and* $\mathrm{x}''(t)$ *for all* t *in* $0 \le t \le T$.

(3) x(t) *satisfies the equation for all* t *in* $0 \le t \le T$.

(4) $\mathrm{x}'(t) \in D(\mathrm{B})$ *and* $\mathrm{Bx}'(t)$ *is continuous in* $0 \le t \le T$.

If initial values $\mathrm{x}(0) \in D(\mathrm{B}^2)$ *and* $\mathrm{x}'(0) \in D(\mathrm{B})$ *are determined,* (4.3.38) *has the unique solution in* $0 \le t < \infty$ *which depends continuously on the initial values.* >>

For a differential equation of any matric operational function, the following theorem holds by the above fact.

<< *For given operators* $\mathrm{W}, \mathrm{K}_0, \mathrm{K}_1$ *whose elements are defined in the Banach space and a real number* λ_0, *there exists the unique operational function* x(λ) *satisfying the equation for any real number* λ

$$\mathrm{x}''(\lambda) = \mathrm{W}^2\mathrm{x}(\lambda) \qquad (4.3.39)$$

and the conditions

$$\mathrm{x}(\lambda_0) = \mathrm{K}_0 \qquad \mathrm{x}'(\lambda_0) = \mathrm{K}_1 \qquad (4.3.40)$$

if each of its elements lies on the same space as W^2. >>

4.4 Analysis of telegraphic equations

4.4.1 Telegraphic equations The properties of distributed constants systems such as airial transmission lines, electrical or communication cables and pulse transmission lines are described by the set of the partial differential equations of the following form called the telegraphic equation

$$\left.\begin{aligned} -\left\{\frac{\partial v(\lambda,t)}{\partial \lambda}\right\} &= L\left\{\frac{\partial i(\lambda,t)}{\partial t}\right\} + R\{i(\lambda,t)\} \\ -\left\{\frac{\partial i(\lambda,t)}{\partial \lambda}\right\} &= C\left\{\frac{\partial v(\lambda,t)}{\partial t}\right\} + G\{v(\lambda,t)\} \end{aligned}\right\} \qquad (4.4.1)$$

where λ means the distance, $\{v(\lambda,t)\}$ and $\{i(\lambda,t)\}$ mean the voltage and current vectors at the point (λ,t) and L,R,G and C mean the indactance, resistance,condactance and capacitance matrices per unit length.

When the initial values are given as

$$v(\lambda,0)=0 \qquad i(\lambda,0)=0 \qquad (4.4.2)$$

the following set of simultaneous ordinary differential equations is given by applying the operational calculus.

$$\left.\begin{aligned} v'(\lambda) &= -(Ls+R)\,i(\lambda) \\ i'(\lambda) &= -(Cs+G)\,v(\lambda) \end{aligned}\right\} \qquad (4.4.3)$$

Speperate the unknown vectors, then

$$\left.\begin{aligned} v''(\lambda) &= [LCs^2+(RC+LG)s+RG]\,v(\lambda) \\ i''(\lambda) &= [CLs^2+(CR+GL)s+GR]\,i(\lambda) \end{aligned}\right\}. \qquad (4.4.4)$$

4.4.2 Wave type equations If R and G can be neglected, the voltage equation in (4.4.4) becomes the following wave equation.

$$v''(\lambda)=A^2s^2v(\lambda) \qquad A^2=LC \qquad (4.4.5)$$

Suppose that the boundary values are given by

$$\{v(0,t)\}=\{v_1(t)\} \qquad \{v(\lambda_0,t)\}=\{v_2(t)\} \qquad (4.4.6)$$

or as the operational form

$$v(0)=v_1 \qquad v(\lambda_0)=v_2. \qquad (4.4.7)$$

From the physical meaning, the coefficient matrix A is positive definite and ~~symmetric~~ can be transformed into the diagonal matrix by a similar transformation, therefore, by the second theorem in 4.3 we have the unique solution $v(\lambda)$ expressed by

$$v(\lambda)=(1-e^{-2\lambda_0 As})^{-1}[(e^{-\lambda As}-e^{-(2\lambda_0-\lambda)As})v_1 + (e^{-(\lambda_0-\lambda)As}-e^{-(\lambda_0+\lambda)As})v_2] \quad 0\le\lambda\le\lambda_0. \qquad (4.4.8)$$

As was shown in 4.3 $e^{-\lambda As}$ is the matrix of translation operators, therefore by the property of the series of translation operators

$$(1-e^{-2\lambda_0 As})^{-1}=\sum_{k=0}^{\infty} e^{-2k\lambda_0 As} \qquad (4.4.9)$$

$v(\lambda)$ becomes

$$v(\lambda)=\sum_{k=0}^{\infty}[(e^{-(2k\lambda_0+\lambda)As}-e^{-(\overline{2k+1}\lambda_0-\lambda)As})v_1 + (e^{-(\overline{2k+1}\lambda_0-\lambda)As}-e^{-(\overline{2k+1}\lambda_0+\lambda)As})v_2] \quad 0\le\lambda\le\lambda_0. \qquad (4.4.10)$$

Suppose $a_1, a_2, \ldots, a_n$ eigen-values of A, then

$$e^{As}=Q^{-1}\operatorname{diag}[e^{a_1 s}, e^{a_2 s}, \ldots, e^{a_n s}]Q \qquad (4.4.11)$$

where Q is a non-singular numerical matrix and the time solution of $\{v(\lambda,t)\}$ is determined analytically.

Specially if $v_2=0$, it becomes

$$\{v(\lambda,t)\}=Q^{-1}\sum_{k=0}^{\infty}\left[\begin{cases}0: & 0\le t<\overline{2k\lambda_0+\lambda}a_1\\ u_1(t-\overline{2k\lambda_0+\lambda}a_1): & \overline{2k\lambda_0+\lambda}a_1<t\end{cases}, \ldots\ldots, \begin{cases}0: & 0\le t<\overline{2k\lambda_0+\lambda}a_n\\ u_n(t-\overline{2k\lambda_0+\lambda}a_n): & \overline{2k\lambda_0+\lambda}a_n<t\end{cases}\right]^t$$

$$0\le\lambda\le\lambda_0 \qquad t\ge 0 \qquad (4.4.12)$$

where $[u_1, u_2, \ldots, u_n]^t=Qv_1$.

For the lines with infinite length, it becomes

$$\{v(\lambda,t)\}=Q^{-1}\left[\begin{cases}0: & 0\le t<\lambda a_1\\ u_1(t-\lambda a_1): & \lambda a_1<t\end{cases}, \ldots\ldots, \begin{cases}0: & 0\le t<\lambda a_n\\ u_n(t-\lambda a_n): & \lambda a_n<t\end{cases}\right]^t$$

$$\lambda\ge 0 \quad t\ge 0. \qquad (4.4.13)$$

The current vector can also be solved in the same way as the voltage vector

4.4.3 Heat conduction type equations If L and G can be neglected, the voltage equation becomes the following heat equation

$$v''(\lambda)=A^2 s v(\lambda) \quad A^2=RC. \tag{4.4.14}$$

Since the coefficient matrix A is positive definite, if the boundary values are given as

$$v(0)=v_1 \quad v(\lambda_0)=v_2 \tag{4.4.15}$$

the following unique solution is determined.

$$v(\lambda)=(1-e^{-2\lambda_0 A\sqrt{s}})^{-1}[(e^{-\lambda A\sqrt{s}}-e^{-2(\lambda_0-\lambda)A\sqrt{s}})v_1$$
$$+(e^{-(\lambda_0-\lambda)A\sqrt{s}}-e^{-(\lambda_0+\lambda)A\sqrt{s}})v_2] \quad 0\le\lambda\le\lambda_0 \tag{4.4.16}$$

As was shown in 4.3 the operator $e^{-\lambda A\sqrt{s}}$ is given as

$$e^{-\lambda A\sqrt{s}}=\{F(\lambda,t)\}=\{\frac{\lambda A}{2\sqrt{\pi t^3}}\exp(-\frac{\lambda^2 A^2}{4t})\} \quad 0\le\lambda<\infty \tag{4.4.17}$$

and F is the matrix function of $C^{m\times m}$ for any fixed value of λ, therefore, by the theorem in 3.3.2, the following infinite series holds.

$$(1-e^{-2\lambda_0 A\sqrt{s}})^{-1}=\sum_{k=0}^{\infty} e^{-2k\lambda_0 A\sqrt{s}} \tag{4.4.18}$$

Substitute this infinite series into (4.4.16) and use (4.4.17), then

$$\{v(\lambda,t)\}=\sum_{k=0}^{\infty}\{\int_0^t[F(2k\lambda_0+\lambda,t-\tau)-F(\overline{2k+1}\lambda_0-\lambda,t-\tau)]v_1(\tau)d\tau$$
$$+\int_0^t[F(\overline{2k+1}\lambda_0-\lambda,t-\tau)-F(\overline{2k+1}\lambda_0+\lambda,t-\tau)]v_2(\tau)d\tau\}$$
$$0\le\lambda\le\lambda_0 \quad t\ge 0. \tag{4.4.19}$$

For lines with infinite length, it becomes

$$\{v(\lambda,t)\}=\{\int_0^t F(\lambda,t-\tau)v_1(\tau)d\tau\} \quad \lambda\ge 0 \quad t\ge 0 \tag{4.4.20}$$

and specially $\{v_1(t)\}=\{V_0\}$ (V_0 constant) it becomes

$$\{v(\lambda,t)\}=\{\mathrm{cerf}(\frac{\lambda A}{2\sqrt{t}})V_0\} \quad \lambda\ge 0 \quad t\ge 0. \tag{4.4.21}$$

4.4.4 General case If all the coefficient matrices cannot be neglected, the following unique operational solution yiels for the voltage vector if the boundary values are given by

$$v(0)=v_1 \qquad v(\lambda_0)=v_2 \tag{4.3.22}$$

$$v(\lambda)=(1-e^{-2\lambda_0 A(s)})^{-1}[(e^{-\lambda A(s)}-e^{-(2\lambda_0-\lambda)A(s)})v_1$$
$$+(e^{-(\lambda_0-\lambda)A(s)}-e^{-(\lambda_0+\lambda)A(s)})v_2]$$
$$A^2(s)=LCs^2+(RC+LG)s+RG \qquad 0\le\lambda\le\lambda_0. \tag{4.3.23}$$

Since the matric polynomial $A^2(s)$ has more than one coefficient matrix, the time function arising from the corresponding scalar function can be given only when all the coefficient matrices are commutative.

For a fairly simple case, we shall consider the lines with infinite length where G can be neglected, then operational solution becomes

$$v(\lambda)=e^{-\lambda\sqrt{LCs^2+RCs}}v_1 \tag{4.4.24}$$

and if matrices LC and RC are commutative, we have the following result from the corresponding scalar equation.

$$\{v(\lambda,t)\}=\{e^{-\lambda A_1 A_2}e^{-\lambda A_1 s}v_1(t)-e^{-\lambda A_1 s}\int_0^t \frac{\lambda A_1 A_2}{\sqrt{\tau^2 1-\lambda^2 A_1^2}}e^{-\lambda A_2 \tau}$$
$$iJ_1(iA_2\sqrt{\tau^2 1-\lambda^2 A_1^2})v_1(t-\tau)d\tau\}$$
$$A_1^2=LC \qquad A_2=1/2C^{-1}L^{-1}RC \qquad \lambda\ge 0 \tag{4.4.25}$$

This solution contains the matrix of translation operators $e^{-\lambda A_1 s}$ but we can get the analytical form in the same way as stated in 4.4.2.

4.5 Time functions for operational solutions

For a phenomenon whose properties are described by a set of linear differential equations, we can get the operational solution in a matrix or vector form satisfying the given conditions easily, and we have shown the operational method to get the time function of matrix form arising from the corresponding scalar function for given matric operational solution. Here we shall consider the problem whether we can get the time function of the matrix form arising from the scalar function when any operational solution corresponding to the scalar form is given.

When the matric operational solution has one coefficient matrix and each of its eigenvalues satisfies the condition required for the coefficient of the scalar solution, we can determine the time function arising from the scalar function.

But in such a case we cannot accept that the problem can be solved because the time function is merely formal and its analytical form is influenced by eigen values of coefficient matrix, so we shall prescribe that the problem is solved only when either of the following conditions is satisfied.

(1) Every element of the time function is determined analytically.

(2) Every element of the time function is determined numerically.

For the first case we can get the analytical solution when all the eigen values of the coefficient matrix and a numerical matrix of transforming it to the canonical form are known.

For the second case it is sufficient that we can get the numerical solution in any way.

In applying the operational calculus, we shall often treat exponential or transcendential functions in the time domain, and in such cases if all the eigen values lie in the convergence circle of the infinite series of the corresponding scalar function, we can get the numerical solution approximately by a finite sum of matrix series arising from the scalar series.

In the theoretical treatments, we must use the first method, but it is fairly cumbersome in case of many variable system. But in engineering applications it is preferred to get the numerical solution rather than the analytical solution and by the progress of numerical technique to the computations, the second method has been used generally.

In such a case, in order that the solution must be stable numerically, it is caused frequently that for eigen values of the coefficient matrix, a more rigorous restriction is necessary in numerically than theoreticlly. When a matric operational solution has more than one coefficient matrix we can get a time solution arising from the scalar function only when everyone of them is commutative with another and all the eigen values of them satisfy the condition required for the coefficients of the scalar function, and if not so it is difficult to get the time function by either of above methods.

A numerical inversion method of the Laplace transform that will be shown in Chapter 6 does not need numerical restrictions for coefficient matrices of the operational solution and gives a numerical solution directly from the operational solution, thererfore, it must be used suitably to the problems stated here.

4.6 Summary

In this chapter we gave the definition of matric operational functions, their derivatives and integrals, and then studied the behaviors of differential equations of matric operational functions in a Banach space. Some operational forms of matrix functions which are derived from the differential equations of the operational functions were given.

For an example of their applications, we analysed distributed constants systems, specially telegraphic equations. The given results have already been known, but they are derived by the operational method here.

5. Multi-dimensional operational calculus

5.1 Preface

The main advantage in analysing the differential equations by the operational calculus lies in the fact that it decreases one variable with respect to differentiation such that a partial differential equation of two variables is transformed into an ordinary differential equation of one variable and analysed by the method stated hitherto.

Therefore, if the operational calculus is extended to the dimension of all the variables, the partial differential equation can be solved only by the algebraic operations.

Hence it is the purpose of this chapter to show the principle of the multi-dimensional operational calculus and its applications to the initial value problems of multi-dimensional partial differential equations, specially the telegraphic equation as an example of its applications.

Here the discussions are limited for simplicity to the scalar equations.

So far, all the problems necessary in analysing the partial differential equations by the operational calculus have not been given, therefore, we shall show the fundamental problems of higher order partial differential equations of two variables necessary to the discussions of this chapter in Appendix D.

5.2 Multi-dimensional operational calculus

5.2.1 Principle of multi-dimensional operational calculus   As was shown in the discussions hitherto, the reason that the operational calculus is used as the powerful tool in analysing the differential equations is that it decreases one variable with respect to differentiation.

Therefore, the initial value problems of partial differential equations with more than one variable can be solved algebraically by extending the operational calculus to the dimension of all the variables.

Here we shall show the principle of the multi-dimensional operational calculus introduced by Gutterman[10].

Let us consider two continuous functions $\{u(x_1,x_2,\ldots,x_n)\}$ and $\{v(x_1,x_2,\ldots,x_n)\}$ in the domain $D(x_i\geq 0;\ i=1,2,\ldots,n)$ and define the convolution $\{w(x_1,x_2,\ldots,x_n)\}$ of them by

$$w(x_1,x_2,\ldots,x_n)=\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_n}u(x_1-t_1,x_2-t_2,\ldots,x_n-t_n)\,v(t_1,t_2,\ldots,t_n)\,dt_1dt_2..dt_n \quad (5.2.1)$$

and denote it by

$$\{w(x_1,x_2,\ldots,x_n)\}=\{u(x_1,x_2,\ldots,x_n)\}\{v(x_1,x_2,\ldots,x_n)\} \quad (5.2.2)$$

then a set of such functions has no divisors of zero with respect to the convolution as in the case of the set of functions of one variables.

By this definition, the following operators are introduced.

Integral operator l is defined by $l=\{1\}$ and causes

$$l\{u(x_1,x_2,\ldots,x_n)\}=\{\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_n}u(t_1,t_2,\ldots,t_n)\,dt_1dt_2\ldots dt_n\}. \quad (5.2.3)$$

Differentiation operator s is defined by the inverse of l and given $s=\frac{1}{l}$.

Numerical operator is defined by

$$[c]=\frac{\{c\}}{l}=c.$$

Now let us introduce the following operator

$$l_i=\frac{\{x_i\}}{l} \qquad i=1,2,\ldots,n \quad (5.2.4)$$

which causes

$$l_i\{u(x_1,x_2,\ldots,x_n)\}=\{\int_0^{x_i}u(x_1,x_2,\ldots,t_i,\ldots,x_n)\,dt_i\} \quad (5.2.5)$$

and then

$$l=l_1l_2\ldots l_n. \quad (5.2.6)$$

Let us denote the inverse operator of l_i by

$$s_i = \frac{1}{l_i} \qquad i=1,2,\ldots,n \tag{5.2.7}$$

then we have

$$s = s_1 s_2 \ldots s_n. \tag{5.2.8}$$

By the relation

$$\{u(x_1,x_2,\ldots,x_n)\} = \{\int_0^{x_i} u_{x_i}(x_1,x_2,\ldots,t_i,\ldots,x_n)\,dt_i\} + \{u(x_1,x_2,\ldots,0,\ldots,x_n)\}$$

follows the formula

$$\{u_{x_i}(x_1,x_2,\ldots,x_n)\} = s_i\{u(x_1,x_2,\ldots,x_n)\} - s_i\{u(x_1,x_2,\ldots,0,\ldots,x_n)\} \tag{5.2.9}$$

and more generally

$$\left.\begin{aligned}
&\{\frac{\partial^r u}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}}\} = s_1^{r_1}\ldots s_n^{r_n}\{u(x_1,\ldots,x_n)\} \\
&\quad -(\sum_{k_1=0}^{r_1}\ldots\sum_{k_n=0}^{r_n} s_1^{k_1}\ldots s_n^{k_n}\{a^r_{k_1\ldots k_n}\} - \{a^r_{0\ldots 0}\}) \\
&\qquad r_1+\ldots+r_n=r \\
&\{a^r_{k_1\ldots k_n}\} = \left\{\frac{\partial^{r-k} u}{\partial x_1^{r_1-k_1}\ldots\partial x_n^{r_n-k_n}}\right\}(x_1\delta_0^{k_1},\ldots,x_n\delta_0^{k_n}) \\
&\qquad k_1+\ldots+k_n=k
\end{aligned}\right\} \tag{5.2.10}$$

where δ_i^j is Kronecker's delta.

5.2.2 Application to multi-dimensional differential equations Here we shall consider the following partial differential equation of n variables

$$\{F(u,u_{x_1},\ldots,u_{x_n},u_{x_1x_2},\ldots)\} = \{\phi(x_1,\ldots,x_n)\} \tag{5.2.11}$$

where F is a linear function with respect to every variable.

For the function satisfying the given initial values in the domain $D(x_i \geq 0; i=1,2,\ldots,n)$, the following operational solution is given by applying (5.2.10)

$$u=\frac{F[0,s_1\{u(0,x_2,\ldots,x_n)\},\ldots,s_n\{u(x_1,\ldots,0)\},s_1\{u_{x_2}(0,x_2,\ldots,x_n)\}+s_2\{u_{x_1}(x_1,0,\ldots,x_n)\}+s_1s_2\{u(0,0,\ldots,x_n)\},\ldots]+\{\phi(x_1,x_2,\ldots,x_n)\}}{F(1,s_1,\ldots,s_n,s_1s_2,\ldots)}. \tag{5.2.12}$$

However there exist severer restrictions among the initial values, external function ϕ and function F than stated in Appendix D in order that u is the unique solution satisfying the given conditions in the domain D and if these restrictions are not satisfied, u has no solution[11].

It is very difficult to give them in the general forms and we must investigate them in every given case.

In case of two variables, these conditions must be given so as to remove non-logarithmic factors and inadequate factors to be the function in the domain D and if they exist in the dominator of (5.2.12), they must be canceled by the numerator and be removed entirely from the operational solution. In such a case, initial conditions are not independent and number of them is necessary to remove the unexpected factors.

5.3 Application to telegraphic equations

5.3.1 Operational solutions of telegraphic equations Here we shall consider the following equations in the domain $D(\lambda\geq 0, t\geq 0)$.

$$\left.\begin{aligned} -\frac{\partial\{v(\lambda,t)\}}{\partial\lambda}&=L\frac{\partial\{i(\lambda,t)\}}{\partial t}+R\{i(\lambda,t)\}\\ -\frac{\partial\{i(\lambda,t)\}}{\partial\lambda}&=C\frac{\partial\{v(\lambda,t)\}}{\partial t}+G\{v(\lambda,t)\}\end{aligned}\right\}. \tag{5.3.1}$$

THe voltage equation becomes

$$\frac{\partial^2}{\partial t^2}\{v(\lambda,t)\}=LC\frac{\partial^2}{\partial t^2}\{v(\lambda,t)\}+(RC+LG)\frac{\partial}{\partial t}\{v(\lambda,t)\}+RG\{v(\lambda,t)\} \tag{5.3.2}$$

and let us consider the equation with the conditions

$$\left.\begin{array}{ll}\{v(0,t)\}=\{v_1(t)\} & \{v_\lambda(0,t)\}=\{v_2(t)\} \\ \{v(\lambda,0)\}=\{v_3(\lambda)\} & \{v_t(\lambda,0)\}=\{v_4(\lambda)\}\end{array}\right\}. \quad (5.3.3)$$

Denote suffix with respect to λ, t by 1,2, then the operational equation becomes

$$s_1^2 v - s_1\{v_2(t)\} - s_1^2\{v_1(t)\} = LC[s_2^2 v - s_2\{v_4(\lambda)\} - s_2^2\{v_3(\lambda)\}] + (RC+LG)$$
$$[s_2 v - s_2\{v_3(\lambda)\}] + RGv \quad (5.3.4)$$

and the voltage solution becomes

$$v = \frac{s_1^2\{v_1(t)\} + s_1\{v_2(t)\} + [LCs_2^2 + (RC+LG)s_2]\{v_3(\lambda)\} - LCs_2\{v_4(\lambda)\}}{s_1^2 - [LCs_2^2 + (LG+RC)s_2 + RG]}. \quad (5.3.5)$$

The denominator is solved into the factors as follows

$$[s_1 + \sqrt{LCs_2^2 + (LG+RC)s_2 + RG}][s_1 - \sqrt{LCs_2^2 + (LG+RC)s_2 + RG}]$$

and the second factor is inadequate in order that v becomes the function in D, therefore, it must be removed from (5.3.5) by substituting the conditions

$$\left.\begin{array}{ll}\{v_1(t)\}=\ell V_1(s_2) & \{v_2(t)\}=\ell V_2(s_2) \\ \{v_3(\lambda)\}=\ell V_3(s_1) & \{v_4(\lambda)\}=\ell V_4(s_1)\end{array}\right\}. \quad (5.3.6)$$

These conditions must be given so as to remove the inadequate factor

$$s_1 - \sqrt{LCs_2^2 + (LG+RC)s_2 + RG}$$

but, it is pretty difficult to give them in general forms because in every pair of the coefficient matrices, they are not always commutative, so we shall consider some special cases.

To remove the unexpected factors, more than one non-zero initial condition ~~is~~ are necessary and one of them is determined by others.

5.3.2 Wave type equations When R and G can be neglected, (5.3.5) becomes

$$\left.\begin{array}{l} v = \dfrac{s_1^2\{v_1(t)\} + s_1\{v_2(t)\} - Q^2 s_2^2\{v_3(\lambda)\} - Q^2 s_2\{v_4(\lambda)\}}{s_1^2 - Q^2 s_2^2} \\ Q^2 = LC \end{array}\right\}. \quad (5.3.7)$$

The matrix Q is positive definite and ~~symmetric~~ can be transformed into the diagonal matrix by a similar transformation, then it can be written as

$$Q=P^{-1}\mathrm{diag}[\alpha_1,\alpha_2,\ldots\ldots,\alpha_n]P \tag{5.3.8}$$

therefore, we have

$$\left.\begin{aligned} v&=P^{-1}\mathrm{diag}\Big[\frac{1}{(s_1+\alpha_1 s_2)(s_1-\alpha_1 s_2)},\frac{1}{(s_1+\alpha_2 s_2)(s_1-\alpha_2 s_2)},\ldots\ldots\\ &\quad\ldots,\frac{1}{(s_1+\alpha_n s_2)(s_1-\alpha_n s_2)}\Big]V(s_1,s_2)\\ V(s_1,s_2)&=\ell P[s_1^2V_1(s_2)+s_1V_2(s_2)-Q^2s_2^2V_3(s_1)-Q^2s_2V_4(s_1)]\end{aligned}\right\}. \tag{5.3.9}$$

The restriction among the initial values is given by

$$V_i(\alpha_i s_2,s_2)=0 \text{ or } V_i(s_1,s_1/\alpha_i)=0 \quad i=1,2,..,n \tag{5.3.10}$$

where V_i is the i-th element of V shown as

$$V(s_1,s_2)=[V_1(s_1,s_2),V_2(s_1,s_2),\ldots\ldots,V_n(s_1,s_2)]^t.$$

Let us write it

$$\left.\begin{aligned} V(s_1,s_2)&=\ell[s_1^2A(s_2)+s_1B(s_2)+s_2^2C(s_1)+s_2D(s_1)]\\ A=PV_1 \quad B&=PV_2 \quad C=-PQ^2V_3 \quad D=-PQ^2V_4\end{aligned}\right\}. \tag{5.3.11}$$

then we have

$$\left.\begin{aligned} &\alpha^2s_2A(s_2)+\alpha B(s_2)+s_2C(\alpha s_2)+D(\alpha s_2)=0\\ \text{or}\\ &s_1A(s_1/\alpha)+B(s_1/\alpha)+s_1C(s_1)/\alpha^2+D(s_1)/\alpha=0\end{aligned}\right\} \tag{5.3.12}$$

for every i-th element where suffix i is omitted and the initial conditions must be given so as to satisfy either of the above restrictions.

Case 1. A, B and C are given.

From the second relation of (5.3.12), D is determined by

$$D(s_1)=-\alpha[s_1A(s_1/\alpha)+B(s_1/\alpha)+s_1C(s_1)/\alpha^2]$$

and v becomes

$$v=\frac{l}{(s_1+\alpha s_2)(s_1-\alpha s_2)}[s_1^2A(s_2)+s_1B(s_2)+s_2^2C(s_1)-\alpha s_1s_2A(s_1/\alpha)$$
$$-\alpha s_2B(s_1/\alpha)-s_1s_2C(s_1)/\alpha]. \tag{5.3.13}$$

Expand it to partial fraction, then

$$l\frac{s_1^2A(s_2)-\alpha s_1s_2A(s_1/\alpha)}{(s_1+\alpha s_2)(s_1-\alpha s_2)}=\frac{l}{2}\frac{s_1A(s_2)}{s_1+\alpha s_2}+\frac{l}{2}\frac{\alpha s_2A(s_1/\alpha)}{s_1+\alpha s_2}+\frac{l}{2}\frac{s_1A(s_2)-\alpha s_1A(s_1/\alpha)}{s_1-\alpha s_2}.$$

For a function $\{A(x)\}$ which is equal to 0 in $x<0$, let $x=t-\alpha\lambda$, then for $t>\alpha\lambda$

$$\frac{\partial}{\partial\lambda}\{A(t-\alpha\lambda)\}=-\alpha\frac{\partial}{\partial t}\{A(t-\alpha\lambda)\}$$

or operationally

$$s_1\{A(t-\alpha\lambda)\}+s_1\{A(t)\}=-\alpha s_2\{A(t-\alpha\lambda)\}$$

therefore,

$$\frac{s_1\{A(t)\}}{s_1+\alpha s_2}=\{A(t-\alpha\lambda)\} \qquad t>\alpha\lambda. \tag{5.3.14}$$

Let $x=\alpha\lambda-t$, then for $t<\alpha\lambda$

$$\frac{\partial}{\partial\lambda}\{A(\alpha\lambda-t)\}=-\alpha\frac{\partial}{\partial t}\{A(\alpha\lambda-t)\}$$

or operationally

$$s_1\{A(\alpha\lambda-t)\}=-\alpha s_2\{A(\alpha\lambda-t)\}+\alpha s_2\{A(\alpha\lambda)\}$$

therefore,

$$\frac{\alpha s_2\{A(\alpha\lambda)\}}{s_1+\alpha s_2}=\{A(\alpha\lambda-t)\} \quad t<\alpha\lambda. \tag{5.3.15}$$

Let $x=\alpha\lambda+t$, then

$$s_1\{A(\alpha\lambda+t)\}+s_1\{A(t)\}=\alpha s_2\{A(\alpha\lambda+t)\}+\alpha s_2\{A(\alpha\lambda)\}$$

therefore,

$$\frac{s_1\{A(t)\}-\alpha s_2\{A(\alpha\lambda)\}}{s_1-\alpha s_2}=\{A(\alpha\lambda+t)\}. \tag{5.3.16}$$

By these results, we have

$$\frac{l}{2}\,\frac{s_1A(s_2)+\alpha s_2A(s_1/\alpha)}{s_1+\alpha s_2}=\frac{1}{2}\{A(|t-\alpha\lambda|)\}$$

$$\frac{l}{2}\,\frac{s_1A(s_2)-\alpha s_2A(s_1/\alpha)}{s_1-\alpha s_2}=\frac{1}{2}\{A(t+\alpha\lambda)\}.$$

$$l\frac{s_1B(s_2)-\alpha s_2B(s_1/\alpha)}{(s_1+\alpha s_2)(s_1-\alpha s_2)}=\frac{l}{2}\,\frac{l_1[s_1B(s_2)+\alpha s_2B(s_1/\alpha)]}{s_1+\alpha s_2}+\frac{l}{2}\,\frac{l_1[s_1B(s_2)-\alpha s_2B(s_2/\alpha)]}{s_1-\alpha s_2}$$

$$=\frac{1}{2\alpha}\left\{\int_{t-\alpha\lambda}^{t}B(\eta)\,d\eta+\int_0^{\alpha\lambda-t}B(\eta)\,d\eta+\int_t^{t+\alpha\lambda}B(\eta)\,d\eta\right\}=\frac{1}{2\alpha}\left\{\int_{t-\alpha\lambda}^{t+\alpha\lambda}B(\eta)\,d\eta+\int_0^{\alpha\lambda-t}B(\eta)\,d\eta\right\}$$

where $B(0)=0$ and

$$l\frac{s_2^2C(s_1)-s_1s_2C(s_1)/\alpha}{(s_1+\alpha s_2)(s_1-\alpha s_2)}=-l\frac{s_2C(s_1)}{\alpha(s_1+\alpha s_2)}=\frac{-1}{\alpha^2}\{C(\lambda-t/\alpha)\} \qquad \lambda>\alpha/t.$$

Finally we have the following result.

$$\{v(\lambda,t)\}=\frac{1}{2\alpha^2}\{\alpha^2A(|t-\alpha\lambda|)+\alpha^2A(t+\alpha\lambda)+\alpha\int_{t-\alpha\lambda}^{t+\alpha\lambda}B(\eta)d\eta$$

$$+\alpha\int_0^{\alpha\lambda-t}B(\eta)d\eta-2C(\lambda-t/\alpha)\}. \qquad (5.3.17)$$

Case 2. A, C and D are given.

From the first relation of (5.3.12), B is determined by

$$B(s_2)=-\frac{1}{\alpha}[\alpha^2 s_2A(s_2)+s_2C(\alpha s_2)+D(\alpha s_2)]$$

and v becomes

$$v=\frac{l}{(s_1+\alpha s_2)(s_1-\alpha s_2)}[s_1^2A(s_2)+s_2^2C(s_1)+s_2D(s_1)-\alpha s_1s_2A(s_2)$$

$$-s_1s_2C(\alpha s_2)/\alpha-s_1D(\alpha s_2)/\alpha]. \qquad (5.3.18)$$

In the same way as Case 1, we have

$$\frac{l}{(s_1+\alpha s_2)(s_1-\alpha s_2)}[s_1^2A(s_2)-\alpha s_1s_2A(s_2)]=l\frac{s_1A(s_2)}{s_1+\alpha s_2}=\{A(t-\alpha\lambda)\}$$

$$\frac{l}{(s_1+\alpha s_2)(s_1-\alpha s_2)}[s_2^2C(s_1)-s_1s_2C(\alpha s_2)/\alpha]$$

$$=-\frac{l}{2\alpha^2}\,\frac{s_2C(s_1)-s_1C(\alpha s_2)/\alpha}{s_2+s_1/\alpha}-\frac{l}{2\alpha^2}\,\frac{s_2C(s_2)-s_1C(\alpha s_2)/\alpha}{s_2-s_1/\alpha}$$

$$=-\frac{1}{2\alpha^2}\{C(|t/\alpha-\lambda|)+C(t/\alpha+\lambda)\}$$

and

$$\frac{l}{(s_1+\alpha s_2)(s_1-\alpha s_2)}[s_2D(s_1)-s_1D(\alpha s_2)/\alpha]$$

$$=-\frac{l}{2\alpha^2}\,\frac{l_2[s_2D(s_1)-s_1D(\alpha s_2)/\alpha]}{s_2+s_1/\alpha}-\frac{l}{2\alpha^2}\,\frac{l_2[s_2D(s_1)-s_1D(\alpha s_2)/\alpha]}{s_2-s_1/\alpha}$$

$$=-\frac{1}{2\alpha}\{\int_{t/\alpha-\lambda}^{t/\alpha+\lambda}D(\eta)d\eta+\int_0^{\lambda-t/\alpha}D(\eta)d\eta\}. \quad D(0)=0.$$

Finally we have the following result.

$$\{v(\lambda,t)\}=-\frac{1}{2\alpha^2}\{C(|t/\alpha-\lambda|)+C(t/\alpha+\lambda)+\alpha\int_{t/\alpha-\lambda}^{t/\alpha+\lambda}D(\eta)d\eta+\alpha\int_0^{\lambda-t/\alpha}D(\eta)d\eta$$

$$-2\alpha^2A(t-\alpha\lambda)\} \qquad (5.3.19)$$

In the remaining cases we can get the solutions in the same way by using either relation of (5.3.12).

Specially when the initial values are given only on the λ-axis as

$$\{v(0,t)\}=\{v_1(t)\} \qquad \{v_\lambda(0,t)\}=\{v_2(t)\}$$

then (5.3.7) becomes

$$v=\ell\frac{s_1^2 V_1(s_2)-s_1 V_2(s_2)}{s_1^2-Q^2 s_2^2}. \qquad (5.3.20)$$

To remove the inadequate factor s_1-Qs_2, the following relation must be held

$$Q^2 V_1(s_2)-s_1 V_2(s_2)=0$$

and (5.3.20) becomes

$$v=\frac{s_2 V_1(s_2)}{s_1+Qs_2} \qquad (5.3.21)$$

and

$$v(\lambda,s_2)=e^{-\lambda Q s_2}V_1(s_2)$$

and finally it becomes the same result as (4.4.13).

5.3.3 Heat conduction type equations When L and G can be neglected (5.3.5) becomes

$$\left.\begin{aligned} v&=\frac{s_1^2\{v_1(t)\}+s_1\{v_2(t)\}+Q^2 s_2\{v_3(\lambda)\}}{s_1^2-Q^2 s_2} \\ Q^2&=RC \end{aligned}\right\}. \qquad (5.3.22)$$

can be transformed into the diagonal matrix by a similar transformation

The matrix Q is positive definite and ~~symmetric~~, then it can be written as (5.3.8), therefore we have

$$\left.\begin{aligned} v=P^{-1}\mathrm{diag}\Big[&\frac{1}{(s_1+\alpha_1\sqrt{s_2})(s_1-\alpha_1\sqrt{s_2})},\frac{1}{(s_1+\alpha_2\sqrt{s_2})(s_1-\alpha_2\sqrt{s_2})},\cdots \\ &\cdots,\frac{1}{(s_1+\alpha_n\sqrt{s_2})(s_1-\alpha_n\sqrt{s_2})}\Big]V(s_1,s_2) \\ V(s_1,s_2)&=\ell[s_1^2 A(s_2)+s_1 B(s_2)+s_2 C(s_1)] \\ A&=PV_1 \qquad B=PV_2 \qquad C=PQ^2V_3 \end{aligned}\right\}. \qquad (5.3.23)$$

The restriction among the initial values is given as follows to remove the factor $s_1-Q\sqrt{s_2}$ for every element of V.

$$\alpha^2 s_2 A(s_2)+\alpha\sqrt{s_2}B(s_2)+s_2 C(\alpha\sqrt{s_2})=0 \qquad (5.3.24)$$

Case 1. A and C are given.

From the relation (5.3.24) B is determined by

$$B(s_2)=-[\alpha\sqrt{s_2}A(s_2)-\sqrt{s_2}C(\alpha\sqrt{s_2})/\alpha]$$

and v becomes

$$v=\frac{l}{(s_1+\alpha\sqrt{s_2})(s_1-\alpha\sqrt{s_2})}[(s_1^2-\alpha s_1\sqrt{s_2})A(s_2)+s_2C(s_1)$$

$$-s_1\sqrt{s_2}C(\alpha\sqrt{s_2})/\alpha]. \qquad (5.3.25)$$

Expanding it to the partial fractions, we have

$$\frac{l}{(s_1+\alpha\sqrt{s_2})(s_1-\alpha\sqrt{s_2})}[(s_1^2-\alpha s_1\sqrt{s_2})A(s_1)]=l\frac{s_1A(s_1)}{s_1+\alpha\sqrt{s_2}}$$

and

$$\frac{l}{(s_1+\alpha\sqrt{s_2})(s_1-\alpha\sqrt{s_2})}[s_2C(s_1)-s_1\sqrt{s_2}C(\alpha\sqrt{s_2})/\alpha]=-\frac{l}{2}\,\frac{\sqrt{s_2}C(s_1)}{\alpha(s_1+\alpha\sqrt{s_2})}$$

$$-\frac{l}{2}\,\frac{\sqrt{s_2}C(\alpha\sqrt{s_2})}{\alpha(s_1+\alpha\sqrt{s_2})}+\frac{l}{2}\,\frac{\sqrt{s_2}C(s_1)}{\alpha(s_1-\alpha\sqrt{s_2})}-\frac{l}{2}\,\frac{\sqrt{s_2}C(\alpha\sqrt{s_2})}{\alpha(s_1-\alpha\sqrt{s_2})}\,.$$

Let us consider the operational function $F(\lambda)=\{e^{-\lambda\alpha\sqrt{s_2}}A(t)\}$, then

$$\frac{\partial}{\partial\lambda}F=-\alpha\sqrt{s_2}F$$

or operationally

$$s_1F-s_1\{A(t)\}=-\alpha\sqrt{s_2}F$$

therefore, we have

$$\frac{s_1\{A(t)\}}{s_1+\alpha\sqrt{s_2}}=F=\{\int_0^t\phi(\lambda\alpha,t-\tau)A(\tau)d\tau\} \qquad (5.3.26)$$

where

$$\phi(\lambda\alpha,t)=\frac{\lambda\alpha}{2\sqrt{\pi t^3}}\exp\left(-\frac{\lambda^2\alpha^2}{4t}\right).$$

Let F the following function

$$F=\{\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{\lambda^2\alpha^2}{4t}\right)\}\{C(\lambda)\}$$

then from the law of differentiation for the convolution*

$$\frac{d}{dx}\{\int_0^x Z(x-\tau)f(\tau)d\tau\}=\{\int_0^x Z'(x-\tau)f(\tau)d\tau\}+\{Z(0)f(x)\} \qquad (5.3.27)$$

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*See I-5 Chapter III

we have

$$\frac{\partial}{\partial\lambda}\{F(\lambda\alpha,t)\}=\{-\frac{\lambda\alpha}{2\sqrt{\pi t^3}}\exp(-\frac{\lambda^2\alpha^2}{4t})\}\{C(\lambda)\}+\{\frac{1}{\sqrt{\pi t}}C(\lambda)\}$$

or operationally

$$s_1F=-\alpha\sqrt{s_2}F+\sqrt{s_2}\{C(\lambda)\}$$

where $C(0)=0$, therefo~~er~~re, we have

$$\frac{\sqrt{s_2}\{C(\lambda)\}}{s_1+\alpha\sqrt{s_2}}=F=\{\int_0^\lambda \chi(\overline{\lambda-\xi}\alpha,t)C(\xi)d\xi\} \tag{5.3.28}$$

where

$$\chi(\lambda\alpha,t)=\frac{1}{\sqrt{\pi t}}\exp(-\frac{\lambda^2\alpha^2}{4t}).$$

Let us consider the function

$$F=\{\int_0^\infty \chi(\overline{\lambda+\xi}\alpha,t)C(\xi)d\xi\}$$

then we have

$$\frac{\partial F}{\partial\lambda}=\{\int_0^\infty \frac{\partial}{\partial\lambda}\chi(\overline{\lambda+\xi}\alpha,t)C(\xi)d\xi\}=\alpha\{\int_0^\infty \phi(\overline{\lambda+\xi}\alpha,t)C(\xi)d\xi\}$$

or operationally

$$s_1F+s_1\{\int_0^\infty \chi(\xi\alpha,t)C(\xi)d\xi\}=\alpha\sqrt{s_2}F.$$

From the relation

$$l\sqrt{s_2}\,C(\alpha\sqrt{s_2})=s_1\{\int_0^\infty e^{-\alpha\sqrt{s_2}\xi}C(\xi)d\xi\} \tag{5.3.29}$$

we have

$$l\frac{\sqrt{s_2}C(\alpha\sqrt{s_2})}{s_1+\alpha\sqrt{s_2}}=\{\int_0^\infty \chi(\overline{\lambda+\xi}\alpha,t)C(\xi)d\xi\}. \tag{5.3.30}$$

In the same way we have

$$l\frac{\sqrt{s_2}[C(s_1)-C(\alpha\sqrt{s_2})]}{s_1-\alpha\sqrt{s_2}}=\{\int_\infty^\lambda \chi(\overline{\xi-\lambda}\alpha,t)C(\xi)d\xi\} \tag{5.3.31}$$

and the solution becomes as follows.

$$\{v(\lambda,t)\}=-\frac{1}{2\alpha}[\int_0^\infty [\chi(\overline{\lambda-\xi}\alpha,t)+\chi(\overline{\lambda+\xi}\alpha,t)]C(\xi)d\xi$$
$$+2\alpha\int_0^t \phi(\lambda\alpha,t-\tau)A(\tau)d\tau\} \tag{5.3.32}$$

case 2. B and C are given.

From the relation (5.3.24) A is determined by

$$A(s_2)=-C(\alpha\sqrt{s_2})/\alpha^2-B(s_2)/\alpha\sqrt{s_2}$$

and v becomes

$$v=\frac{l}{(s_1+\alpha\sqrt{s_2})(s_1-\alpha\sqrt{s_2})}[s_2C(s_2)-s_1^2C(\alpha\sqrt{s_2})/\alpha^2+(s_1-s_1/\alpha\sqrt{s_2})$$
$$B(s_2)]. \qquad (5.3.33)$$

In the same way as Case 1, we have

$$\{v(\lambda,t)\}=-\frac{1}{2\alpha^2}\{\int_0^\infty[\chi(\overline{\lambda+\xi}\alpha,t)+\chi(\overline{\lambda-\xi}\alpha,t)]C(\xi)d\xi$$
$$+2\alpha\int_0^t\chi(\lambda\alpha,t-\tau)B(\tau)d\tau\}. \qquad (5.3.3\cancel{7}4)$$

Specially when the initial values are given only on the λ-axis as

$$\{v(0,t)\}=\{v_1(t)\} \quad \{v_\lambda(0,t)\}=\{v_2(t)\}$$

(5.3.22) becomes

$$v=\ell\frac{s_1^2V_1(s_2)-s_1V_2(s_2)}{s_1^2-Q^2s_2}. \qquad (5.3.35)$$

To remove the inadequate factor $s_1-Q\sqrt{s_2}$, the following relation must be held.

$$Q\sqrt{s_2}V_1(s_2)-V_2(s_2)=0$$

and (5.3.35) becomes

$$v=\frac{s_2V_1(s_2)}{s_1+Q\sqrt{s_2}} \qquad (5.3.36)$$

and

$$v(\lambda,s_2)=e^{-\lambda Q\sqrt{s_2}}V_1(s_2)$$

and finally becomes the same result as (4.4.20).

5.4 ~~s~~Summary

In this chapter the method of multi-dimensional operational calculus and its application to the analysis of the multi-dimensional partial differential equations were given.

The procedure to remove the unexpected factors from the given operational solutions by the multi-dimensional operational calculus is shown for fairly simple cases, for examples the telegraphic equations.

6. Numerical inversion of Laplace transform

6.1 Preface

As is known from the results given hitherto, the reason why the operational calculus is used as a powerful tool in analysing the physical system is the fact that it decreases the degree of transcendency of the system equations and makes it easy to analyse them.

The method of the operational calculus has its origin to the intuitive symbolic method constructed by Heaviside.

In the engineering applications the method of the Laplace transform has been used widely, but there were some doubts whether the Laplace transform is mathematically strict or not until the theory of distribution was made by Schwartz.

The operational calculus by Miksinsky contains the results given by the theory of distribution and is a more general method than the Laplace transform.

In applying the operational calculus, it is pretty easy to get the operational solution for a physical system, but it is always not so easy to get the corresponding time solution by inverting the operational solution. But in such a case, it is sufficient for the engineering applications that the numerical values of the time solution can be given from the operational solution by any way instead of getting it analytically

For such a problem as to get a numerical solution the Laplace transform is more suitable than the operational calculus of Miksinsky because the numerical correspondence between the time t and the operator s is given.

Hence it is the purpose of this chapter to show a method of numerical inversion of the Laplace transform as the tool for such a problem.

First we shall show the relation between the operational calculus and the Laplace transform and show the fact that the relation is mathematically isomorphism under certain conditions and show that the operator s corresponds to a certain set of complex numbers.

For such a set of complex numbers, the Laplace transform is converted into the Fourier transform, therefore, we shall show some fundamental matters which are necessary in the Fourier analysis and then we shall show the principal method of inverting the Laplace transform numerically by the Fourier series technique and give some numerical examples.

The week point of this method is that it needs long executive time and large size memory in digital computation, but it can be avoided by introducing the Fast Fourier Transform method.

Finally we shall compare this method with other numerical inversion methods of the Laplace transform and show that this method is the most practical method with respect to accuracy and generality and keeps the sprit of the operational calculus.

The principle of the method and the algorithm in calculation of the Fast Fourier Transform will be shown in Appendix E.

6.2 Relation between operational calculus and Laplace transform

Let us consider a locally integrable complex valued function $\{x(t)\}$ of real variable t which is equal to 0 in $t<0$. For this function, we shall introduce the analytical function of complex variable s defined by the following integral transformation

$$X(s)=\int_0^{\infty} x(t)e^{-st}dt \qquad (6.2.1)$$

which is called the Laplace transform of $\{x(t)\}$.

The theory of the Laplace transform is completed by the theory of distribution made by Schwartz and so far as the integral (6.2.1) is convergent, the complex number s in (6.2.1) corresponds to the differential operator of Miksinsky and almost every result given by the operational calculus can be adapted to the Laplace transform.

This fact is called mathematically isomorphism and in formal calculations the function of the variable s can be treated in the same way whether it denotes an analytical function or an operator.

But in applying the Laplace transform, it is not ascertained whether the obtained solution is unique or not without the assumption that every function treated during the calculation must be transformable by (6.2.1). In such a case that the existence and the uniqueness of the solution cannot be proved, it may be regarded as the solution if the result given formally by the Laplace transform is practically appropriate in engineering applications.

Now we shall give a more detailed discussion to the relation between the operational calculus and the Laplace transform.

As is shown in Appendix A, in the operational calculus we have treated the operators which are operationally convergent to a set C of continuous functions of real variable t in the interval $0\le t<\infty$, which is given by the mapping b consisting of multiplication of a positive continuous function $b(t)$ to every element of the set B of bounded continuous functions.

In the Laplace transform we treat the functions which are operationally convergent to the set C' of continuous functions that is given by the mappinf b' consisting of multiplication of a function $b'(t)=e^{at}$ (a positive) to every element of the set B' satisfying the condition

$$|x(t)|<Me^{-at}. \qquad (6.2.2)$$

Then for a function $\{x(t)\}$ deduced from C', the integral (6.2.1) is convergent in the domain

$$\mathrm{Re}(s)<a$$

and the following relations hold.

$$B'\subset B \quad C'\subset C \quad b'\subset b \tag{6.2.3}$$

Therefore the Laplace transform is included thoroughly in the operational calculus of Miksinsky and for the transformable functions, all the results given by the operational calculus are also held.

6.3 Properties in Fourier analysis

6.3.1 Fourier transform For a function $\{x(t)\}$ of real variable t which is equal to 0 in $t<0$, the Laplace transform and its inversion formula are defined as follows

$$X(s)=\int_0^\infty x(t)e^{-st}dt \tag{6.3.1}$$

$$\{x(t)\}=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} X(s)e^{st}ds \tag{6.3.2}$$

where a is an arbitrary positive number chosen so that it is greater than the real part of the singularity of $X(s)$.

Here we shall omit the function braces { } in the discussion hereafter.

Put $s=a+i\omega$, then

$$X(a+i\omega)=\int_0^\infty x(t)e^{-at}e^{-i\omega t}dt \tag{6.3.3}$$

$$x(t)=\frac{e^{at}}{2\pi}\int_{-\infty}^{\infty} X(a+i\omega)e^{i\omega t}d\omega. \tag{6.3.4}$$

These equations are the Forier transform of $x(t)e^{-at}$ and its inversion, therefore, we shall show some basic properties of the Fourier transform which are necessary in our discussions.

First we shall show the following theorems*.

<< *Let $L^1(-\infty,\infty)$ is a set of integrable functions in the interval $-\infty<t<\infty$ in the sens of Lebesgue and consider a function $x(t)$ of L^1 which has bounded variation in any finite interval. Define*

$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \tag{6.3.5}$$

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*See I-8 Chapter I

then

$$\frac{x(t+0)+x(t-0)}{2}=\lim_{A\to\infty}\frac{1}{2\pi}\int_{-A}^{A}X(\omega)e^{i\omega t}d\omega \tag{6.3.6}$$

and if $X(\omega)$ is summable and $x(t)$ is continuous

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{i\omega t}d\omega. \tag{6.3.7}$$ >>

<< *Consider a summable function $x(t)$ which has two continuous summable derivatives $x'(t)$ and $x''(t)$, then*

$$\int_{-\infty}^{\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2d\omega. \tag{6.3.8}$$ >>

This property is correct only on the assumption $x(t)\in L^2(-\infty,\infty)$ where $L^2(-\infty,\infty)$ is the set of functions whose square is integrable in the sense of Lebesgue in $-\infty<t<\infty$. In the case of $x(t)\in L^1(-\infty,\infty)$, the Fourier transform $X(\omega)$ is not always summable, but if we interpret the limit in (6.3.6) in the sense of distribution, there exist Fourier transform and its inversion.

In our case the function to be considered is confined to the class of

$$\lim_{t\to\infty}x(t)e^{-at}=0$$

therefore, the Fourier transform $X(a+i\omega)$ in (6.3.3) is summable.

6.3.2 Fourier series As will be shown below, the Laplace transform and its inversion are approximated by the Fourier series of the infinite series of functions with period $2T$, therefore we shall show some basic properties of the Fourier series which are necessary in our discussions.

First we shall show the following theorems*.

<< *Consider a periodic function $x(t)$ of period $2T$ which is summable in $0\leq t\leq 2T$ and has a bounded variation in the neighborhood of $t=t$, then for the partial sum of Fourier series*

$$S_n(t)=\sum_{k=-n}^{n}X_k e^{ik(\pi/T)t} \tag{6.3.9}$$

where

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* See I-8 Chapter I

$$X_k=\frac{1}{2T}\int_0^{2T} x(t)e^{-ik(\pi/T)t}dt \tag{6.3.10}$$

the following relation yields.

$$\lim_{t\to\infty} S_n(t)=\frac{x(t+0)+x(t-0)}{2} \tag{6.3.11}$$ >>

Denote

$$x(t)=\sum_{k=-\infty}^{\infty} X_k e^{ik(\pi/T)t} \tag{6.3.12}$$

then by the above theorem it becomes as follows.

$$x(t)=\begin{cases} x(t) & t\text{: continuous point of } x(t) \\ \frac{x(t+0)+x(t-0)}{2} & t\text{: discontinuous point of } x(t) \end{cases}$$

<< *For* (6.3.12) *the relation*

$$\|x\|_{L^2}=\|X\|_{l^2} \tag{6.3.13}$$

holds where

$$\|x\|_{L^2}=[\int_0^{2T}|x(t)|^2dt]^{1/2}$$

$$\|X\|_{l^2}=[\sum_{k=-\infty}^{\infty}|X_k|^2]^{1/2}.$$ >>

By the above theorem, the Fourier series expansion of $x(t)$ defines an isometric operator from the L^2-space to l^2-space and all the properties of the topological space related to the algebraic operations and the norm.

6.3.3 Finite Fourier approximation Here we shall consider the case where a function to be integrated is arbitrary and we cannot get its Fourier coefficients analytically.

In such a case the Fourier coefficients must be given numerically. Then let us consider a function $x(t)$ which is sectionally continuous except finite points in the interval $0\le t\le 2T$, then Lebesgue integral becomes equivalent to Riemann integral and can be approximated by the infinite series.

In this case the function given by the Fourier series expansion is not $x(t)$ but $\frac{1}{2}[x(t-0)+x(t+0)]$ at discontinuous points.

The relation (6.3.10) is given by an integral, therefore, numerical value of $x(t)$ at an arbitrary fixed point t does not give any influence, but in approximating the integral by the infinite series of $2N$ samples, the following relation must be used to make correspondence numerically to the inverse transform

$$X_k=\lim_{N\to\infty}\frac{1}{2N}\sum_{n=0}^{2N-1}\frac{x(nT/N+0)+x(nT/N-0)}{2}e^{-ink\pi/N}. \qquad (6.3.14)$$

When $x(t)$ is equal to 0 in $t<0$ and continuous in $0\le t<2T$, it becomes

$$X_k=\lim_{N\to\infty}\frac{1}{2N}\Big[\sum_{n=0}^{2N-1}x(nT/N)e^{-ink\pi/N}-\frac{1}{2}x(0)\Big] \qquad (6.3.15)$$

and approximating it by finite sum of the infinite series, we have

$$X_k=\frac{1}{2N}\Big[\sum_{n=0}^{2N-1}x(nT/N)e^{-ink\pi/N}-\frac{1}{2}x(0)\Big]. \qquad (6.3.16)$$

If $x(t)$ is a real function, from the periodicity of complex exponential function $e^{-ink\pi/N}$, we have

$$X_{2N-k}=X_k^{*} \qquad (6.3.17)$$

where X_k^{*} is the conjugate of X_k, and we can get only N coefficients X_k $(k=1,2,\ldots.,N-1)$.

This effect is called aliase* and for band limited signals this phenomenon is famous in the sampling theory.

6.4 Numerical inversion of Laplace transform

6.4.1 Principle of numerical inversion[12),13)]

When a matrix of functions $x(t)$ is given, the Laplace transform and its inversion are given as follows

$$X(a+i\omega)=\int_0^{\infty}x(t)e^{-at}e^{-i\omega t}dt \qquad (6.4.1)$$

$$x(t)=\frac{e^{at}}{2\pi}\int_{-\infty}^{\infty}X(a+i\omega)e^{i\omega t}d\omega \qquad (6.4.2)$$

When the Laplace transform $X(s)$ is known, its time solution at any point t can be given numerically by evaluating the integral (6.4.2).

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*See I-9 Chapter 4

Approximately it becomes

$$x(t) \simeq \frac{e^{at}}{2\pi} \sum_{k=-\infty}^{\infty} X(a+ik\Delta\omega)\, e^{ik\Delta\omega t} \Delta\omega \qquad (6.4.3)$$

and by substituting $\Delta\omega=\pi/T$, it becomes

$$x(t) \simeq \frac{e^{at}}{2T} \sum_{k=-\infty}^{\infty} X(a+ik\pi/T)\, e^{ik(\pi/T)t}. \qquad (6.4.4)$$

Here we shall introduce the following sequence of periodic functions of period $2T$.

$$\left.\begin{array}{ll} x_n(t)=h(t+2nT) & h(t)=x(t)e^{-at} \\ 0\le t<2T & n=0,1,2,\ldots. \end{array}\right\} \qquad (6.4.5)$$

$$\left.\begin{array}{ll} x_n(t)=x_n(t-2mT) & 2mT<t<2(m+1)T \\ m=0,1,2,\ldots. & \end{array}\right\} \qquad (6.4.6)$$

Every function $x_n(t)$ is a periodic function, therefore, it can be expanded to the Fourier series of the following form

$$x_n(t)=\sum_{k=-\infty}^{\infty} C_{n,k} e^{ik(\pi/T)t} \qquad (6.4.7)$$

and Fourier coefficient $C_{n,k}$ is given as

$$C_{n,k}=\frac{1}{2T}\int_0^{2T} x_n(t)\, e^{-ik(\pi/T)t} dt. \qquad (6.4.8)$$

Next we shall introduce the following infinite series.

$$x_p(t)=e^{at}\sum_{n=0}^{\infty} x_n(t) \qquad (6.4.9)$$

Substitute (6.4.5),(6.4.7) and (6.4.8) into (6.4.9), then it becomes

$$x_p(t)=\frac{e^{at}}{2T}\sum_{k=-\infty}^{\infty}\left(\int_0^{\infty} x(t)\, e^{-at} e^{-ik(\pi/T)t} dt\right) e^{ik(\pi/T)t}$$

and by (6.4.1) it becomes

$$x_p(t)=\frac{e^{at}}{T}\sum_{k=-\infty}^{\infty} X(a+ik\pi/T)\, e^{ik(\pi/T)t} \qquad (6.4.10)$$

and by (6.4.4) the following relation yields.

$$x(t) \simeq x_p(t)$$

The relation between $x(t)$ and $x_p(t)$ is given precisely as follows.

$$\left.\begin{array}{l} x_p(t)=e^{at}\sum\limits_{n=0}^{\infty} x_n(t)= x(t)+\sum\limits_{n=1}^{\infty} e^{-2anT} x(t+2nT) \\ 0\le t<2T \end{array}\right\} \quad (6.4.11)$$

By considering the relation

$$X(a+ik\pi/T)e^{ik(\pi/T)t}+X(a-ik\pi/T)e^{-ik(\pi/T)t}$$
$$=2\mathrm{Re}[X(a+ik\pi/T)e^{ik(\pi/T)t}]$$

and truncating the infinite series by K terms, the following numerical inversion formula yields.

$$\left.\begin{array}{l} x(t)\simeq\frac{e^{at}}{T}[\mathrm{Re}\sum\limits_{k=1}^{K} X(a+ik\pi/T)e^{ik(\pi/T)t}+\frac{1}{2}X(a)] \\ 0\le t<2T \end{array}\right\} \quad (6.4.12)$$

This relation has a simular form as (6.3.16), but because $X(a+ik\pi/T)$ is a complex, the aliase effect is not so severe and a numerical value of $x(t)$ at any point t in $0\le t<2T$ can be given approximately if the operational solution $X(s)$ is known.

Specially at $t=T$ it becomes

$$x(T)\simeq\frac{e^{aT}}{T}[\mathrm{Re}\sum_{k=1}^{K} X(a+ik\pi/T)(-1)^k+\frac{1}{2}X(a)] \quad (6.4.13)$$

and by this formula the numerical value of $x(T)$ is given approximately at any point T.

6.4.2 Error in numerical inversion formula When a numerical value of $x(t)$ is computed by the formula (6.4.13), the numerical error is consisted in the following two kinds. One is caused in approximating the infinite integral (6.4.2) by the infinite series (6.4.4) and is given theoretically as follows by the relation (6.4.11).

$$\sum_{n=1}^{\infty} e^{-2naT} x(\overline{2n+1}T) \quad (6.4.14)$$

The other is caused in truncating the sum of the infinite series (6.6.4) by K terms in (6.4.13) and is given as follows.

$$\frac{e^{aT}}{T}\mathrm{Re}\sum_{k=K+1}^{\infty} X(a+ik\pi/T)(-1)^k \quad (6.4.15)$$

In (6.4.13) the values of the parameters a and K will be determined as follows.

It is preferable that the value of a is large to reduce the error given by (6.6.14), but it is important that a be made no greater than is necessary, since as indicated in (6.4.15) the truncation error is magnified by the factor e^{aT} and the value of K must be taken too large to reduce the truncation error and too long executive time is necessary in computation.

Therefore, it is necessary to choose the optimum value of a considering the above contradictory requirements for required accuracy.

In such a case it is proper to take the value of aT constant for any T because the value which affects the numerical error is the value aT in the factor e^{aT}.

In practical applications, it is sufficient that the value of aT is selected equal to $4\sim5$, because the error introduced by (6.4.14) is as small as $e^{-8}\sim e^{-10}$ $(3.4\times10^{-4}\sim4.5\times10^{-5})$ and can be recognized sufficiently small.

The series of (6.4.13) is an alternating series and converges slowly, and besides it is magnified by the factor e^{aT}, fairly large value of K is necessary to reduce the truncation error.

Here we shall consider the truncation error in some cases.

If $X(s)$ is an operational solution derived from a lamped constant system, each of its elements becomes

$$\left.\begin{array}{l} [X(s)]_{ij}=\dfrac{N_{ij}(s)}{M_{ij}(s)} \\ N_{ij}(s) \quad \text{polynomial of } s \text{ with degree } n \\ M_{ij}(s) \quad \text{polynomial of } s \text{ with degree } m \\ \qquad\qquad m\geq n+1 \end{array}\right\} \quad (6.4.16)$$

and if K is large

$$\mathrm{Re}[X(a+ik\pi/T)]_{ij} \propto \frac{1}{K^2} \qquad (6.4.17)$$

and the error caused by (6.4.15) is given approximately by

$$\varepsilon_K\left(\frac{1}{(K+1)^2}-\frac{1}{(K+2)^2}+\frac{1}{(K+3)^2}-\cdots\cdots\right)<\frac{K+1}{K}\varepsilon_K \qquad (6.4.18)$$

where

$$\frac{e^{aT}}{T}\max|\mathrm{Re}[X(a+ik\pi/T)_{ij}]|=\varepsilon_K \qquad (6.4.19)$$

therefore, if the formula (6.4.13) is used, the truncation error is nearly equat to the absolute value of the last term.

If $X(s)$ is an operational solution derived from a distributed constant system such as a transmission line, each of its elements becomes

$$[X(s)]_{ij}=e^{-Q_{ij}(s)}\frac{N_{ij}(s)}{M_{ij}(s)} \qquad (6.4.20)$$

and it decreases more rapidly as K increases than in the case of a lamped constant system because of the factor $e^{-Q_{ij}(s)}$ and the truncation error becomes smaller at the same value of K in the previous case.

6.4.3 Algorithm of numerical inversion formula In Fig 6-4-1, the flow chart in digital computation of the inversion formula (6.4.13) is shown where the upper limit of the value of K is determined by the following inequality

$$\frac{e^{aT}}{T}\max|\mathrm{Re}X(a+ik\pi/T)_{ij}|<\varepsilon$$

then by (6.4.18) the truncation error is smaller than a small value ε given arbitrary.

Here we shall show some numerical examples by this method.

Numerical example 1 $\quad X(s)=\frac{1}{(s+A)}x(0)$

$$A=\begin{pmatrix}1 & 0 & 3\\ 1 & 2 & 1\\ -3 & 0 & 1\end{pmatrix} \qquad x(0)=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

The numerical values of $x(T)$ and values of K are shown in Table 6-4-1

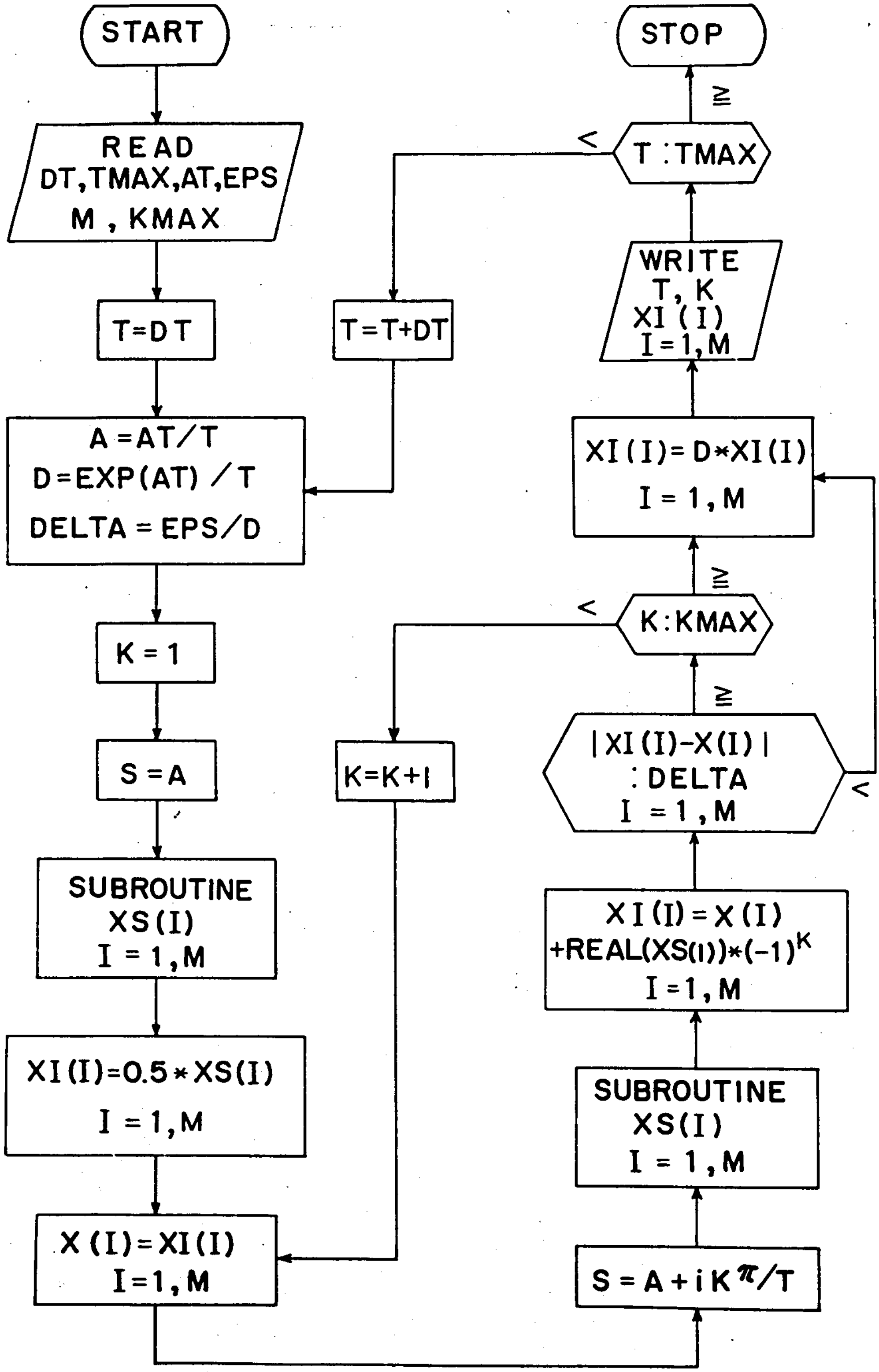


Fig 6-4-1 Flow chart of the numerical inversion method by (6.4.13)

in comparision with the exact values, and the relation between the truncation error ε and the upper limit K of the series is shown in Fig 6-4-2 where the value of a is selected as aT=5.

T	Approximate value				Exact value		
	$x_1(T)$	$x_2(T)$	$x_3(T)$	K	$x_1(T)$	$x_2(T)$	$x_3(T)$
0.1	0.5995	0.6516	1.1341	128	0.597027	0.649153	1.131822
.2	.2110	.3894	1.1361	133	.213438	.391864	1.138018
.3	- .1223	.2158	1.1039	137	- .119803	.218292	1.040803
.4	- .3844	.1133	.8661	141	- .381869	.115757	.867670
.5	- .5597	.0723	.6493	146	- .562107	.069868	.647916
.6	- .6567	.0682	.4110	150	- .659150	.065696	.409768
.7	- .6769	.0913	.1791	154	- .679356	.088862	.177958
.8	- .6324	.1289	- .0268	158	- .634838	.126439	- .027827
.9	- .5438	.1651	- .1947	161	- .541328	.167617	- .193809
1.0	- .4186	.2016	- .3131	165	- .416113	.204093	- .312283
.1	- .2787	.2278	- .3820	169	- .276195	.230210	- .381212
.2	- .1343	.2453	- .4027	172	- .136814	.242852	- .403383
.3	- .0079	.2436	- .3847	176	- .010402	.241159	- .385278
.4	.0916	.2236	- .3363	179	.094031	.226108	- .335825
.5	.1736	.2025	- .2647	182	.171082	.200021	- .265151
.6	.2213	.1685	- .1831	186	.218788	.166055	- .183457
.7	.2357	.1253	- .1004	189	.238181	.127717	- .100081
.8	.2351	.0909	- .0225	192	.232651	.088448	- .022823
.9	.2047	.0488	.0422	195	.207212	.051295	.042482
2.0	.1702	.0212	.0923	198	.167760	.018679	.092130
.1	.1179	- .0102	.1244	201	.120380	- .007728	.124498
.2	.0732	- .0246	.1400	204	.070769	- .027037	.139809
.3	.0213	- .0416	.1397	207	.023790	- .039091	.139777
.4	- .0143	- .0419	.1272	210	- .016812	- .044362	.127118
.5	- .0510	- .0463	.1055	213	- .048542	- .043803	.105449
.6	- .0677	- .0362	.0781	216	- .070158	- .038681	.078173
.7	- .0791	- .0279	.0488	218	- .081549	- .030416	.048814
.8	- .0860	- .0229	.0250	221	- .083546	- .020428	.020390
.9	- .0752	- .0075	- .0048	224	- .077672	- .010016	- .004714

Table 6-4-1 Numerical solutions of Example 1

The solution of this example is given formally by

$$x(T)=e^{-AT}x(0)$$

and the value of e^{-AT} is given by the partial sum of the following infinite series[14].

$$e^{-AT}=1-AT+\frac{A^2T^2}{2!}-\frac{A^3T^3}{3!}+\ldots. \tag{6.4.21}$$

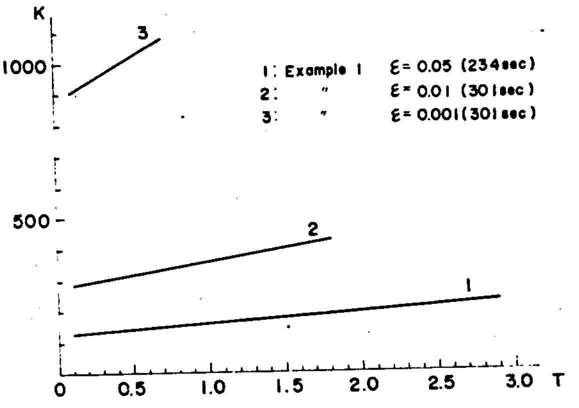


Fig 6-4-2 Relation between K and ε

In such a case this series converges by few terms, but the following restriction between the maximum value of eigen values of matrix A and T is necessary to be stable numerically[15].

$$|\lambda_{max}|T<C \qquad C\simeq 2 \text{ constant} \tag{6.4.22}$$

In example 1, the eigen values of A are

$$-2 \qquad -1\pm 3i$$

and $|\lambda_{max}|=\sqrt{10}$, then the numerical value of $x(T)$ can be given stablly until $T\simeq 2/3$ by (6.4.21).

However our method need not the restriction (6.4.22) and the value of T can be chosen independently on the eigen values of A although hundreds of terms of the series are necessary in (6.4.13).

Numerical example 2 $\quad X(s)=\frac{1}{s}\exp(-As)x_0$

$$A=\begin{pmatrix}4 & 0 & 2/3\\ 0 & 4 & 4/3\\ 2 & 2 & 4\end{pmatrix} \qquad x_0=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

The numerical values of $x(T)$ are shown compa~~i~~ring with the exact values in Table 6-4-2 where the value of K is fixed to 150 and $aT=5$.

The value of $x(T)$ is given by

$$x(T)=\frac{x(T-0)+x(T+0)}{2}$$

at the discontinuous points of T marked by the symbol * and this fact is caused by the intrinsic restriction of the Fourier analysis.

But in the operational calculus no information is given for the value of the particular point, and this is the merit that the discontinuous function can be treated as well as the continuous functions.

Furthermore, the value of $x(T)$ at the discontinuous point can be corrected theoretically, therefore, this method can be used without interference in practical applications.

T	Approximate value			Exact value		
	$x_1(T)$	$x_2(T)$	$x_3(T)$	x_1	x_2	x_3
1.0	0.0000	0.0000	-0.0000	0	0	0
2.0*	.0830	.1660	- .2489	1/6	1/3	-1/2
3.0	.1652	.3300	- .4951	1/6	1/3	-1/2
4.0*	.3344	.1721	- .5066	1/2	0	-1/2
5.0	.4951	- .0010	- .4903	1/2	0	-1/2
6.0*	.7422	.4942	.2539	1	1	1
7.0	1.0030	1.0201	.9899	1	1	1
8.0	.9987	.9844	1.0281	1	1	1
9.0	.9966	.9712	1.0029	1	1	1
10.0	.9861	.9879	.9935	1	1	1

Table 6-4-2 Numerical solutions of example 2

By this method, the numerical values can be determined if the operational solution is known, therefore, in higher order equations the order of them need not be decreased to the first order by introducing the new variables as were done in 2.5.2.

6.5 Applications of Fast Fourier Transform[16),17),18)]

6.5.1 Applications to inversion formula In the numerical inversion formula of the Laplace transform given above, hundreds of complex frequency spectra must be computed at every point T and most of the executive time in computation is occupied by this operation, therefore, this method is not practical because it consumes too long time in computation except when the numerical values are necessary at a few points.

In such a case, the ex~~c~~ecutive time in computation can be reduced extremely by taking as many points as frequency spectra in the interval $0 \le t < 2T$ and calculating them by Fast Fourier Transform.

Then we shall show the method below.

Rewrite (6.4.12) by replacing the upper limit K of the series by $K-1$, then

$$\left.\begin{array}{l} x(t)=\dfrac{e^{at}}{T}\left[\mathrm{Re}\displaystyle\sum_{k=0}^{K-1} X(a+ik\pi/T)e^{ik(\pi/T)t}-\frac{1}{2}X(a)\right] \\ \qquad 0 \le t < 2T \end{array}\right\}. \quad (6.5.1)$$

Substitute

$$t_n = n(2T/K) \qquad n=0,1,2,\ldots\ldots,K-1$$

then it becomes

$$\left.\begin{array}{l} x(t_n)=\dfrac{e^{at_n}}{T}\left[\mathrm{Re}\displaystyle\sum_{k=0}^{K-1} X(a+ik\pi/T)e^{i2\pi nk/K}-\frac{1}{2}X(a)\right] \\ \qquad n=0,1,2,\ldots\ldots,K-1 \end{array}\right\}. \quad (6.5.2)$$

Let

$$x_n = \sum_{k=0}^{K-1} X(a+ik\pi/T)e^{i2\pi nk/K} \qquad (6.5.3)$$

then, Fast Fourier Transform can be adapted to x_n and the numerical values of $x(t_n)$ are given by (6.5.2).

By doing so, we can get K numerical values of $x(t)$ in the interval $0 \le t < 2T$ in nearly equal time to be necessary in computing a few value $x(T)$ by the previous method. Therefore, this method is more practical than the previous method and (6.5.2) is used as a practical formula of numerical inversion method of the Laplace transform.

In Fig. 6-5-1, the flow chart in digital computation by (6.5.2) applying Fast Fourier transform to (6.5.3) is shown.

The theory and an algorithm in computing the Fast Fourier Transform will be summarised in Appendix E.

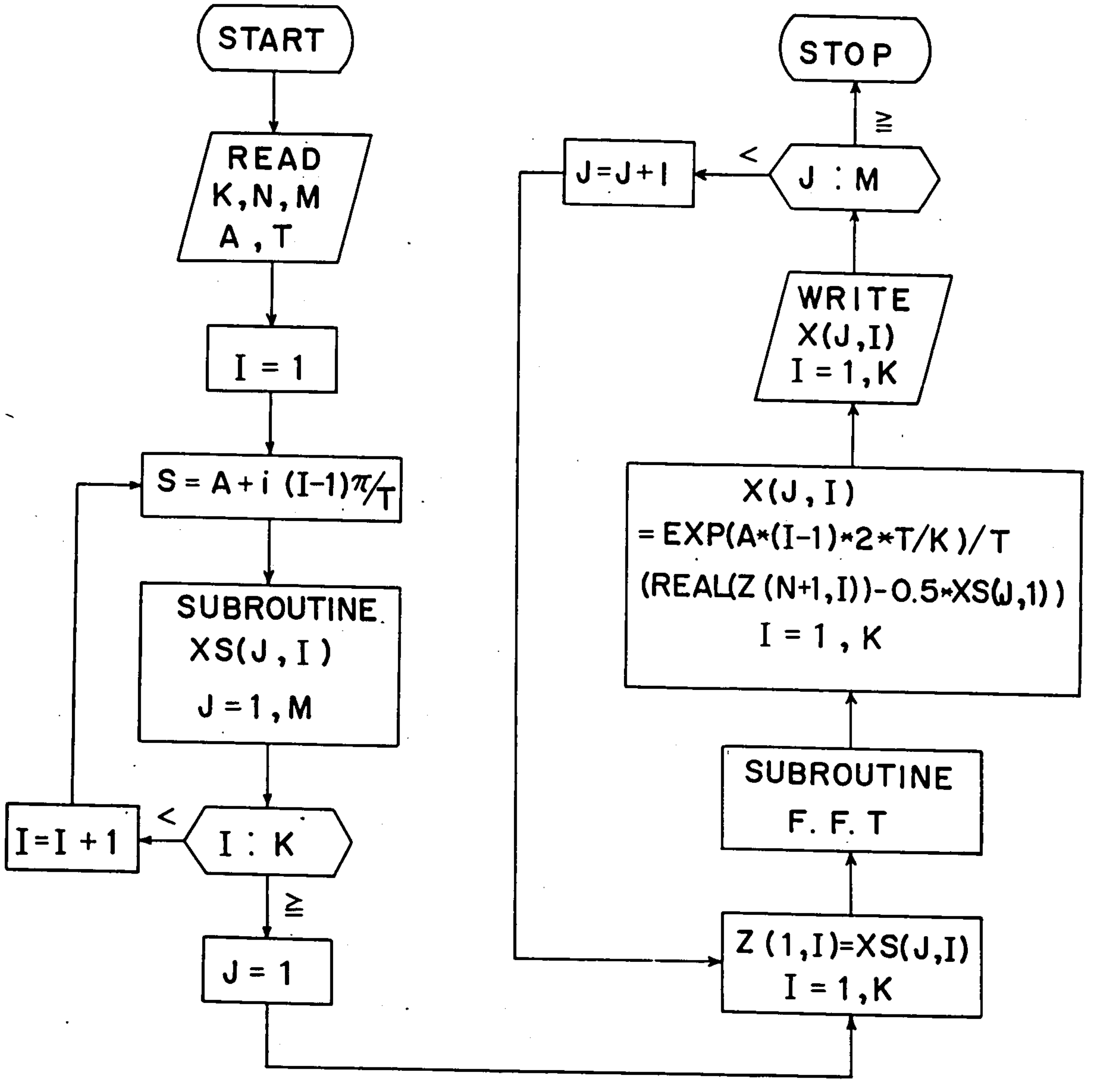


Fig 6-5-1 Flow chart of the numerical inversion method by (6.2.5)

6.5.2 Error in numerical inversion formula Here we shall consider the numerical error caused in this method.

What differs of this method from the previous method is that the real and imaginary parts of every spectrum $X(a+ik\pi/T)$ give complicated influence to the numerical solution and it is fairly difficult to get the evaluation of the error theoretically, therefore, we shall show some results given numerically.

In Fig 6-5-2 the behavior of numerical error to various functions are shown at certain values of the parameters a,T and K. The behavior depends on the nature of the function in the time domain and the absolute value of the error increases rapidly as t increases by the factor e^{at} specially in $T<t<2T$.

Next we shall consider the relation between the numerical error and the value of the parameter K at fixed values of a and T.

As is shown in Fig 6-5-3, the absolute value of the error is not reduced too much in the interval $T<t<2T$ as the value of K increases, therefore, it gives a little advantage in numerical analysis to increase the value of K because by the fact that needs too long an executive time in computation in spite of the accuracy is not so improved.

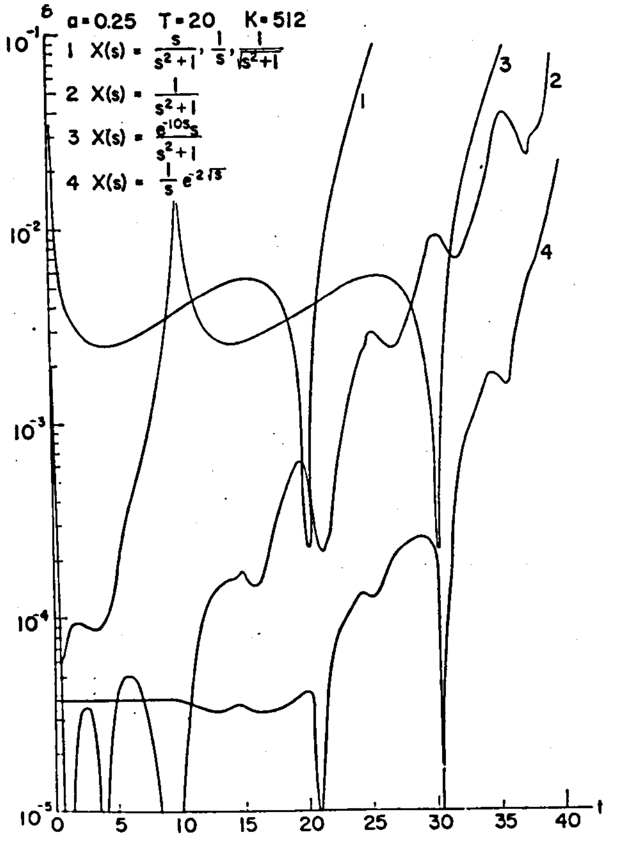


Fig 6-5-2 Absolute value of the error to various functions

In applications, it may be practical to get the numerical solutions with desired accuracy in the following way.

(1) Give a value of T as the upper limit of the time interval where the numerical solution is necessary.

(2) For above T give a so that $aT=4\sim5$.

(3) For an arbitrary small value ε give the value of K as the minimum value of powers of 2 satisfying the following inequality.

$$\frac{e^{aT}}{T}\max|\mathrm{Re}[X(a+iK\pi/T)]_{ij}|<\varepsilon$$

Practically above inequality is always satisfied by $K=512$ if ε is chosen about 10^{-3} as are shown in Fig 6-5-2.

(4) For given parameters, get the numerical values of $x(t)$ only in the interval $0\leq t<T$.

Here we shall show a numerical example by this method using the value of parameters determined by the above manner.

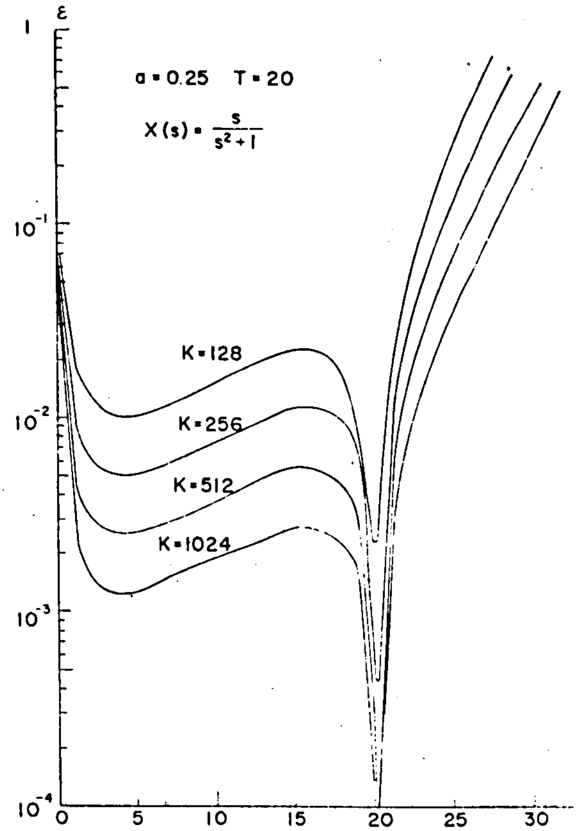


Fig 6-5-3 Absolute value of the error to the value of K

Numerical example $\quad X(s)=\frac{1}{(s+A)}x(0)$

$$A=\begin{pmatrix}1 & 0 & 3\\ 1 & 2 & 1\\ -3 & 0 & 1\end{pmatrix} \qquad x(0)=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

This example is the same as Example 1 in 6.4.3 and when $aT=5, T=3$ and $K=256$ the executive time in computation is reduced from 234 seconds to 14 seconds although the time step has gone from 0.1 to 6/256 where the numerical computation is carried out by FACOM 230-60.

6.6 Numerical evaluation of Laplace transform

Here we shall consider a linear equation

$$L(d/dt)x(t)=Bu(t) \tag{6.6.1}$$

where L is a matrix of the differential operator.

The Laplace transform of this equation becomes

$$L(s)X(s)=L_0(s)+BU(s) \tag{6.6.2}$$

where L_0 is the term caused by the initial value of the variable $x(t)$ and the following operational solution yields.

$$X(s)=L^{-1}(s)[L_0(s)+BU(s)] \tag{6.6.3}$$

The numerical solution can be given by applying the inversion method to the operational solution $X(s)$.

Here we shall consider a case where the input function $u(t)$ is arbitrary and its operational form cannot be given analytically.

In such a case it is necessary to know not the operational function but its complex frequency spectra at the particular points of s, therefore, it is sufficient to get them numerically.

Then we shall show the method to get the frequency spectra from the sampled values of $u(t)$ in the time domain.

For simplicity, let us assume that the input function is a scalar function and denote it by $u(t)$ in the time domain and $U(s)$ operational domain.

From the definition

$$\left.\begin{aligned} &U(a+i\omega)=\int_0^\infty u(t)e^{-at}e^{-i\omega t}dt=\int_0^{2T}u_p(t)e^{-at}e^{-i\omega t}dt \\ &u_p(t)=u(t)+\sum_{n=1}^{\infty}e^{-2anT}u(t+2nT) \\ &0\le t<2T \end{aligned}\right\} \tag{6.6.4}$$

and by approximating it by the finite series, it becomes

$$U(a+i\omega)\simeq\frac{2T}{N}\left[\sum_{n=1}^{K-1}u_p\left(n\frac{2T}{N}\right)e^{-2anT/N}e^{-i\omega(2nT/N)}-\frac{1}{2}u(0)\right] \tag{6.6.5}$$

and

$$U(a+ik\pi/T)\simeq\frac{2T}{N}\left[\sum_{n=0}^{N-1}u_p(2nT/N)e^{-2anT/N}e^{-i2\pi nk/N}-\frac{1}{2}u(0)\right]. \qquad (6.6.6)$$

Neglect the term

$$\sum_{n=0}^{\infty} e^{-2anT}u(t+2nT)$$

in $u_p(t)$, then

$$U(a+ik\pi/T)\simeq\frac{2T}{N}\left[\sum_{n=0}^{N-1}u(2nT/N)e^{-2anT/N}e^{-i2\pi nk/N}-\frac{1}{2}u(0)\right] \qquad (6.6.7)$$

and the numerical value of $U(a+ik\pi/T)$ can be given in a short time by applying the Fast Fourier Transform to the term of $\sum$.

In this case, N must be determined as $N\geq 2K$ to avoid the aliase effect if K frequency spectra are necessary.

The absolute values of the spectra become very small as K increases, therefore, it must be given as precisely as possible.

The numerical error caused by (6.6.7) is consisted of the following two kinds

ε_1 caused in approximating $u_p(t)$ by $u(t)$

ε_2 caused in approximating the integral by the finite series

and each of them must be made as small as possible because by the inversion formula, they are magnified and given at $t=t_n$ by

$$\varepsilon'(t_n)=\frac{e^{2anT/K}}{T}(\varepsilon_1+\varepsilon_2). \qquad (6.6.8)$$

The upper limit of ε_1 is bounded by the following inequality

$$\left.\begin{aligned}&|\varepsilon_1|=\left|\int_{2T}^{\infty}u(t)e^{-at}e^{-i\omega t}dt\right|\leq\int_{2T}^{\infty}|u(t)|e^{-at}dt\leq\frac{M}{a}e^{-2aT}\\&M=\sup_{t\in(2T,\infty)}|u(t)|\end{aligned}\right\} \qquad (6.6.9)$$

and in the time domain

$$|\varepsilon_1'(t_n)|\leq\frac{M}{aT}e^{-2aT(K-n)/K}. \qquad (6.6.10)$$

Specially at $t_n=T$, it becomes

$$|\varepsilon_1'(T)|\simeq Me^{-aT}/aT$$

and it must be made as small as desirable.

The purpose is attained by making the value a large but it demands a large value of K and gives a heavy influence in the time domain to do so, therefore, attention must be paid to determine the value of a as in the case of the numerical inversion.

To reduce ε_1 we must use $u_p(t)$ instead of $u(t)$, but it is pretty difficult in the general case because the value of $u(t)$ cannot be known at large value of t from the sampled values of $u(t)$ in the interval $0 \le t < 2T$.

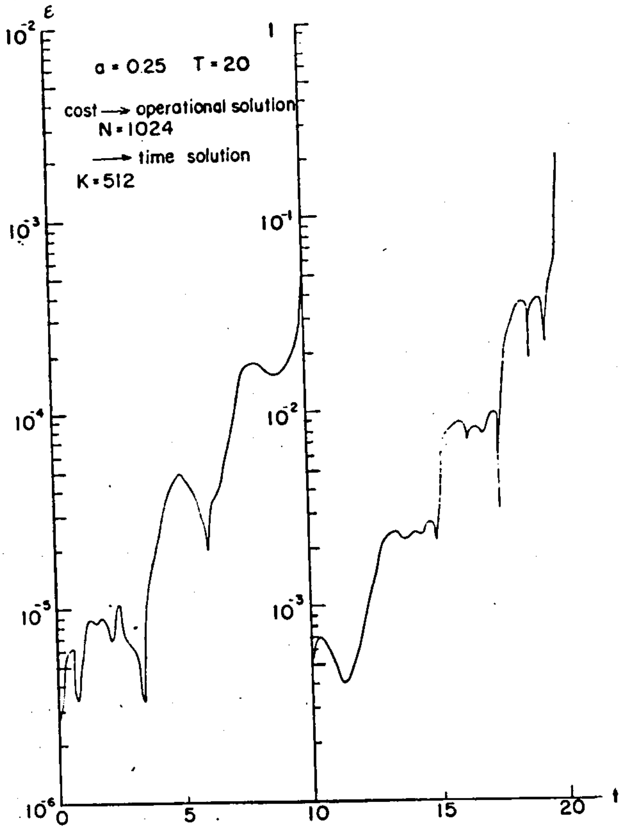


Fig 6-6-1 Absolute value of the numerical error of the example

To reduce ε_2 the number of the sampled points N must be increased and the value of $u(t)$ at the discontinuous point must be corrected as follows if exists.

$$\left.\begin{array}{l} u(t)=u(t-0)+\delta/2=u(t+0)-\delta/2 \\ \delta=x(t+0)-x(t-0) \end{array}\right\} \quad (6.6.11)$$

Numerical example In Fig 6-6-1 a numerical example of the error given by the following procedures is shown.

First for $u(t)=\cos t$ get 512 spectra by (6.6.7) where every parameter is selected as a=0.25, T=20 and N=1024, and then get the numerical values by the inversion formula using these given spectra.

6.7 Comparision of various inversion formulas

In a case where the time solution cannot be known analytically for a give operational solution, various numerical inversion methods of the Laplace transform have been conceived hitherto.

Here we shall show some of them which are fairly wide in use. These methods give numerical quadrature formulas by approximating the inverse or direct transformations properly.

In the former methods, the numerical quadrature formulas are given by evaluating the inverse transform

$$x(t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}X(s)e^{st}ds \qquad (6.7.1)$$

making use of the proper approximation for a given $X(s)$ and a typical example is the Salzer's method[19].

His method gives the following numerical quadrature formula

$$x(t)\simeq\sum_{k=1}^{m}A_k^{(m)}(t)X(k) \qquad (6.7.2)$$

by approximating $X(s)$ by a polynomial with the order m in the variable $1/s$ which causes the polynomial with order m-1 in the variable t.

The coefficients of this series are determined by Lagrange's interpolation formula at the points $s=1\sim m$ and by inverting them, each of the weighting coefficients is determined by

$$A_k^{(m)}(t)=\frac{k^{m+1}}{m!}\sum_{j=1}^{m}a_j\frac{t^{m-j}}{(m-j)!}$$

where a_j is the coefficient of the explicit expression of the $(m+1)$-points Lagrange's polynomial $L_k^{(m+1)}(s)$ and given as follows.

$$L_k^{(m+1)}(s)=\sum_{j=1}^{m}a_js^j$$

In the latter methods, the numerical quadrature formulas are given by evaluating the direct transform

$$X(s)=\int_0^{\infty} x(t)e^{-st}dt \tag{6.7.3}$$

for some values of s at the interpolated points making use of the numerical integration formula. A typical example of them ~~are~~ is given by Bellman[20).
His method gives the following munerical quadrature formula

$$\left.\begin{aligned} &x(t_i)\simeq\sum_{k=1}^{m} A_k^i X(k) \\ &i=1,2,\ldots\ldots,m \end{aligned}\right\} \tag{6.7.4}$$

which has been derived as follows.
Substitute $y=e^{-t}$, (6.7.3) becomes

$$X(s)=\int_0^1 x(-\log y)y^{s-1}dy$$

and by the numerical integration formula

$$\left.\begin{aligned} &X(i)\simeq\sum_{k=1}^{m} W_k^i y_i^{s-1} x(-\log y_i) \\ &i=1,2,\ldots\ldots,m \end{aligned}\right\} \tag{6.7.5}$$

where y_i is the zero point of the shifted L~~a~~egendre polynomial of order i.
By inverting (6.7.5) he gave the quadrature formula (6.7.4).
The principles of these methods are complicated, but the coefficients matrices of the quadrature formulas are given by the numerical tables, therefore, these methods can be used conveniently in certain cases. However these methods have the following defects.

(1) Every element of the coefficient matrix demands many digits and each of its elements differs extremly in the order.

(2) Coefficient matrix becomes often ill-conditioned and accuracy of the given results is not so good in spite of demanding many digits.

(3) These methods are based on the interpolation formula, the class of the functions to be treated is limited and cannot give the sufficient results for certain functions such as oscillatory functions or functions containing time laggings.

Other methods are based on the numerical quadrature of the infinite expansion of certain polynomials and various quadrature formulas have been conceived.

Papoulis gave the quadrature formulas by the trigonometric, Legendre and Laguerre sets[21]. But these methods are very complicated although accuracy of the given results is not so good.

The values of the coefficient matrices of the various numerical inversion formulas are summarised in the tables[22], and in some cases it is convenient to ~~reffer~~ refer either of them when above defects can be avoided and the numerical values at some points are necessary.

The method stated in this chapter is based on the numerical integration based on the Fourier analysis and is the most simple and useful method without the above defects and has the following advantages.

(1) It gives sufficient numerical solutions in a short executive time and needs a little programing effort in digital computations.

(2) The class of functions to be treated is not limited.

(3) Scalar operations can be applied easily to corresponding matrix or vector operations.

6.8 Summary

In this chapter the method to get the numerical solution of the time function from the corresponding operational solution was given.

In this method, the Laplace transform is converted into the Fourier transform and the expected aim could be attained by applying the Fourier series technique.

For this purpose, the relation between the operational calculus stated in Chapters 2 - 4 and the Laplace transform, the principle and the algorithm of the numerical inversion method of the Laplace transform and some numerical examples were given.

This method makes the most of the spirit of the operational calculus in comparison with other inversion formulas and must be a powerful tool in the numerical analysis in the case where the time function cannot be known analytically or is given very complicatedly or formally when the operational solution is given.

7. Analysis of traveling waves on transmission systems

7.1 Preface

In Chapter 4 the method to analyse the telegraphic equations by the operational calculus has been shown. But for the analysis of traveling waves such as surges on the transmission lines, the coefficient matrices cannot be regarded to constant with respect to the differential operator s as in the case of the telegraphic equations.

This phenomenon is due to the skin effect of the transmission systems and the absorption effect of the dielectric substance, and differential equations of operational functions to the voltage and current variables become very complicated.

Hence it is the purpose of this chapter to show the operational differential equations for typical transmission systems such as airial lines, underground lines and co-axial cables.

Detailed procedures in deducing the equations will be shown in Appendix F and numerical treatment of them will be shown in the successive chapter.

7.2 Airial transmission lines

7.2.1 Coefficient matrices of airial transmission lines For the behavior of the traveling waves on the airial transmission lines, the operational differential equations for the voltage and current variables are fairly different from those for the telegraphic equations.

There the influence of the coefficient matrices R and G to the phenomena is very small comparing with that of L and C, therefore, it is possible to neglect R and G.

Here we shall consider an airial system with the uniform ground return and consisting of lines streched parallel above the surface of the ground as shown in Fig 7-2-1. Every element of the coefficient matrices L and C is given approximately as follows in the region of low frequency waves by regarding the ground as the perfect conductor and using the method of the electrical images on the assumption $h_i, h_j, d_{ij} \gg r_i, r_j$.

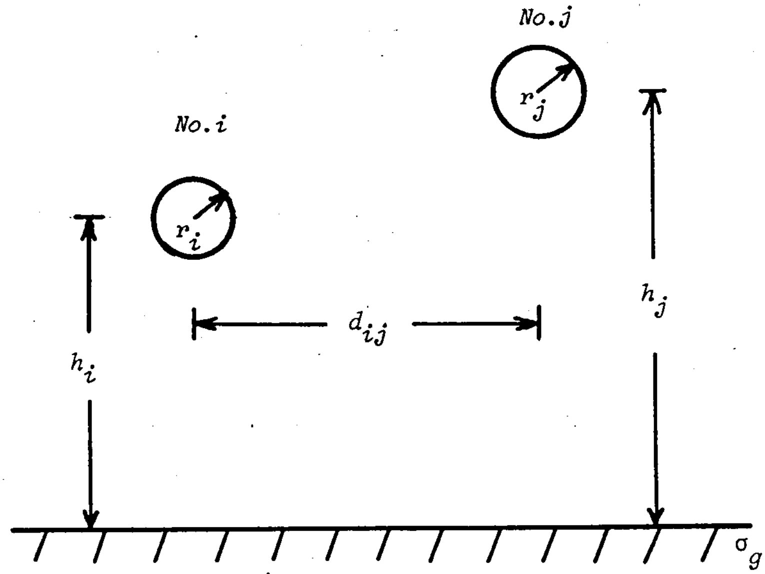


Fig 7-2-1 Airial transmission system

$$\left.\begin{aligned} L_{ii} &= \frac{\mu_c}{8\pi} + \frac{\mu_0}{2\pi}\log\frac{2h_i}{r_i} \\ L_{ij} &= L_{ji} = \frac{\mu_0}{\pi}\log\frac{(h_i+h_j)^2+d_{ij}^2}{(h_i-h_j)^2+d_{ij}^2} \end{aligned}\right\} \qquad (7.2.1)$$

and

$$\left.\begin{aligned} P_{ii} &= \frac{1}{2\pi\varepsilon_0}\log\frac{2h_i}{r_i} \\ P_{ij} &= P_{ji} = \frac{1}{\pi\varepsilon_0}\log\frac{(h_i+h_j)^2+d_{ij}^2}{(h_i-h_j)^2+d_{ij}^2} \end{aligned}\right\} \qquad (7.2.2)$$

where the matrix $P=[P_{ij}]$ is the electrostatic induction matrix and the capacitance matrix yiels

$$C=P^{-1}. \qquad (7.2.3)$$

Every letter means the following quantity.

r_i, r_j: radius of conductor i, j

h_i, h_j: height of conductor i, j

d_{ij} : horizontal distance between conductors i and j

μ_c, μ_0: magnetic permeabilities of conductor and vacu~~me~~um

ε_0 : permeability of vacu~~me~~um

7.2.2 Skin effect of conductors The coefficient matrices L and C shown above are determined on the assumption that the conductivites of the ground and the conductors are infinite, but for real systems, the conductivities are finite and the influence of the electromagnetic induction becomes remarkable in the region of high frequency waves.

This influence appears heavily on the impeadance matrix but little on the admittance matrix.

The term $\mu_c/8\pi$ in the first relation of (7.2.1) shows the influence of the magnetic field inside the conductor on the assumption that the current density is constant, but for a high frequency current this assumption is inadequate and the current flows almost near the surface of the conductor.

This phenomenon is called the skin effect of the conductor and the equivalent impedance matrix $Z_c(s)$ for a conductor with circular section given as follows by solving the Maxwell's equations[23].

$$\left.\begin{aligned} Z_{cii}(s) &= \frac{\mu s}{2\pi\sqrt{\sigma\mu s}r_i}\,\frac{I_0(\sqrt{\sigma\mu s}r_i)}{I_1(\sqrt{\sigma\mu s}r_i)} \\ Z_{cij}(s) &= Z_{cji}(s) = 0 \end{aligned}\right\} \qquad (7.2.4)$$

Every new letter means the follwing quantity.

σ : conductivity of conductor

s : differential operator

I_0: first kind modified Bessel function of order 0

I_1: first kind modified Bessel function of order 1

This result means that the impedance of the conductor varies from $\mu s/8\pi$ to $Z_c(s)$.

7.2.3 Skin effect of uniform ground In considering the phenomena in the region of high frequency waves the equivalent impedance of the ground return is affected remarkably by the electromagnetic induction the same as in the case of the conductor.

This phenomenon is called the skin effect of the ground and the equivalent impedance of it can be given by solving the Maxwell's equations.

For a system shown in Fig 7-2-1, every element of the equivalent impedance matrix $Z_g(s)$ is given as follows on the assumption that the current flows pararell to the conductors in the ground.

$$\left.\begin{aligned} Z_{gii}(s)&=\frac{\mu_0}{\pi}\int_0^\infty(\sqrt{\xi^2+s}-\xi)e^{-2h_i'\xi}d\xi \\ Z_{gij}(s)&=Z_{gji}(s)=\frac{\mu_0}{\pi}\int_0^\infty(\sqrt{\xi^2+s}-\xi)\cos d_{ij}'\xi e^{-(h_i'+h_j')\xi}d\xi \\ h_i'&=\sqrt{\mu_0\sigma_g}h_i \quad h_j'=\sqrt{\mu_0\sigma_g}h_j \quad d_{ij}'=\sqrt{\mu_0\sigma_g}d_{ij} \end{aligned}\right\} \quad (7.2.5)$$

The new letter means the following quantity.

σ_g: conductivity of the ground

The operational differential equations for this system considering the skin effects of the conductors and the ground return are given as follows

$$\left.\begin{aligned} -v'(\lambda)&=[L_0 s+Z_c(s)+Z_g(s)]i(\lambda) \\ -i'(\lambda)&=Csv(\lambda) \end{aligned}\right\} \quad (7.2.6)$$

where every element of the matrices $L_0, C, Z_c(s)$ and $Z_g(s)$ is given by (7.2.1) (eliminate the term $\mu_0/8\pi$), (7.2.3), (7.2.4) and (7.2.5).

7.2.4 Skin effect of two-layer ground Here we shall consider a ground consisting of pararell two layers as shown in Fig 7-2-2.

By solving the Maxwell's equations on the same assumption as the uniform ground, we have the equivalent impedance matrix $Z_g(s)$ whose elements are given as follows.

$$\left.\begin{aligned} Z_{gii}(s)&=\frac{\mu_0 s}{\pi}\int_0^\infty F(\xi)e^{-2h_i'\xi}d\xi \\ Z_{gij}(s)&=Z_{gji}(s)=\frac{\mu_0 s}{\pi}\int_0^\infty F(\xi)\cos d_{ij}'\xi e^{-(h_i'+h_j')\xi}d\xi \\ F(\xi)&=\frac{\sqrt{\xi^2+s}+\sqrt{\xi^2+r's}+(\sqrt{\xi^2+s}-\sqrt{\xi^2+r's})e^{-2d'\sqrt{\xi^2+s}}}{(\sqrt{\xi^2+s}+\sqrt{\xi^2+r's})(\sqrt{\xi^2+s}+\xi)+(\sqrt{\xi^2+s}-\sqrt{\xi^2+r's})(\xi-\sqrt{\xi^2+s})e^{-2d'\sqrt{\xi^2+s}}} \\ h_i'&=\sqrt{\sigma_1\mu_0}h_i \quad h_j'=\sqrt{\sigma_1\mu_0}h_j \quad d_{ij}'=\sqrt{\sigma_1\mu_0}d_{ij} \\ d'&=\sqrt{\sigma_1\mu_0}d \quad r'=\sigma_2/\sigma_1 \end{aligned}\right\} \quad (7.2.7)$$

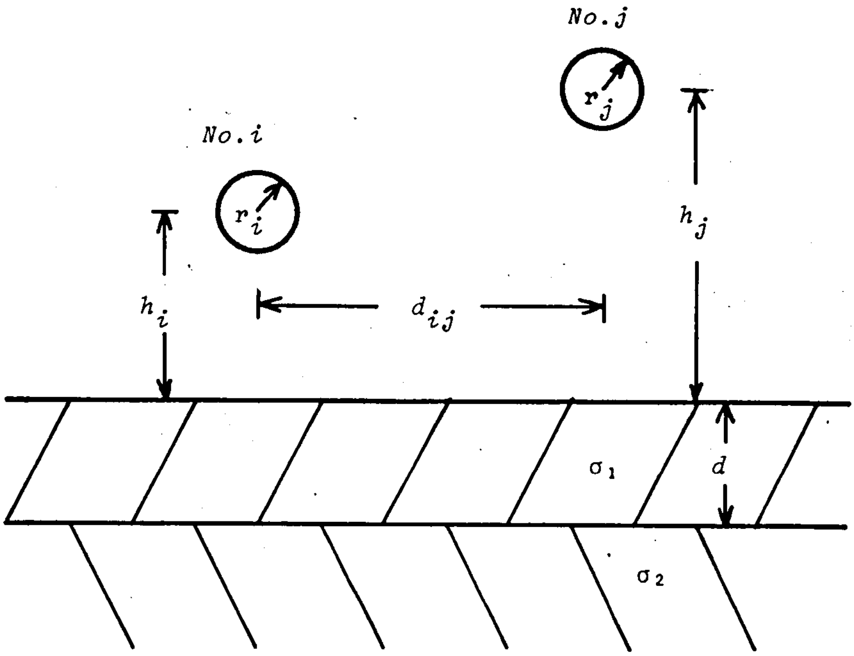


Fig-7-2-2 Airial transmission system with two-layer ground

Every new letter means the following quantity.

σ_1: conductivity of the upper layer

σ_2: conductivity of the lower layer

d : depth of the boundary plane

The set of the operational differential equations is given in the same form as (7.2.6).

7.3 Underground transmission lines

7.3.1 Coefficient matrices of underground transmission lines Here we shall consider an insulated underground system with the uniform ground return and consisting of insulated lines buried parallel under the surface of the ground as shown in Fig 7-3-1.

Every element of the coefficient matrices L and C in the region of low

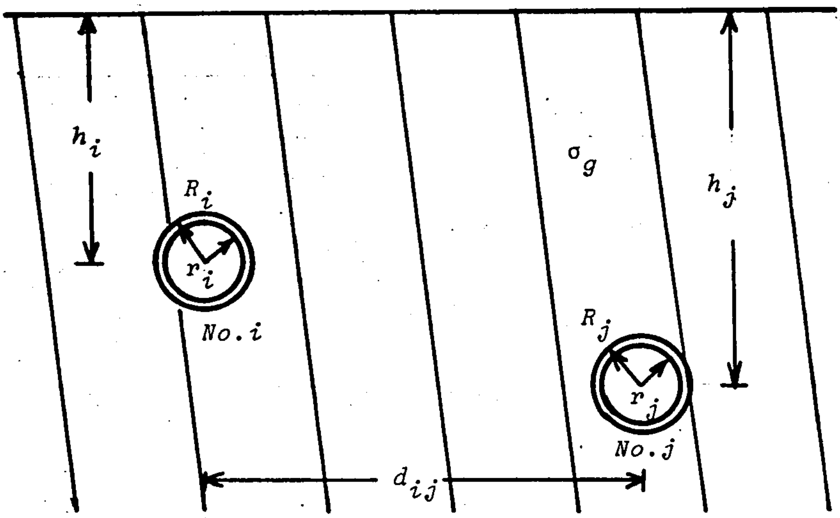


Fig 7-3-1 Underground transmission system

frequency waves is given as follows on the assumption $h_i, h_j, d_{ij} >> r_i, r_j, R_i, R_j$.

$$\left.\begin{aligned} L_{ii} &= \frac{\mu_0}{2\pi}\log\frac{2h_i}{r_i} \\ L_{ij} = L_{ji} &= \frac{\mu_0}{\pi}\log\frac{(h_i+h_j)^2+d_{ij}^2}{(h_i-h_j)^2+d_{ij}^2} \end{aligned}\right\} \qquad (7.3.1)$$

$$\left.\begin{aligned} P_{ii} &= \frac{1}{2\pi\varepsilon_1}\log\frac{2h_i}{R_i} + \frac{1}{2\pi\varepsilon_2}\log\frac{(h_i+h_j)^2+d_{ij}^2}{(h_i-h_j)^2+d_{ij}^2} \\ P_{ij} = P_{ji} &= \frac{1}{\pi\varepsilon_1}\log\frac{(h_i+h_j)^2+d_{ij}^2}{(h_i-h_j)^2+d_{ij}^2} \end{aligned}\right\} \qquad (7.3.2)$$

$$C = P^{-1} \qquad (7.3.3)$$

Every letter means the following quantity.

r_i, r_j: radius of conductor i, j

h_i, h_j: depth of the conductor i, j

d_{ij} : horizontal distance between conductors i and j

R_i, R_i: outer radius of insulator enclosing conductor i, j

μ_0 : magnetic permeability of vacu~~me~~um

ε_1 : permeability of ground

ε_2 : permeability of insulator

7.3.2 Skin effect of ground The equivalent impeadance matrix $Z_c(s)$ for the skin effect of conductors in the region of high frequency waves is given by the same equation as (7.2.4).

For the equivalent impeadance matrix $Z_g(s)$, we have the following results by solving the Maxwell's equations.

$$\left.\begin{aligned} Z_{gii}(s) &= \frac{\mu_0 s}{\pi}\int_0^\infty \frac{e^{-2h_i\sqrt{u^2+k_1^2}}}{\sqrt{u^2+k_1^2}+u}du \\ Z_{gij}(s) = Z_{gji}(s) &= \frac{\mu_0 s}{\pi}\int_0^\infty \frac{e^{-(h_i+h_j)\sqrt{u^2+k_1^2}}}{\sqrt{u^2+k_1^2}+u}\cos d_{ij}u\,du \\ k_1^2 &= (\sigma_g+\varepsilon_1 s)\mu_0 s \end{aligned}\right\} \qquad (7.3.4)$$

The set of the operational differential equations is given in the same form as (7.2.6).

7.4 Co-axial cables

7.4.1 Line constants of co-axial cables Here we shall consider a single conductor co-axial cable with circular section.

The line constants of this cable is given as follows.

$$L = \frac{\mu}{2\pi}\log\frac{R}{r} \qquad (7.4.1)$$

$$C = \frac{2\pi\varepsilon}{\log(R/r)} \qquad (7.4.2)$$

Every letter means the following quantity.

r: radius of inner conductor

R: inner radius of outer conductor

ε: permeability of insulator

μ: magnetic permeability of insulator

7.4.2 Skin effect of co-axial cables The equivalent impeadance for the skin effect of conductors of a co-axial cable is given as follows by solving the Maxwell's equations.

$$
\left.
\begin{aligned}
&Z_c(s)=Zc_1(s)+Zc_2(s)\\
&Zc_1(s)=\frac{\mu_1 s}{2\pi\sqrt{\sigma_1\mu_1 s}\,r}\,\frac{I_0(\sqrt{\sigma_1\mu_1 s}\,r)}{I_1(\sqrt{\sigma_1\mu_1 s}\,r)}\\
&Zc_2(s)=\frac{\mu_2 s}{2\pi}\,\frac{K_0(\sqrt{\sigma_2\mu_2 s}D)I_0(\sqrt{\sigma_2\mu_2 s}R)-I_0(\sqrt{\sigma_2\mu_2 s}D)K_0(\sqrt{\sigma_2\mu_2 s}R)}{K_0(\sqrt{\sigma_2\mu_2 s}D)C_1-I_0(\sqrt{\sigma_2\mu_2 s}D)C_2}\\
&\quad C_1=\sqrt{\sigma_2\mu_2 s}DI_1(\sqrt{\sigma_2\mu_2 s}D)-\sqrt{\sigma_2\mu_2 s}RI_1(\sqrt{\sigma_2\mu_2 s}R)\\
&\quad C_2=-\sqrt{\sigma_2\mu_2 s}DK_1(\sqrt{\sigma_2\mu_2 s}D)+\sqrt{\sigma_2\mu_2 s}RK_1(\sqrt{\sigma_2\mu_2 s}R)
\end{aligned}
\right\}
\qquad (7.4.3)
$$

where $Zc_1(s)$ and $Zc_2(s)$ mean the equivalent impedance of inner and outer conductors.

Every new letter means the following quantity.

μ_1,μ_2: magnetic peameability of inner and outer conductors

σ_1,σ_2: conductivity of inner and outer conductors

D : outer radius of outer conductor

I_0,I_1: first kind modified Bessel function of order 0,1

K_0,K_1: second kind modified Bessel function of order 0,1

7.4.3 Absorption effect of insulators[24),25),26),27)] The peameability of an insulator such as the synthetic polymer cannot be regarded as constant in the region of high frequency waves. It decreases as the frequency increases and this phenomenon is called the absorption effect.

For sinusoidal waves of angular velocity ω, the complex permeability ε of such a substance is given experimentally by Cole and Cole as follows.

$$\varepsilon=\varepsilon^{\infty}+\frac{\varepsilon^{0}-\varepsilon^{\infty}}{1+(i\omega\tau)^{1-\alpha}} \qquad (7.4.4)$$

Every letter means the following quantity.

ε^{∞}: limit permeability in high frequency region

ε^{0}: limit permeability in low frequency region

τ: generalized relaxation time

α: constant value between 0 and 1

In the operational form it becomes

$$\varepsilon(s)=\varepsilon^{\infty}+\frac{\varepsilon^{0}-\varepsilon^{\infty}}{1+(s\tau)^{1-\alpha}} \tag{7.4.5}$$

and the equivalent admittance of the capacitor with such a substance as an insulator is given on the assumption that impulsive responce of $\varepsilon(s)$ is fairly slower than time responce of considered phenomenon

$$Y(s)=C_0\varepsilon(s)s \tag{7.4.6}$$

where C_0 means the capacity in the vacu~~me~~um which is determined only by the geometry of conductors.

The set of operational differential equations for traveling waves for this system is given as follows.

$$\left.\begin{aligned} -v'(\lambda)&=[Ls+Zc(s)]i(\lambda) \\ -i'(\lambda)&=Y(s)v(\lambda) \end{aligned}\right\} \tag{7.4.7}$$

For multi-conductor cables, we have the set of matric differential equations arising from (7.4.7) but it is pretty difficult to get every element of the coefficient matrices and the equivalent impedance and admittance matrices because it is affected heavily by the configuration of conductors and insulators, therefore, we do not give more detailed discussion about that.

7.5 Summary

In this chapter the set of operational differential equations which describe the properties of traveling waves on the transmission systems such as airial lines, underground lines and co-axial cables considering the influences of the skin effect and the absorption effect were given.

The proof of the existence of the operational Bessel functions and the integrals of the operational functions which appear in this chapter is shown in Appendix F.

8. Numerical analysis of traveling waves on transmission systems

8.1 Preface

As was shown in the previous chapter, the analysis of the traveling waves is one of the fields where the operational calculus can be applied effectively In this case the phenomena to be treated continue for a short duration and the equation describing their properties suffer the influences of the skin effect and the absorption of dielectric substance, therefore, the operational solutions become very complicated functions with respect to the operator s and it is impossible to get their time solutions analytically.

In this case such a method have been used hitherto that by bold abbrevation or approximation time solutions are derived analytically for examples error functions, but there if we do not clear up the limit of its adaptability or estimate the theoretical error, we may get the result extremely different from the actual phenomenon.

In physical or engineering applications like above problems, it occurs frequently that we may be satisfied to get a numerical solution with desired accuracy when we cannot get the analytical solution, and for above problems we can get numerical solutions in the time domain by applying the numerical inversion method of the Laplace transform stated in Chapter 6 to the operational solutions of differential equations given in the previous chapter.

In the numerical inversion method, scalar operations can be extended easily to corresponding vector or matrix operations, therefore, we can analyse multi-conductor systems which have been difficult or almost impossible to be analysed hitherto, therefore, we can solve every untouched problem except that caused by the effects based on non-lineality.

Hence it is the purpose of this chapter to study the attenuation and distortion of traveling waves on single or multi conductor transmission systems numerically for various boundary conditions considering the influences of the skin and absorption effects, give some numerical examples and present some numerical techniques used there such as an application of multi-dimensional interpolation formula, an effective numerical integration method and calculation of matrix functions with complex elements.

In this case, an increace in the degree of complexity of mathematical model increases heavily the degree of difficulty in analytical treatments but does not so too much in numerical treatments by our method, therefore, as the fundamental policy in this chapter, we shall calculate important factors as precisely as possible, for example, get the equivalent impedance for the skin effect of the ground return by numerical integration.

On the other hand we shall neglect such a factor positively that gives little influence on the final result.

Here every letter used without explanation is defined in Chapters 6 and 7.

8.2 Single conductor airial lines

8.2.1 Influence of skin effect The wave propagating property of a single conductor airial line streched parallel above the surface of the ground is given by the following set of the differential equations.

$$\left.\begin{aligned} -\frac{dV}{d\lambda} &= [Ls+Zc(s)+Z_g(s)]I \\ -\frac{dI}{d\lambda} &= CsV \end{aligned}\right\} \tag{8.2.1}$$

By eliminating the current I from this equation the voltage equation yields

$$\left.\begin{aligned} -\frac{d^2V}{d\lambda^2} &= Q^2V \\ Q^2 &= [Ls+Zc(s)+Z_g(s)]Cs \end{aligned}\right\}. \tag{8.2.2}$$

Let us consider only a line with infinite length and assume that the initial condition is given by

$$V(0)=V_0(s) \tag{8.2.3}$$

then voltage solution becomes

$$V(\lambda)=e^{-\lambda Q}V_0(s). \tag{8.2.4}$$

By expanding the equivalent impedance of $Z_g(s)$ in few terms by the partial integration, it becomes

$$Z_g(s)=\frac{\mu_0}{\pi}\left(\frac{\sqrt{s}}{2h'}-\frac{1}{4h'^2}+\frac{1}{8h'^3\sqrt{s}}-\frac{3s}{8h'^3}\int_0^\infty \xi(\xi^2+s)^{-5/2}e^{-2h'\xi}d\xi\right) \tag{8.2.5}$$

By asymptotic expansion we have

$$\frac{I_0(\sqrt{\sigma_c\mu_c s}r)}{I_1(\sqrt{\sigma_c\mu_c s}r)}\simeq 1$$

and by comparing $Z_c(s)$ with the first term of $Z_g(s)$ on the assumption of $\mu_c=\mu_0$ we have

$$\frac{Z_c(s)}{Z_g(s)}\simeq\frac{\sqrt{\sigma_g}h}{\sqrt{\sigma_c}r}.$$

For an ordinary airial line, this value is sufficiently small, therefore, it may be safe to neglect the term $Z_c(s)$.

The voltage solution of (8.2.4) is very complicated and it is impossible to get the time solution analytically. To get it in the analytical form various approximations have been used which are the error functions or their modifications.[28),29),30),31)] Then we shall show an approximate method by using the error function.

Elimination of the terms higher than the second yields

$$Q=\sqrt{LCs^2+\mu_0 Cs^{3/2}/2\pi h'}$$

and by the relation $LC=1/c_0^2$ (c_0 light verocity) it becomes approximately

$$Q=\frac{s}{c_0}\left(1+\frac{\mu_0 c_0 Cs^{-1/2}}{2\pi h'}\right)^{1/2}\simeq\frac{s}{c_0}\left(1+\frac{\mu_0 c_0 Cs^{-1/2}}{4\pi h'}\right)=\frac{s}{c_0}+\frac{\mu_0 c_0 C\sqrt{s}}{4\pi h'} \tag{8.2.6}$$

and

$$\left.\begin{aligned} &V=e^{-\lambda(s/c_0+\alpha\sqrt{s})}V_0(s)\\ &\alpha=\frac{\mu_0 c_0 C}{4\pi h'}\end{aligned}\right\} \tag{8.2.7}$$

Here the term $e^{-\lambda s/c_0}$ corresponds to the translation operator and does not give any influence to attenuation and distortion of traveling waves, therefore, by neglecting it the voltage solution is reduced to the heat conduction type operator and can be solved analytically.

Specially when $V_0(s)=1/s$ we have

$$v(t)=\mathrm{cerf}\left(\frac{\alpha\lambda}{2\sqrt{t-\lambda/c_0}}\right)H(t-\lambda/c_0). \qquad (8.2.8)$$

The reason why such a bold approximation has been used is that there have not been any other effective methods and that the shape of a time solution is fairly like that of an observed phenomenon setting aside its absolute value.

In order that above solution is a good approximation, it is necessary that values of more than second terms are sufficiently small in comparision with the value of the first term in (8.2.5) and that the following restriction must be held in (8.2.6).

$$\frac{\mu_0 c_0 C}{4\pi h'\sqrt{s}}<<1$$

In a line with fairly large value of h', the above conditions are satisfied when the value of λ is small and value of s is large, but degree of approximation becomes worse in accordance with increace of the value of λ and decreace of s. This fact shows theoretically that above approximation is good only in a region where distance from the excited point is small and time is sufficiently small.

But by the numerical inversion method of the Laplace transform we can get time solution numerically, therefore, in above case it is possible to get better results by increasing the terms in (8.2.5).

Here we shall show some numerical examples to help the explanations done above. In Fig 8-2-1 we shall show time responses for the step voltage excitation in the following case

h=20m $\quad r$=1cm $\quad \sigma_g$=10mho/km $\quad T$=8μsec $\quad aT$=5 $\quad K$=1024 where factors caused by the translation operators are removed in every case.

Here primary, secondary and third approximations mean the cases where we take to the first, second and third terms in (8.2.5). In case of λ=1km the difference between the error function approximation and the third approximation is small and the third approximation is good, but in case of λ=20km the differences between the approximations are fairly large and it is doubtful to regard whether the third approximation is good or not and to get better result terms more than the fourth must be taken in (8.2.5) by evaluating the integral.

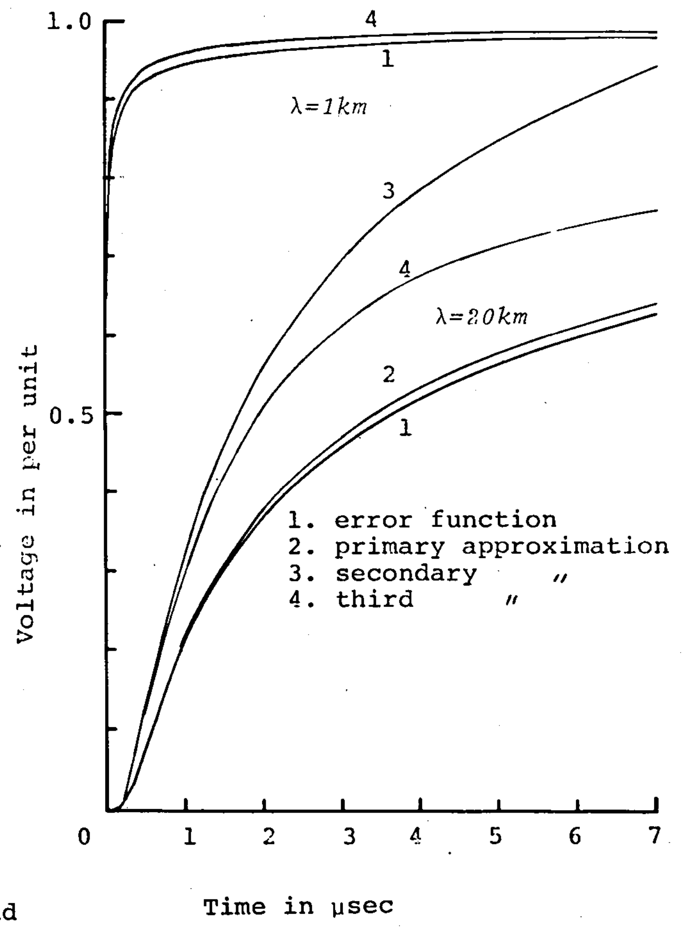


Fig 8-2-1 Approximate solutions for the line with fairly large value of h

However, for a line with small value of h' it is impossible to get a good solution in such a way because in the series of (8.2.5) n-th term is proportional to $1/h'^{n}$ and increases in its absolute value by altering its sign, therefore, the value of $Z_g(s)$ cannot be given by any finite terms as will be shown by the numerical example in Fig 8-2-2 where every value of the parameter is given as follows.

r=1cm σ_g=10mho/km λ=5km T=8μsec aT=5 K=1024

This example shows that as the value of h decreases numerical solution becomes far off from the actual phenomenon by this method.

Therefore, to get the good results in all cases it is necessary to get values of $Z_g(s)$ by numerical integration. In such cases it must be evaluated K times for different value of s and it is necessary to use an effective numerical integration formula to reduce total executive time in digital computation.

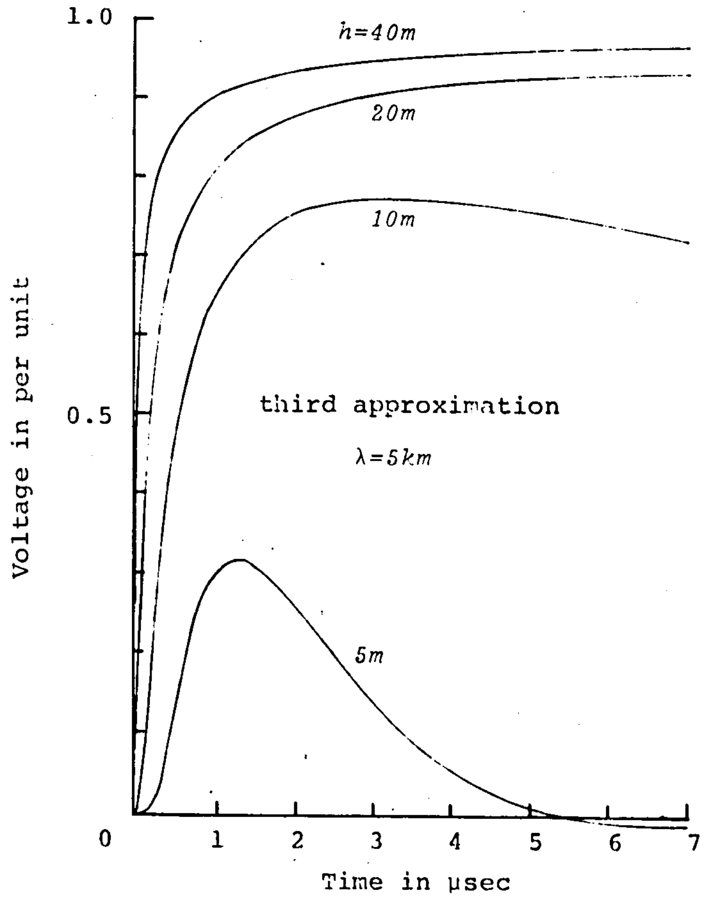


Fig 8-2-2 Approximate solutions for lines with various values of h

Then we shall transform the integral variable by $x=e^{-\xi}$ to get the relation

$$Z_g(s)=\frac{\mu_0}{\pi}\int_0^1[\sqrt{(\log x)^2+s}+\log x]x^{2h'-1}dx \qquad (8.2.9)$$

and apply Gauss's numerical integration formula[20].

Next we must determine how many interpolation points are necessary to acquire a desired accuracy.

Here we have evaluated numerical values of $Z_g(s)$ for some examples by five and ten points integration formula and compared these results.

As are listed in Table 8-2-1 it may be safe to regard that sufficient results can be given by five points integration formula and we shall use it for every numerical integration appearing below.

K	five points		ten points	
	real	imaginary	real	imaginary
1	123.4064	0	123.3323	0
32	558.4059	720.5349	558.1674	719.4575
64	867.0608	1071.859	866.2938	1069.752
96	1111.659	1336.582	1110.240	1333.870
128	1320.506	1557.627	1318.537	1554.531
160	1505.739	1751.236	1503.316	1747.857
192	1673.892	1925.589	1671.085	1921.975
224	1828.952	2085.474	1825.811	2081.655
256	1937.565	2233.979	1970.129	2229.975
288	2109.592	2373.232	2105.888	2369.054
320	2238.400	2504.768	2234.451	2500.427
352	2361.027	2629.743	2356.850	2625.247
384	2478.285	2749.050	2473.896	2744.405
416	2590.822	2863.394	2586.233	2858.606
448	2699.167	2973.349	2694.389	2968.422
480	2806.967	3082.637	2789.157	3064.549
512	2904.951	3181.887	2899.820	3176.697

$h=5\mathrm{m}$

K	five points		ten points	
	real	imaginary	real	imaginary
1	48.4708	0	48.4675	0
32	183.1150	190.9020	183.0965	190.8790
64	265.0154	276.4115	264.9870	276.3780
96	328.1869	341.1601	328.1509	341.1185
128	381.5457	395.4537	381.5032	395.4055
160	428.6034	443.1475	428.5551	443.0934
192	471.1734	486.1862	471.1200	486.1269
224	510.3374	525.7139	510.2791	525.6497
256	546.8015	562.4710	546.7389	562.4023
288	581.0575	596.9694	580.9906	596.8965
320	613.4635	629.5805	613.3927	629.5035
352	644.2904	660.5837	644.2158	660.5029

384	673.7487	690.1957	673.6706	690.1114
416	702.0060	718.5886	701.9244	718.5007
448	729.1982	745.9015	729.1130	745.8103
480	755.4375	772.2491	755.3492	772.1564
512	780.8171	797.7269	780.7258	797.6294

h=20m

Table 8-2-1 Some values of $Z_g(s)$ by numerical integration in the case σ_g=10mho/km,aT=5,T=10μsec and K=512

In Fig 8-2-3 we shall show some numerical examples of time responses of line voltage for the step voltage excitation in the following case

T=10μsec aT=5 K=1024

in comparision with the error function approximations where factors caused by the translation operators are removed.

As is shown from these results it is obvious that the influence of the skin effect becomes greater as distance λ from the excited point increases or as equivalent height h' decreases and in practical applications the error function approximations cannot be used in those cases.

On the contrary, sufficient results can be given in practical applications by the method stated above.

Here the discussions have been restricted to ordinary airial lines with infinite length and any other influences have been neglected than the skin effect of the ground such as the displacement current in the ground, the ohmic losses of the conductor and the ground, the skin effect of the conductor and the leakage current in the free space, but any of them can be considered if necessary without increasing the degree of difficulty in numerical treatment.

Furthermore numerical values of time responses of line voltage can be given for any other excitations than the step voltage.

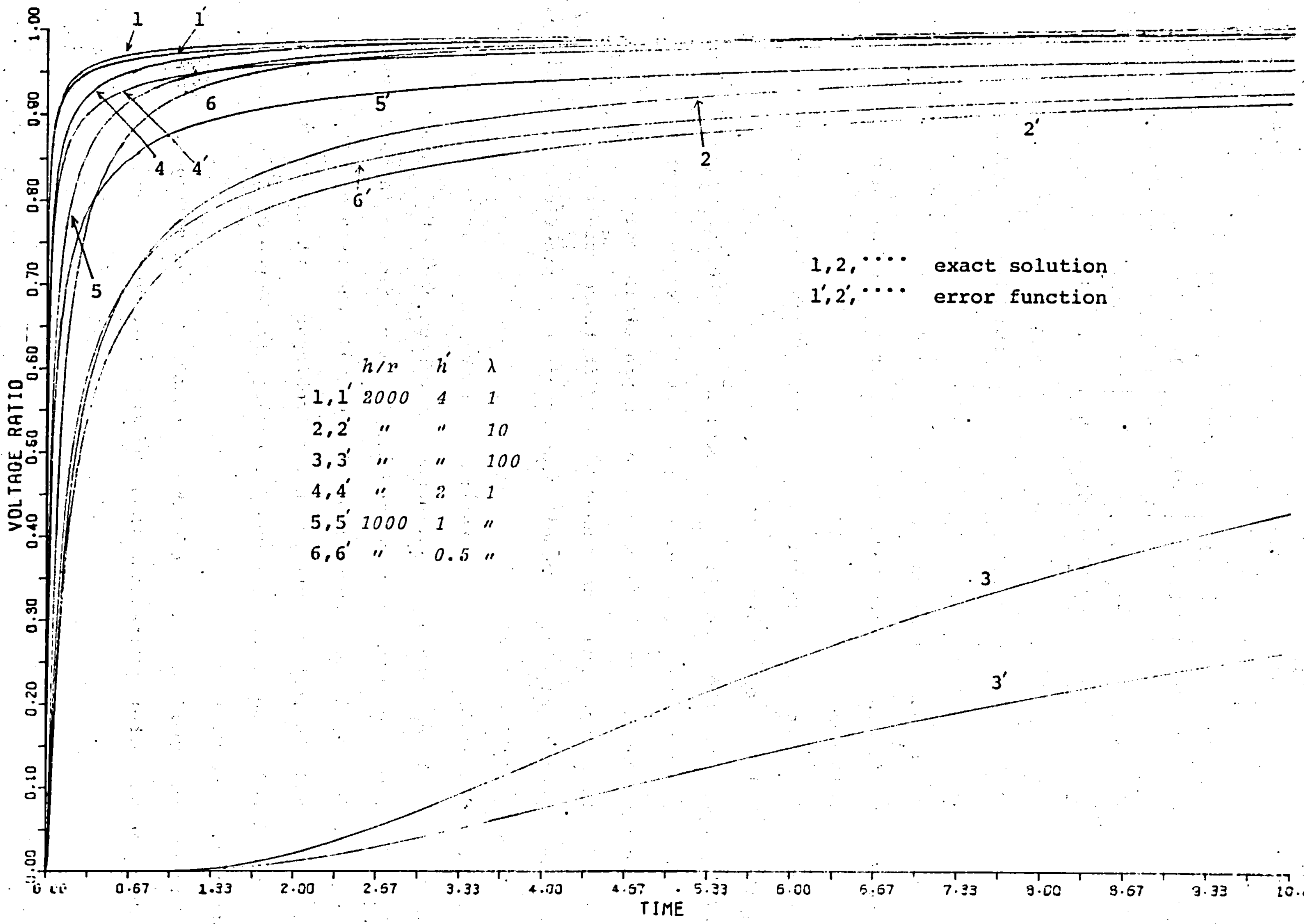


Fig 8-2-3 Numerical solutions for various values of parameters

8.2.2 Application of multi-dimensional interpolation formula In physical or engineering applications, we shall often encounter a case where the numerical value at arbitrary point must be given from the values of selected points by the interpolation method or effective analysis can be performed to do so.

Here we shall investigate this problem in relation to the numerical analysis of traveling waves on the transmission lines with infinite length.

In real lines they are distinguished each other by several parameters, therefore, if numerical values are given at some selected points with respect to them to cover the actual line, a numerical solution for an arbitrary line can be given by the interpolation formula effectively comparing with the method to get it directly.

In such cases the selected points form multi-dimensional lattice points and multi-dimensional interpolation formula must be used there.

Here we shall consider the following function with n variables $f(x_1,x_2,..,x_n)$
Suppose that lattice points are selected for every variable as follws.

$$x_{ij} \qquad i=1,2,...,n \quad j=0,1,2,...,m_i$$

For simplicity we shall assume that m_i is equal to m with respect to every variable x_i.

Let us consider the following m-th order interpolation polynomial with n variables

$$g(x_1,x_2,.....,x_n)=\sum_{i,j,..,k=0}^{m} a_{ij..k}x_1^i x_2^j ... x_n^k \qquad (8.2.10)$$

then every coefficient $a_{ij..k}$ can be determined uniquely by the values of the function at selected points

$$f(x_{1_i},x_{2_j},....,x_{nk}) \quad i,j...,k=0,1,...,m$$

and finally the interpolation formula becomes as follows.

$$g(x_1,x_2,\dots,x_n)=\sum_{i,j,\dots,k=0}^{m} f(x_{1_i},x_{2_j},\dots,x_{nk})\cdot\prod_{\substack{\alpha,\beta,\dots,\gamma=0\\ \alpha\neq i,\beta\neq j,\dots,\gamma\neq k}}^{m}\frac{(x_1-x_{1_\alpha})(x_2-x_{2_\beta})\dots(x_n-x_{n\gamma})}{(x_{1_i}-x_{1_\alpha})(x_{2_j}-x_{2_\beta})\dots(x_{nk}-x_{n\gamma})} \tag{8.2.11}$$

This polynomial is the multi-dimensional Lagrange's interpolation formula and the value of the function at an arbitrary point is obtained from the values at the selected points by this formula.

Time responses of the voltage in airial transmission lines for the step voltage excitation can be represented by the parameters as

$$v(t)=v(h/r,h',\lambda;t)$$

and we get numerical solutions of them at the following 125 lattice points

h/r=2500,200,1500,1000.500

h'=2.5,2.0,1.5,1.0,0.5 $\sqrt{\mu sec}$

λ =10,8,6,4,2 km

under the following conditions.

T=20μsec aT=5 K=1024

From these values at selected points we can get interpolated values at the following three points

h/r=2200	h'=2.2	λ=3
1650	1.65	5
750	0.75	9

by the fourth order interpolation polynomial with respect to every parameter as are shown in Fig 8-2-4 compa~~r~~ring with calculated values and it can be recognized that the interpolated values are in good accord with the calculated values.

From these results we can notice the utility of the interpolation formula to these problems. Ex~~e~~ecutive time in computation can be reduced by increasing the spaces among the lattice points or by decreasing the order of the interpolation polynomial according to the desired accuracy.

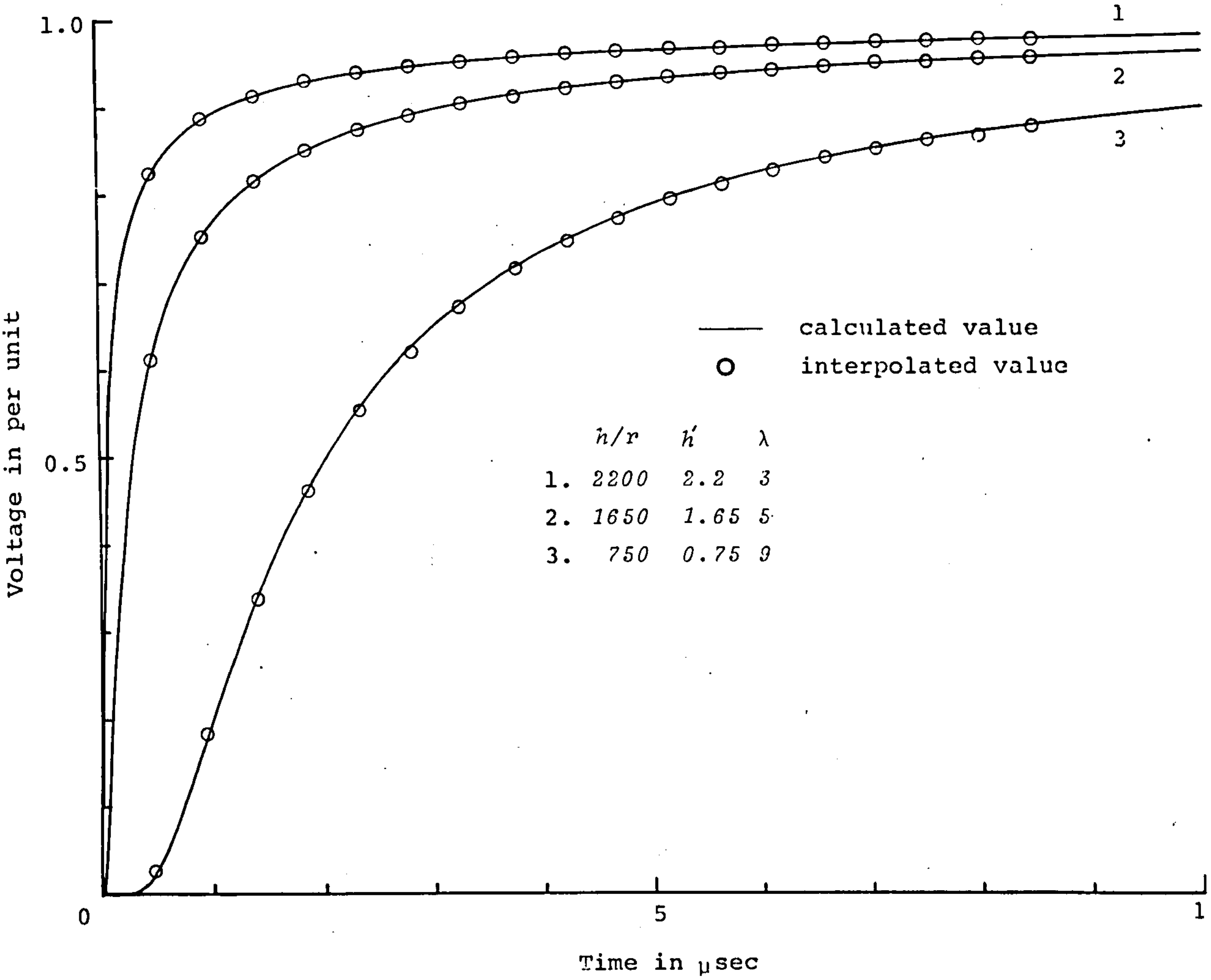


Fig 8-2-4 Numerical solution by interpolation formula

The interpolation formula can be applied to systems with more parameters, but then attention must be paied to the fact that $(m+1)^n$ lattice points are necessary which increase rapidly as n increases and extremely long ex~~e~~ecutive time is necessary in computation if the values of the function at all the lattice points are used.

In such cases the interpolation formula of (8.2.11) cannot be used without modification, but generally weights of the lattice points decrease rapidly as distance from the point to be interpolated increases, therefore, such a method must be used that by the probability technique lattice points with large weights are inspected preferably and inspection is stopped when the interpolated value converges to a value within acceptable accuracy[32).

8.2.3 Influence of two-layer ground In above discussions wave propagation properties of airial lines with uniform ground return were investigated numerically, but we shall often encounter a case where the structure of the ground cannot be regarded as uniform.

Here we shall study wave propagation properties influenced by the structure of the ground for example two-layer ground.

The equivalent impeadance of the parallel two-layer ground is given by (7.2.7) and its numerical value for a certain value of s can be given by the numerical integration, therefore, the time response of the line voltage can be given by the same method stated above.

In Fig 8-2-5 we shall show numerical solutions of the time responses of the line voltage at the points distant 5km and 50km from the excited point under the following conditions.

$$h/r=1500 \quad h'=1.5 \quad d'=1.0 \quad T=10\mu\text{sec} \quad aT=5 \quad K=1024$$

In these cases influence by the structure of the ground appears remarkably. The same calculations were done in case of $d'=10$ under the same conditions as above, but in this case influence of the structure of the ground could not be detected.

As are shown in these examples influence of the two-layer ground to the wave propagation properties becomes remarkable as d' decreases, logarithmic difference between r' and 1 increases and the distance λ from the excited point increases.

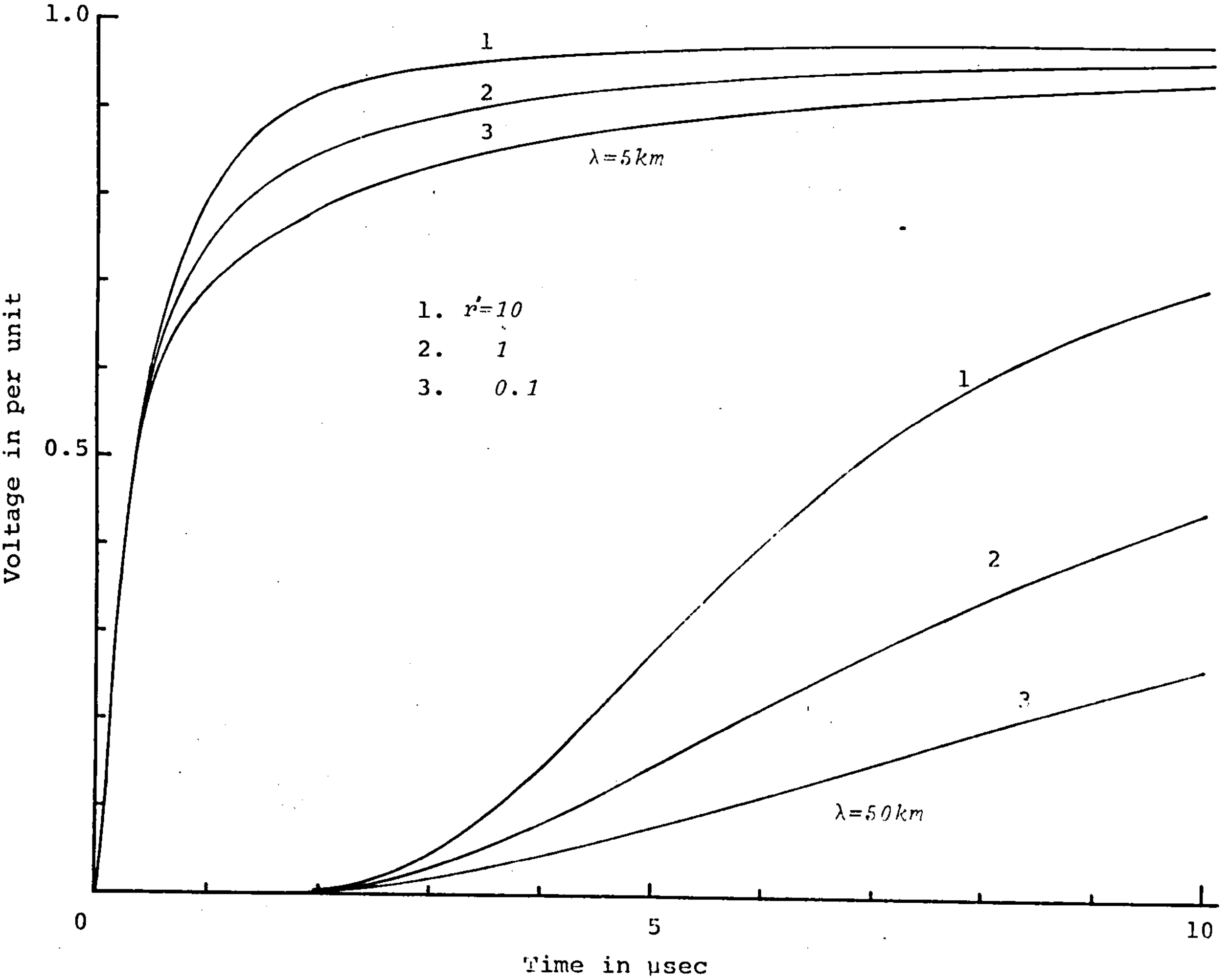


Fig 8-2-5 Influence of two-layer ground

8.2.4 Propagation velocity of traveling waves   Propagation velocity of the traveling waves on a loss less line is equal to the light velocity, but if the influence of the skin effect is considered, time responses of the line voltage become delayed from these of loss less line and it may be regarded that equivalent propagation velocity becomes smaller than the light velocity.

Degree of time delay depends on the values of line parameters, but it is impossible to get the analytical formula to describe it, therefore, we shall get it numerically for some examples.

In Fig 8-2-6 we shall show some numerical results which represent time delay of traveling waves from the loss less cases in which they propagate with the light velocity.

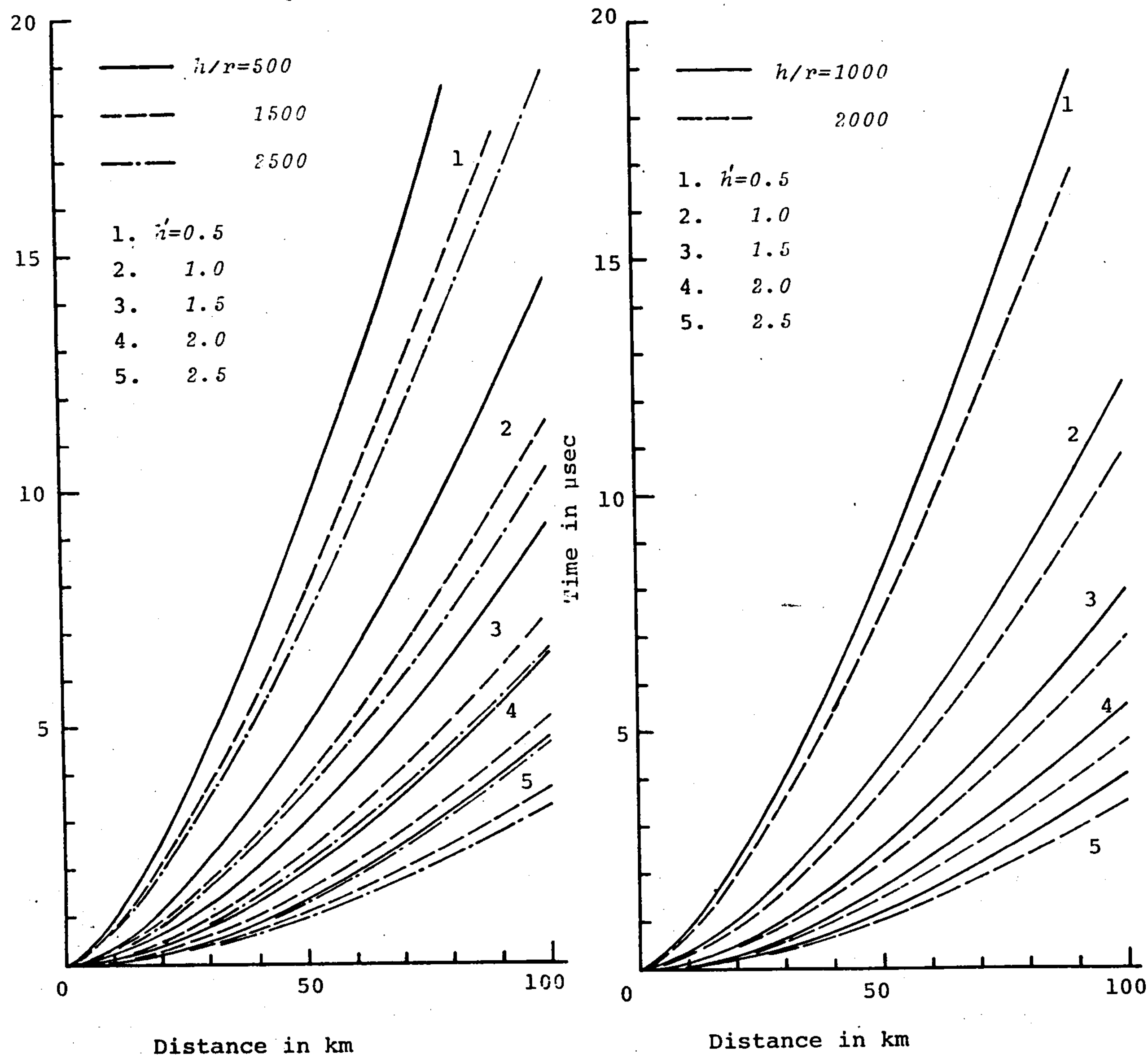


Fig 8-2-6 Time delay from light velocity propagation

Here the values of the parameters are selected as

T=20μsec aT=5 K=1024

and the time which is necessary to become the value of the voltage as large as 1% of the impressed voltage is regarded as the delayed time.

In the region of high frequency current flows almost on the surface of the ground[33] and the equivalent propagation velocity is not influenced practically by the structure of the ground as is shown by the numerical examples of Fig 8-2-5, therefore, the structure of the ground is assumed to be uniform here.

8.3 Single conductor co-axial cables

8.3.1 Influence of skin effect The wave propagating property of the single conductor co-axial cable is given by the following set of differential equations as was shown in the previous chapter.

$$\left.\begin{aligned} -\frac{dV}{d\lambda}&=[Ls+Z_c(s)]I \\ -\frac{dI}{d\lambda}&=CsV \end{aligned}\right\} \qquad (8.3.1)$$

The operational solution for the line voltage with infinite length is given as follows.

$$\left.\begin{aligned} V&=e^{-\lambda Q}V_0(s) \\ Q&=\sqrt{[Ls+Z_c(s)]Cs} \end{aligned}\right\} \qquad (8.3.2)$$

In an ordinary co-axial cable, the conductivity of every conductor is fairly large and the equivalent impedance of the outer conductor can be approximated by

$$Z_{c2}(s)=\frac{\mu_2 s}{2\pi\sqrt{\sigma_2\mu_2 s}R}\,\frac{\mathrm{K}_0(\sqrt{\sigma_2\mu_2 s}R)}{\mathrm{K}_1(\sqrt{\sigma_2\mu_2 s}R)} \qquad (8.3.3)$$

and its numerical values for various values of s can be given by taking the partial sum of the asymtotic series in few terms as well as $Z_{c1}(s)$.

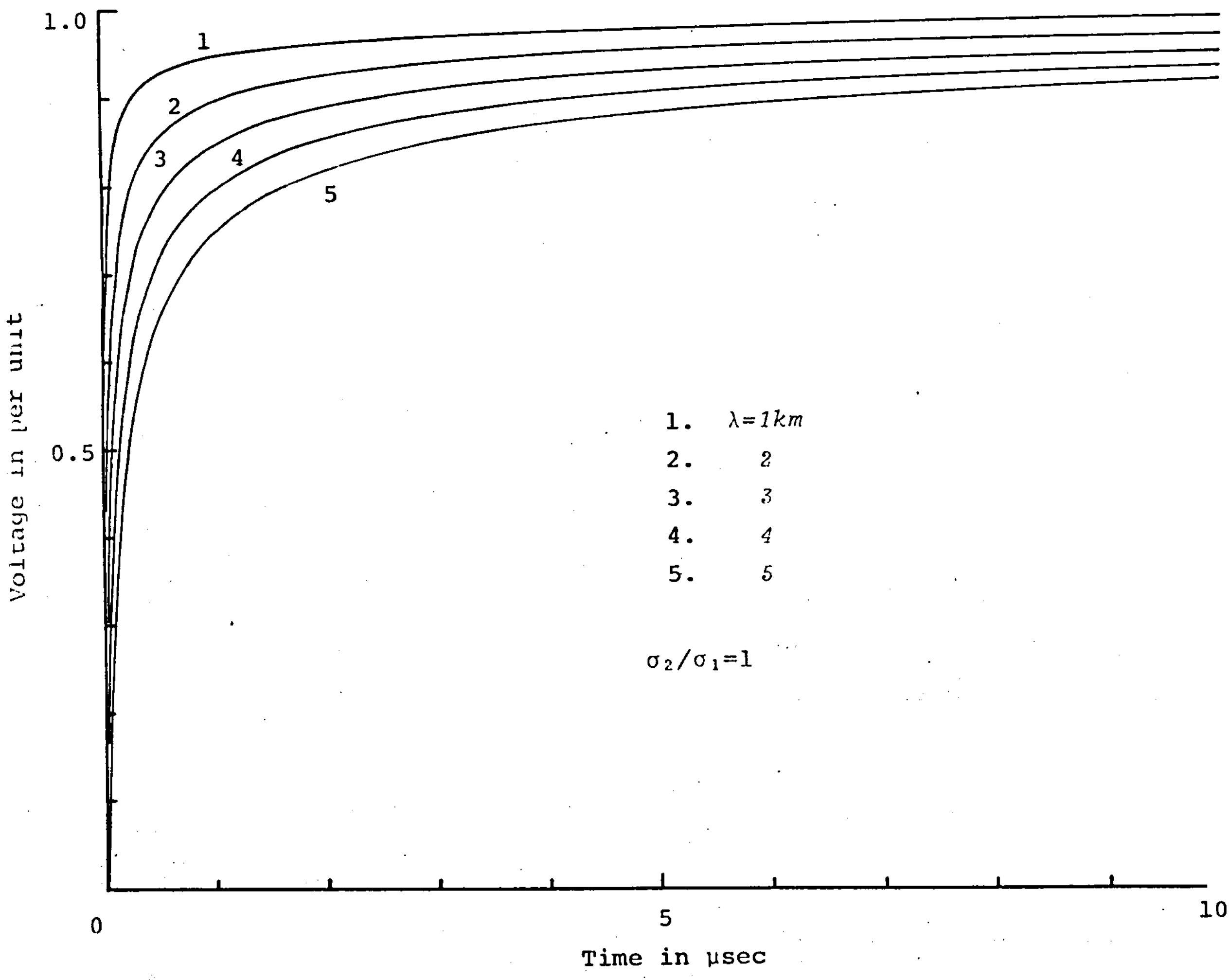
By using these values numerical values of the voltage response in the time domain can be given by the numerical inversion formula.

In Fig 8-3-1 we shall show some numerical examples of time responses of the line voltage for the step excitation under the following conditions

r=1cm R=2cm $\sigma_1=5.92\times10^{10}$mho/km $\varepsilon=4.85$ $\mu_1=\mu_2=\mu_0$

T=10μsec aT=5 K=1024

where factors caused by the translation operator $e^{-\lambda s/\sqrt{LC}}$ are removed in these results.



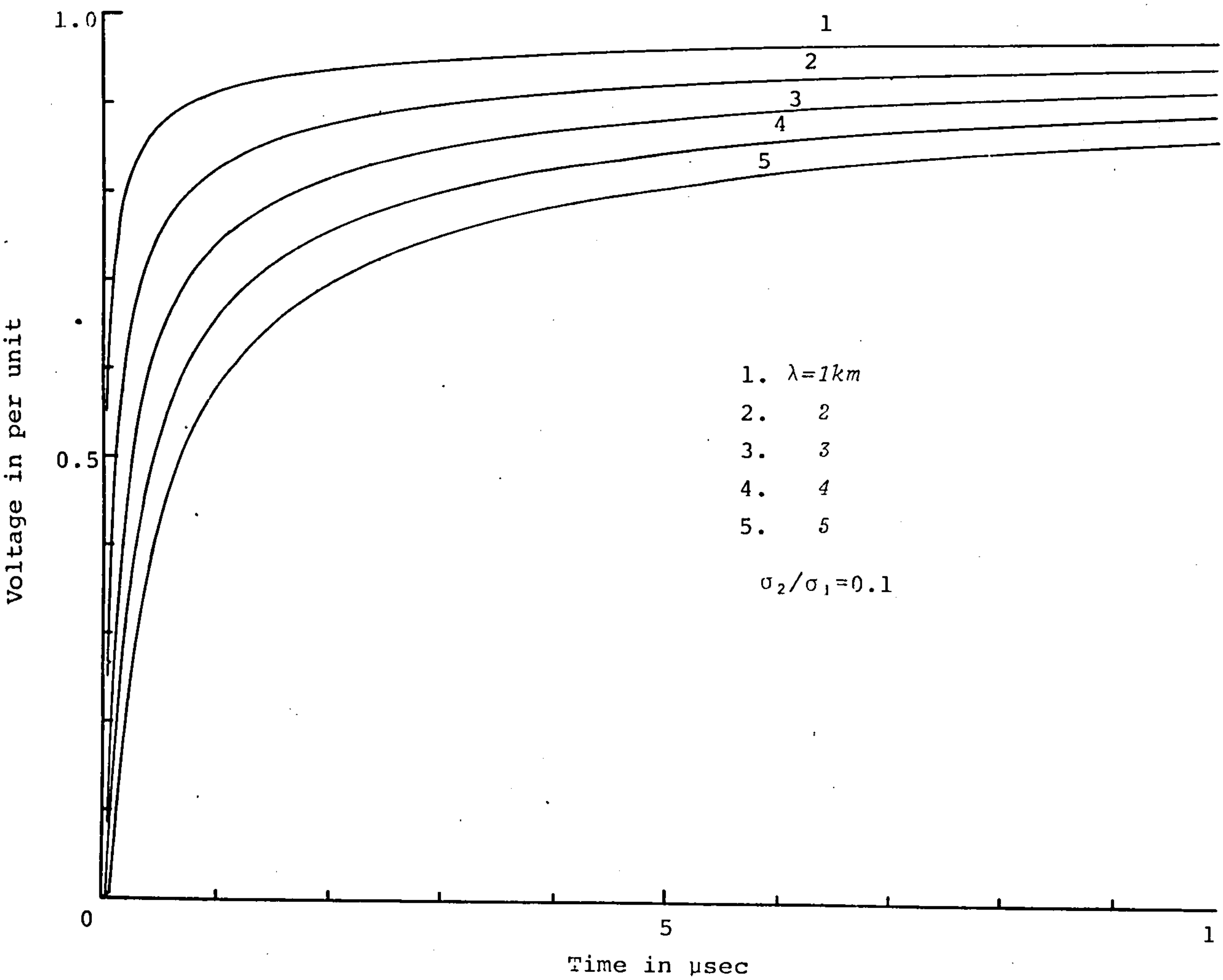


Fig 8-3-1 Numerical solutions of voltage for co-axial cables

8.3.2 Influence of absorption effect In the region of high frequency waves it is necessary to take account of the influence of the absorption phenomena of the insulators. In such a case, a set of the differential equations describing the wave propagation property is shown in the previous chapter and there the propagation term becomes

$$Q=\sqrt{[Ls+Z_C(s)]C_0\varepsilon(s)s} \qquad (8.3.4)$$

and numerical solutions in the time domain for the voltage can be given

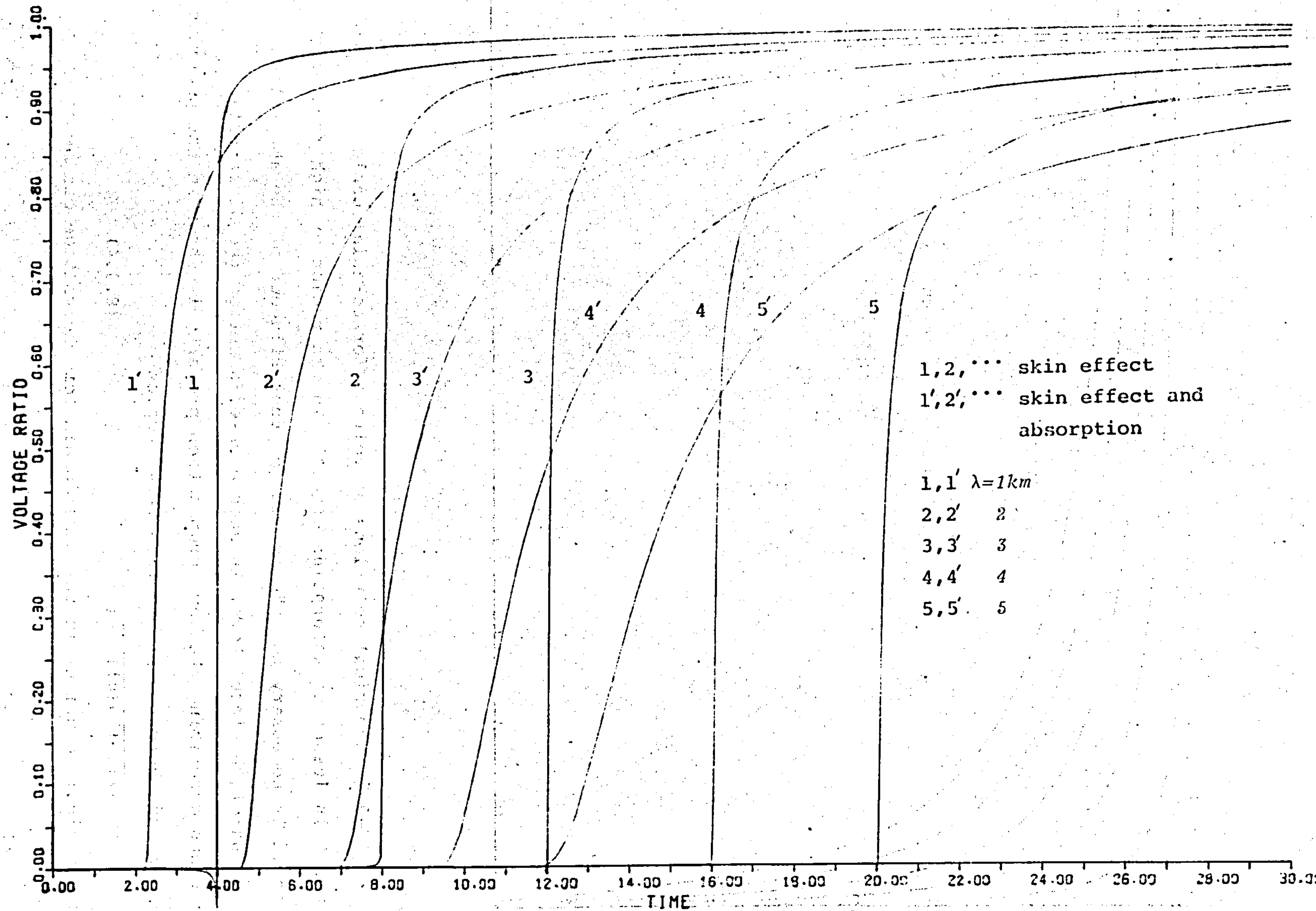


Fig 8-3-2 Influence of absorption for co-axial cable

without increasing the degree of difficulty in the numerical treatment.

In Fig 8-3-2 we shall show numerical solutions of the line voltage for the step voltage excitation given by considering the skin effect and the absorption effect comparing with the results given by considering only the skin effect under the following conditions

r=1cm R=2cm $\sigma_1=\sigma_2=5.92\times10^{10}$mho/km

$\varepsilon^{\infty}=2.7$ $\varepsilon^{0}=4.85$ $\tau=11\mu$sec $\alpha=5/8$

$T=30\mu$sec $aT=5$ $K=1024$

where the factor caused by the translation operator $e^{-\lambda s/c_0}$ caused by the light velocity is removed in every operational solution.

As is known in this example, the attenuation and distortion of the traveling waves become remarkable due to the influence of the absorption effect and because the value of the capacitive coefficient is dominated by ε^{∞} in the region of high frequency waves which is generally smaller than ε^{0}, the propagation velocity becomes faster comparing with the case without considering the absorption effect.

8.4 Lines with finite length

So far the discussions have been limited to the wave propagation properties on lines with infinite length, but for such systems that have shunt loads or composed of different lines, the reflection and transmission of the traveling waves must be considered at the connecting points.

In a complicated system, the voltages at the connecting points become sometimes greater than the imposed voltage by the successive reflections, therefore, it is important to know them for example in considering the insulation design for a power system.

Here we shall consider a system consisted in lines with finite length, then for i-th line the wave propagation property is given by the following set of the differential equations

$$\left.\begin{aligned} -\frac{dV_i}{d\lambda}&=Z_i(s)I_i \\ -\frac{dI_i}{d\lambda}&=Y_i(s)V_i \end{aligned}\right\} \qquad (8.4.1)$$

where every influence caused by the skin effect, absorption effect, line resistance or leakage is contained in $Z_i(s)$ or $Y_i(s)$ if necessary.

General solution of the above equations is given by

$$\left.\begin{aligned} V_i&=e^{-\lambda Q_i}C_i+e^{\lambda Q_i}D_i \\ I_i&=P_i(e^{-\lambda Q_i}C_i-e^{\lambda Q_i}D_i) \\ Q_i&=\sqrt{Z_i(s)Y_i(s)} \qquad P_i=\sqrt{Y_i(s)/Z_i(s)} \end{aligned}\right\} \qquad (8.4.2)$$

where arbitrary constants C_i and D_i are determined by terminal conditions at the connecting points and the operational solutions of the voltage at any points of the system are determined.

By the numerical inversion formula we can get the numerical solution in the time domain if the operational solution is determined, therefore, we can get the numerical values of voltage and current responses by using the above operational solutions.

In order to get the operational solutions from given terminal conditions, it is necessary to solve the set of linear equations with literal coefficients Let us consider a system with n different lines, then the order of the simultaneous equations becomes $2n$ and it is pretty difficult to solve it analytically as n increases, but in such a case every coefficient is determined by a function with respect to the operator s and in the numerical inversion we need spectra of the operational solution for certain values of s , therefore, it is preferable to solve the equations numerically.

In such a case the structure of the coefficient matrix of a set of linear equations depends heavily on the topology of the system. For example, let us assume the number of connection to n_i at the i-th connecting point as shown in Fig 8-4-1, then the number of non-zero coefficients concerned in this point is $4n_i-4$ for the voltage relation and $2n_i$ for the current relation and totally $6n_i-4$ and number of all the non-zero coefficients given by summing up them through all the connecting points is smaller than the number of all the elements of the coefficient matrix $4n^2$ and they are distributed in the neighborhood of the diagonal elements, therefore, the sparce matrix technique[34] can be used effectively in solving the linear equations.

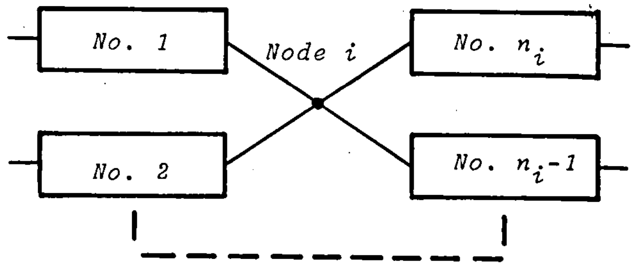


Fig 8-4-1 Connecting point i

Specially for a cascaded system the ratio of non-zero elements in the coefficient matrix is at most $2/n$.

For an example, we shall analyse the voltages at connecting poins of the system shown in Fig 8-4-2

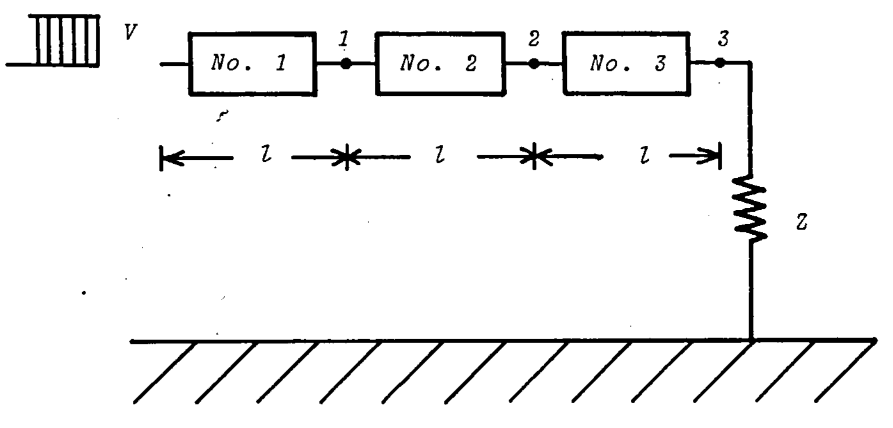


Fig 8-4-2 Model system

where No.1 line is an airial line whose parameters are

h/r=2000 h'=3 l=500m

No.2 line is a co-axial cable whose parameters are

r=1cm R=2cm $\sigma_1=\sigma_2=5.92\times10^{10}$ mho/km $\varepsilon^{\infty}=2.7$ $\varepsilon^{0}=4.85$ τ=11μsec

$\alpha=5/8$ l=500m

and No.3 line is an airial line whose parameters are

h/r=1000 h'=1 l=500m

and Z is the surge impe~~d~~dance of No.3 line given by considering it loss less.

The voltage responses at the junction points for step voltage excitation are shown in Fig 8-4-3 under the numerical conditions

T=30μsec aT=5 K=1024.

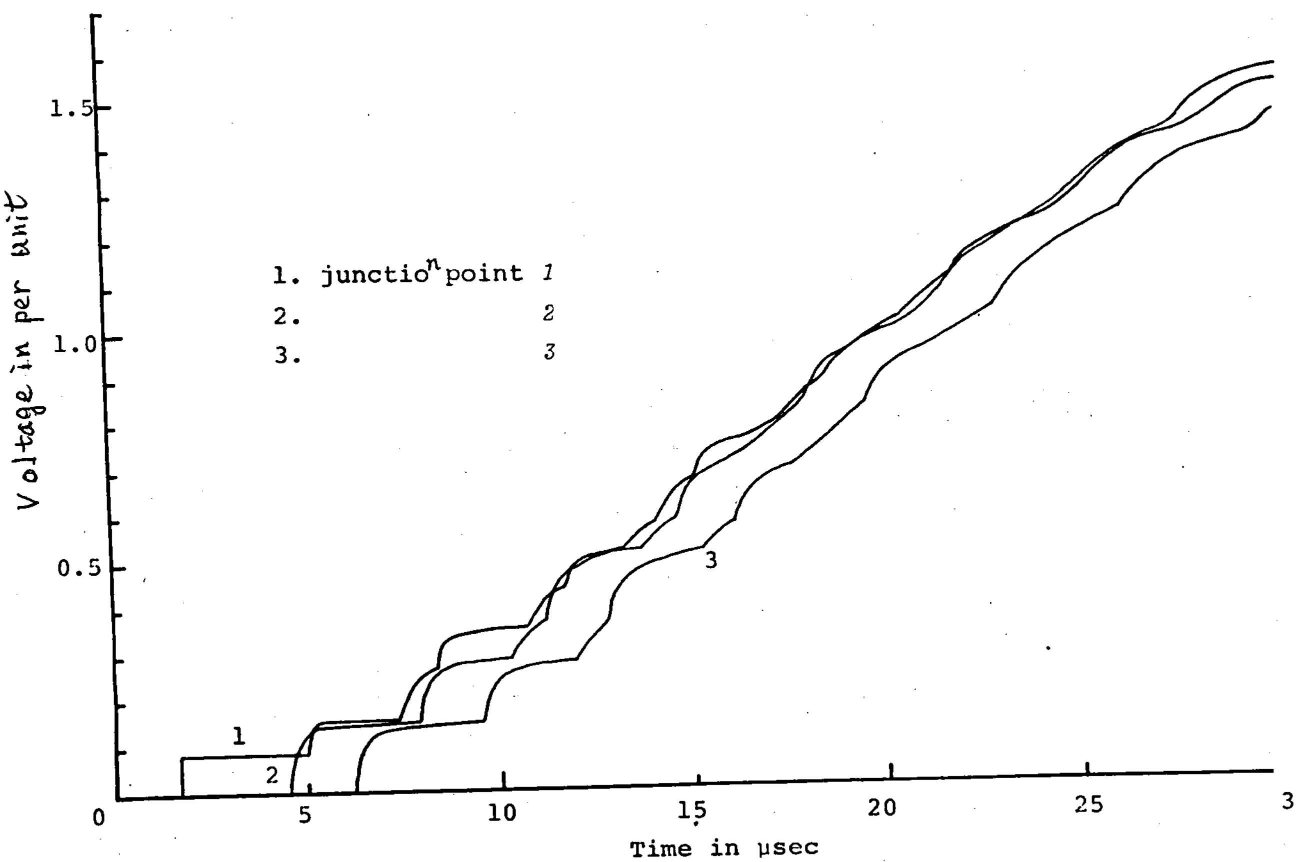


Fig 8-4-3 Numerical solution of the model system

8.5 Multi-conductor transmission lines

8.5.1 General solutions of differential equations The properties of the traveling waves on multi-conductor transmission lines are given by the set of the simultaneous differential equations shown in the previous chapter and the general solutions of the operational forms are given as

$$\left.\begin{aligned} &V=e^{-\lambda Q}C_1+e^{\lambda Q}C_2 \\ &I=Z^{-1}Q(e^{-\lambda Q}C_1-e^{\lambda Q}C_2) \\ &Q^2=ZY \end{aligned}\right\} \qquad (8.5.1)$$

where Z and Y are the equivalent impe~~t~~dance and admittance matrices of the system. C_1 and C_2 are arbitrary constant vectors ans they are determined from the terminal conditions and by doing so the operational solutions at any point λ are determined. By using the set of spectra of them with respect to the operator s numerical responses of the line voltage or current vector in the time domain are given by the numerical inversion formula.

In this case main effort in digital computation is directed to get numerical values of the exponential function of the complex matrix $e^{-\lambda Q}$.

By the definition we have

$$e^{-\lambda Q}=1-\lambda Q+\frac{\lambda^2 Q^2}{2!}-\ldots.. \qquad (8.5.2)$$

and because we can know the matrix Q^2 but not Q from the line constants, we can save labor in computation by the following series.

$$e^{-\lambda Q}=\left(1+\frac{\lambda^2 Q^2}{2!}+\frac{\lambda^4 Q^4}{4!}+\ldots.\right)-\lambda Q\left(1+\lambda^2 Q^2+\frac{\lambda^4 Q^4}{3!}+\ldots.\right) \qquad (8.5.3)$$

The numerical values of $e^{-\lambda Q}$ for certain values of s is given approximately by a finite sum of the above series if the numerical values of the matrix Q is known from those of Q^2 in any way and each of eigen values of the matrix λQ lies within the convergence circle of the corresponding scalar series which does not affect the numerical stability.

8.5.2 Square root of matrix A formula to get the square root of any positive number a is given by solving the following equation by using the Newton's method

$$y^2-a=0 \tag{8.5.4}$$

as the following iterative formula.

$$\left.\begin{aligned} &y_{n+1}=\frac{1}{2}(y_n+a/y_n) \\ &y_1 \quad \text{any positive number} \end{aligned}\right\} \tag{8.5.5}$$

Here we shall consider such a case that a is any complex number.
Let us note $a=Re^{i\theta}$ and choose the initial value as $y_1=re^{i\theta_0}$, then (8.5.5) becomes

$$x_{n+1}=\frac{1}{2}(x_n+1/x_n) \tag{8.5.6}$$

where

$$x_{n+1}=y_{n+1}e^{-i\theta/2}/\sqrt{R} \qquad x_n=y_ne^{-i\theta/2}/\sqrt{R}.$$

Furthermore let us note $x_{n+1}=R_{n+1}e^{i\phi_{n+1}}$ and $x_n=R_ne^{i\phi_n}$, then the following relations yield.

$$\left.\begin{aligned} &R_{n+1}^2=\frac{1}{4}(R_n+1/R_n+2\cos 2\phi_n) \\ &\tan\phi_{n+1}=\frac{R_n^2-1}{R_n^2+1}\tan\phi_n \end{aligned}\right\} \tag{8.5.7}$$

From the second relation by using the inequality $|(R_n^2-1)/(R_n^2+1)|<1$ we have

$$\lim_{n\to\infty}\tan\phi_n=0$$

and

$$\lim_{n\to\infty}\phi_n=\begin{cases} 0: & |\theta_0-\theta/2|<\pi/2 \\ \pi: & |\theta_0-\theta/2|>\pi/2 \end{cases} \tag{8.5.8}$$

and from the first relation by using this result, we have

$$\lim_{n\to\infty}R_n=1$$

therefore, if the initial value y_1 is chosen suitably we have

$$\lim_{n\to\infty} y_n = y = \begin{cases} \sqrt{a}: & |\theta_0-\theta/2|<\pi/2 \\ -\sqrt{a}: & |\theta_0-\theta/2|>\pi/2 \end{cases} \tag{8.5.9}$$

where $y=\sqrt{a}$ if $0\le \arg y<\pi$ and $y=-\sqrt{a}$ if $\pi\le\arg y<2\pi$ and the square root of any complex number can be obtained by this formula.

Next we shall consider a problem to get the square root of any complex matrix A satisfying the following relation.

$$Y^2=A \tag{8.5.10}$$

Let us consider the following special case where the matrix A is reduced to Jordan canonical form

$$\left.\begin{aligned} &A=P^{-1}JP \\ &J=\begin{pmatrix} \lambda_1 & 1 & & & & & & \\ & \lambda_1 & 1 & & & & & \\ & & \lambda_1 & & & & & \\ & & & \lambda_4 & 1 & & & \\ & & & & \lambda_4 & & & \\ & & & & & \lambda_6 & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_m \end{pmatrix} \end{aligned}\right\} \tag{8.5.11}$$

then from the result given in 2.3 we have

$$Y=P^{-1}\begin{pmatrix} \pm\begin{matrix} \sqrt{\lambda_1} & 1/2\sqrt{\lambda_1} & -1/8\lambda_1\sqrt{\lambda_1} \\ & \sqrt{\lambda_1} & 1/2\sqrt{\lambda_1} \\ & & \sqrt{\lambda_1} \end{matrix} & & & & \\ & \pm\begin{matrix} \sqrt{\lambda_4} & 1/2\sqrt{\lambda_4} \\ & \sqrt{\lambda_4} \end{matrix} & & & \\ & & \pm\sqrt{\lambda_6} & & \\ & & & \ddots & \\ & & & & \pm\sqrt{\lambda_m} \end{pmatrix}P. \tag{8.5.12}$$

Here we shall extend (8.5.5) to the following matric relation

$$\left.\begin{aligned} Y_{n+1}&=\tfrac{1}{2}(Y_n+Y_n^{-1}A) \\ Y_1&=\alpha 1 \end{aligned}\right\} \tag{8.5.13}$$

where α is chosen as

$$|\arg\alpha-\frac{1}{2}\arg\lambda i|<\frac{\pi}{2}$$

for every λ_i $(i=1,4,6,\ldots,m)$, then by the same transformation as in (8.5.11) we have

$$\left.\begin{array}{l} X_{n+1}=\frac{1}{2}(X_n+X_n^{-1}A) \\ X_1=\alpha 1 \end{array}\right\} \qquad (8.5.14)$$

where

$$X_{n+1}=P^{-1}Y_{n+1}P \qquad X_n=P^{-1}Y_nP.$$

Let us note

$$J(1)=\begin{pmatrix}\lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1\end{pmatrix} \qquad J(2)=\begin{pmatrix}\lambda_4 & 1 \\ & \lambda_4\end{pmatrix} \qquad J(3)=\begin{pmatrix}\lambda_6 & & \\ & \ddots & \\ & & \lambda_m\end{pmatrix}$$

then J can be represented as

$$J=J(1)\dot{+}J(2)\dot{+}J(3) \qquad (8.5.15)$$

and it may be sufficient to consider the relation of (8.5.14) for J(1), J(2) and J(3) separately.

For J(1) let us note

$$Xn+1=\begin{pmatrix}X_{n+1}^{11} & X_{n+1}^{12} & X_{n+1}^{13} \\ & X_{n+1}^{22} & X_{n+1}^{23} \\ & & X_{n+1}^{33}\end{pmatrix} \qquad Xn=\begin{pmatrix}X_n^{11} & X_n^{12} & X_n^{13} \\ & X_n^{22} & X_n^{23} \\ & & X_n^{33}\end{pmatrix}$$

then from the iterative formula of (8.5.14) we have

$$\left.\begin{array}{l} X_{n+1}^{ii}=\frac{1}{2}(X_n^{ii}+\lambda_1/X_n^{ii}) \qquad i=1,2,3 \\ X_{n+1}^{12}=\frac{1}{2}(X_n^{12}+1/X_n^{11}-\lambda_1 X_n^{12}/X_n^{11}X_n^{22}) \\ X_{n+1}^{23}=\frac{1}{2}(X_n^{23}+1/X_n^{22}-\lambda_1 X_n^{23}/X_n^{22}X_n^{33}) \\ X_{n+1}^{13}=\frac{1}{2}[X_n^{13}-X_n^{12}/X_n^{11}X_n^{22}+\lambda_1(X_n^{12}X_n^{13}-X_n^{13}X_n^{22})/X_n^{11}X_n^{22}X_n^{33}] \end{array}\right\}$$

and finally have the following relation.

$$\lim_{n\to\infty} X_n = X(1) = \begin{bmatrix} \sqrt{\lambda_1} & 1/2\sqrt{\lambda_1} & -1/8\lambda_1\sqrt{\lambda_1} \\ & \sqrt{\lambda_1} & 1/2\sqrt{\lambda_1} \\ & & \sqrt{\lambda_1} \end{bmatrix}$$

In the same way we have

$$X(2) = \begin{bmatrix} \sqrt{\lambda_4} & 1/2\sqrt{\lambda_4} \\ & \sqrt{\lambda_4} \end{bmatrix} \qquad X(3) = \begin{bmatrix} \sqrt{\lambda_6} & & \\ & \ddots & \\ & & \sqrt{\lambda_m} \end{bmatrix}$$

and

$$\lim_{n\to\infty} X_n = X(1) \dot{+} X(2) \dot{+} X(3). \tag{8.5.16}$$

In the arithmetic operations of the matrix algebra the simylar transformatior is held invariant and we can get the square root $\sqrt{A}$ (all signs are + in (8.5.12)) for the given matrix A by the relation of (8.5.13).

Above discussions are valid for any non-singular complex matrices and if we can know the region on the complex plane where each of the eigen values of the given matrix exists, we can get the square root of the matrix by the iterative formula, and by applying this formula to the previous discussions we can get numerical solutions of the line voltage or current vector in the time domain on the multi-conductor line.

8.6 Some examples

8.6.1 Induction voltage in parallel conductors

Here we shall consider a line of two conductors with infinite length shown in Fig 8-6-1 and investigate the voltage response numerically when either of two conductors is excited by a step voltage.

This problem is important to investigate the inductive interference between the power transmission line and the communication line.

The operational solution of the voltage vector of the system at the point distant λ from the excited point is given as

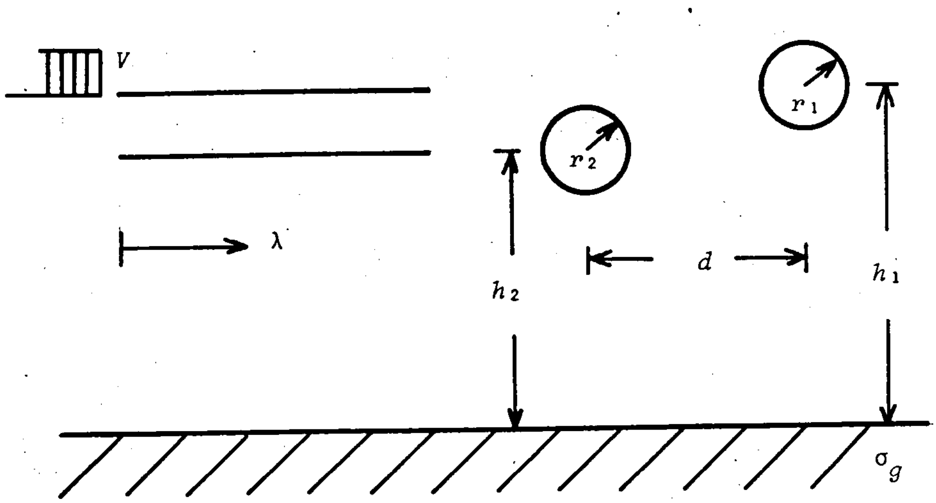
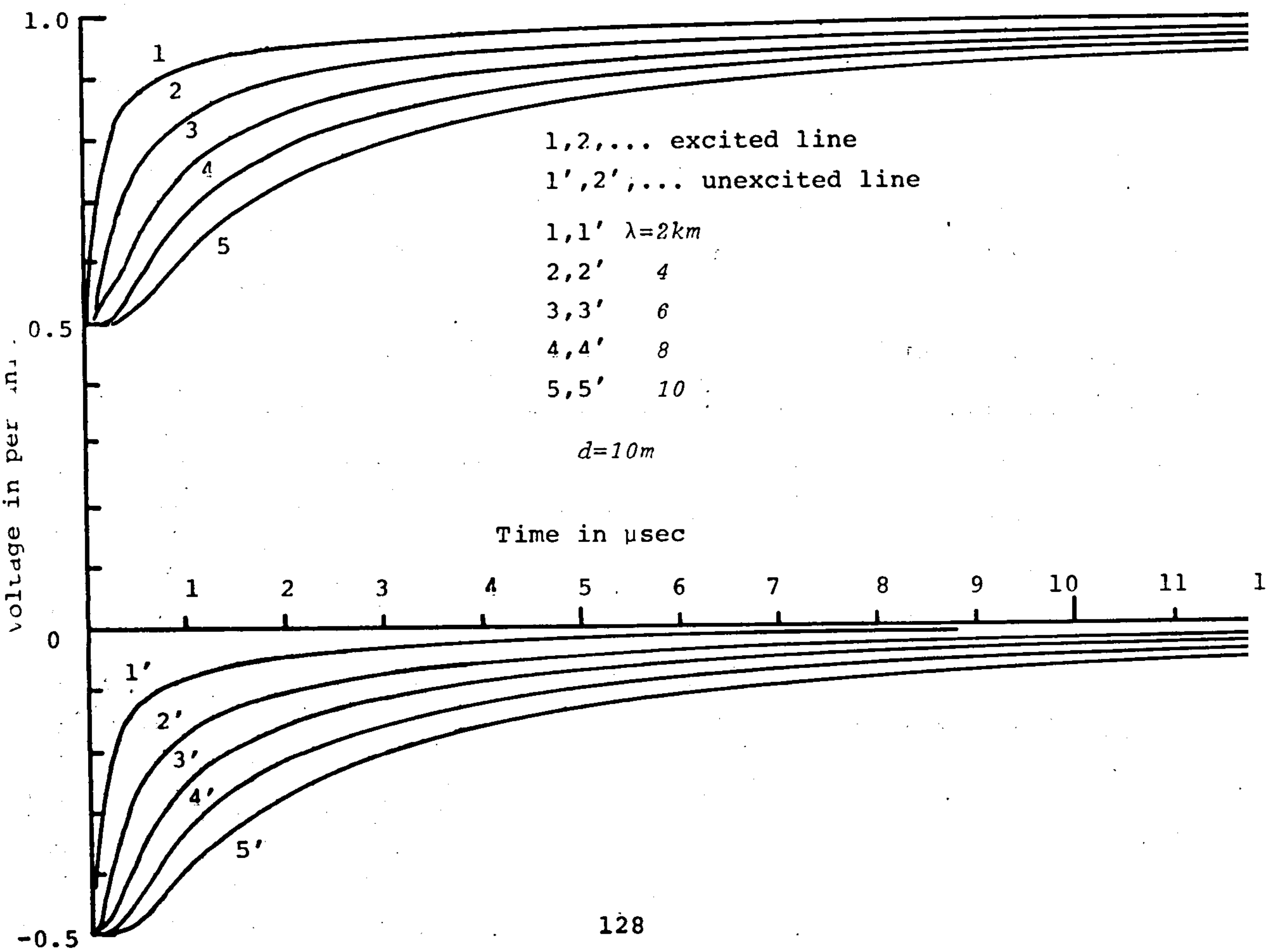


Fig 8-6-1 Model system for numerical example



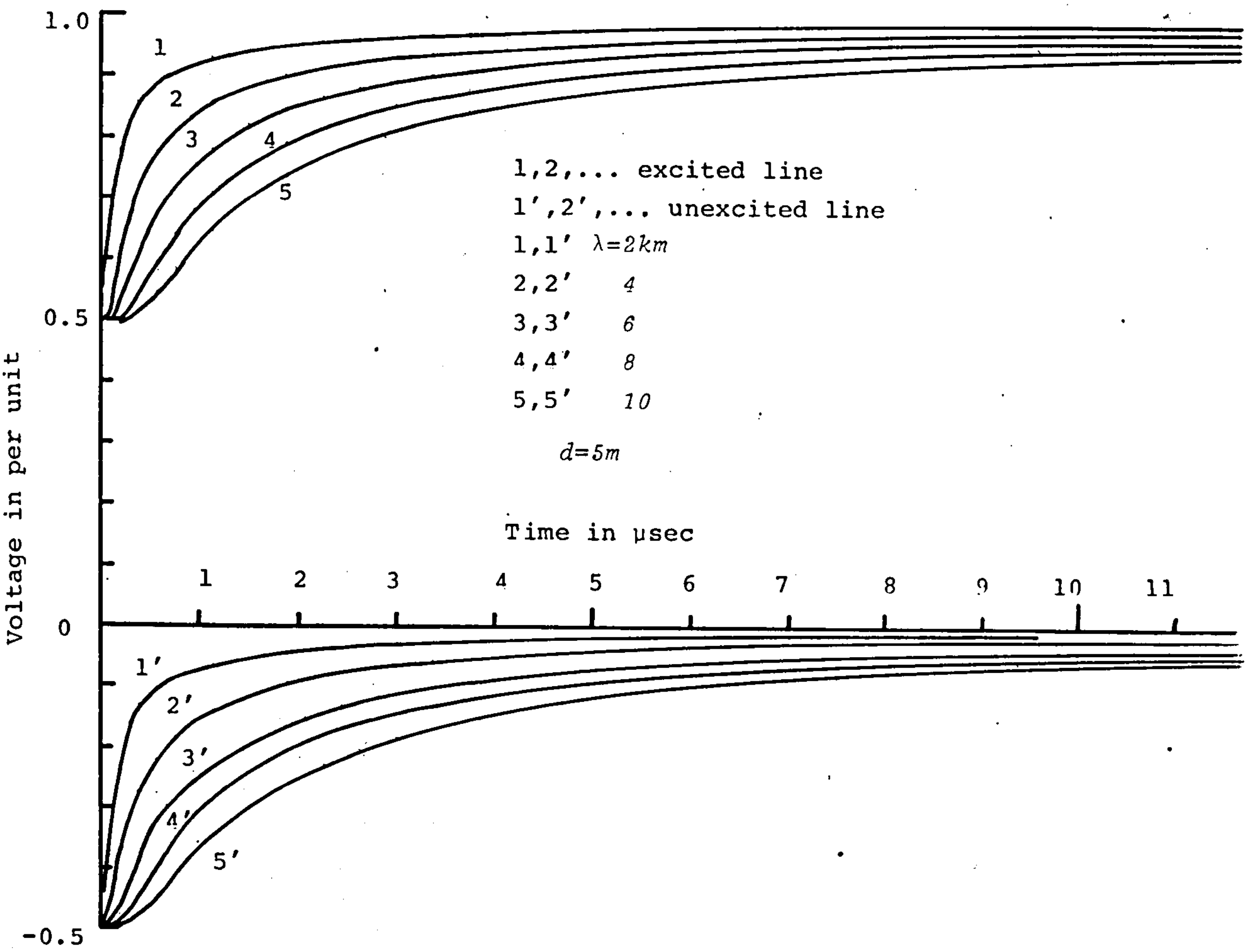


Fig 8-6-2 Numerical solutions for model system

$$V=e^{-\lambda Q}\begin{pmatrix}1/s\\0\end{pmatrix} \tag{8.6.1}$$

where every element of the matrix Q is shown in 7.2.

By applying the numerical inversion formula to this operational solution the time solution can be given numerically.

In Fig 8-6-2 we shall show some numerical examples under the following conditions.

$h_1=h_2=15\text{m}$ $r_1=r_2=1\text{cm}$ $\sigma_g=10\text{mho/km}$ $T=12\mu\text{sec}$ $aT=5$ $K=512$

Here the translation operator is removed in every example.

8.6.2 System with different lines Here we shall consider the following system shown in Fig 8-6-3.

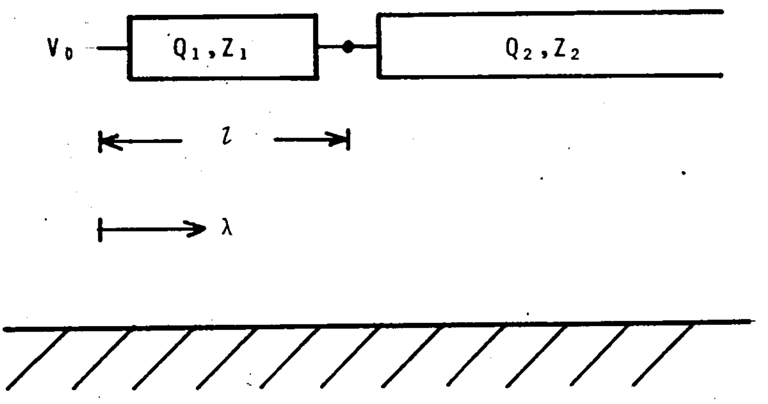


Fig 8-6-3 System with different lines

General solutions of the voltage and current vectors are given operationally by

$$\left.\begin{aligned} &V_1=e^{-\lambda Q_1}C_1+e^{\lambda Q_1}C_2 \\ &I_1=Z_1^{-1}Q_1(e^{-\lambda Q_1}C_1-e^{\lambda Q_1}C_2) \\ &\quad 0\le\lambda<l \end{aligned}\right\} \tag{8.6.2}$$

$$\left.\begin{aligned} &V_2=e^{-\lambda Q_2}C_3 \\ &I_2=Z_2^{-1}Q_2e^{-\lambda Q_2}C_3 \\ &\quad l\le\lambda \end{aligned}\right\} \tag{8.6.3}$$

and the boundary conditions are given by

$$\left.\begin{aligned} &V_1=V_0 \qquad \lambda=0 \\ &V_1=V_2 \qquad I_1=I_2 \qquad \lambda=l \end{aligned}\right\}. \tag{8.6.4}$$

From these results we have

$$\left.\begin{aligned} &C_1=[(Z_1^{-1}Q_1-Z_2^{-1}Q_2)e^{-lQ_1}+(Z_1^{-1}Q_1+Z_2^{-1}Q_2)e^{lQ_1}]^{-1}(Z_1^{-1}Q_1+Z_2^{-1}Q_2)e^{lQ_1}V_0 \\ &C_2=[(Z_1^{-1}Q_1-Z_2^{-1}Q_2)e^{-lQ_1}+(Z_1^{-1}Q_1+Z_2^{-1}Q_2)e^{lQ_1}]^{-1}(Z_1^{-1}Q_1-Z_2^{-1}Q_2)e^{-lQ_1}V_0 \\ &C_3=2e^{lQ_2}(Z_1^{-1}Q_1+Z_2^{-1}Q_2)^{-1}Z_1^{-1}Q_1e^{-lQ_1}C_1 \end{aligned}\right\} \tag{8.6.5}$$

and for these determined results we can get time solutions numerically by applying the numerical inversion formula.

For a numerical example we shall show the influence of transposition of the conductors to the traveling waves on a three-phase transmission line.

The configuration of the conductors is shown in Fig 8-6-4 where

h_1=25.4m h_2=20.4m h_3=14.2m d_{12}=2.5m d_{13}=1m d_{23}=1.5m

r_1=r_2=r_3=1cm σ_g=10mho/km.

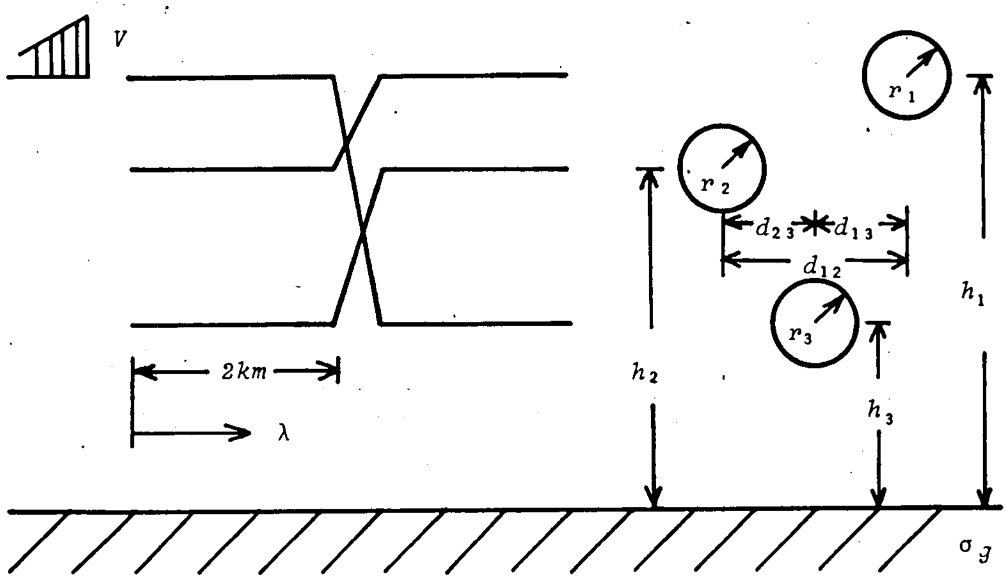


Fig 8-6-4 Model system for numerical example

Numerical solution of the voltage at λ=1km apart from the excited point when No.1 conductor is excited by the exponential wave with the time constant of 20 μsec under the following numerical conditions

T=20μsec aT=5 K=256

is shown in Fig 8-6-5 comparing with the result in untransposed line.

In this case every value of the equivalent impedances for the skin effect of the ground is given not by the numerical integration but by the third approximation stated in 8.2.1.

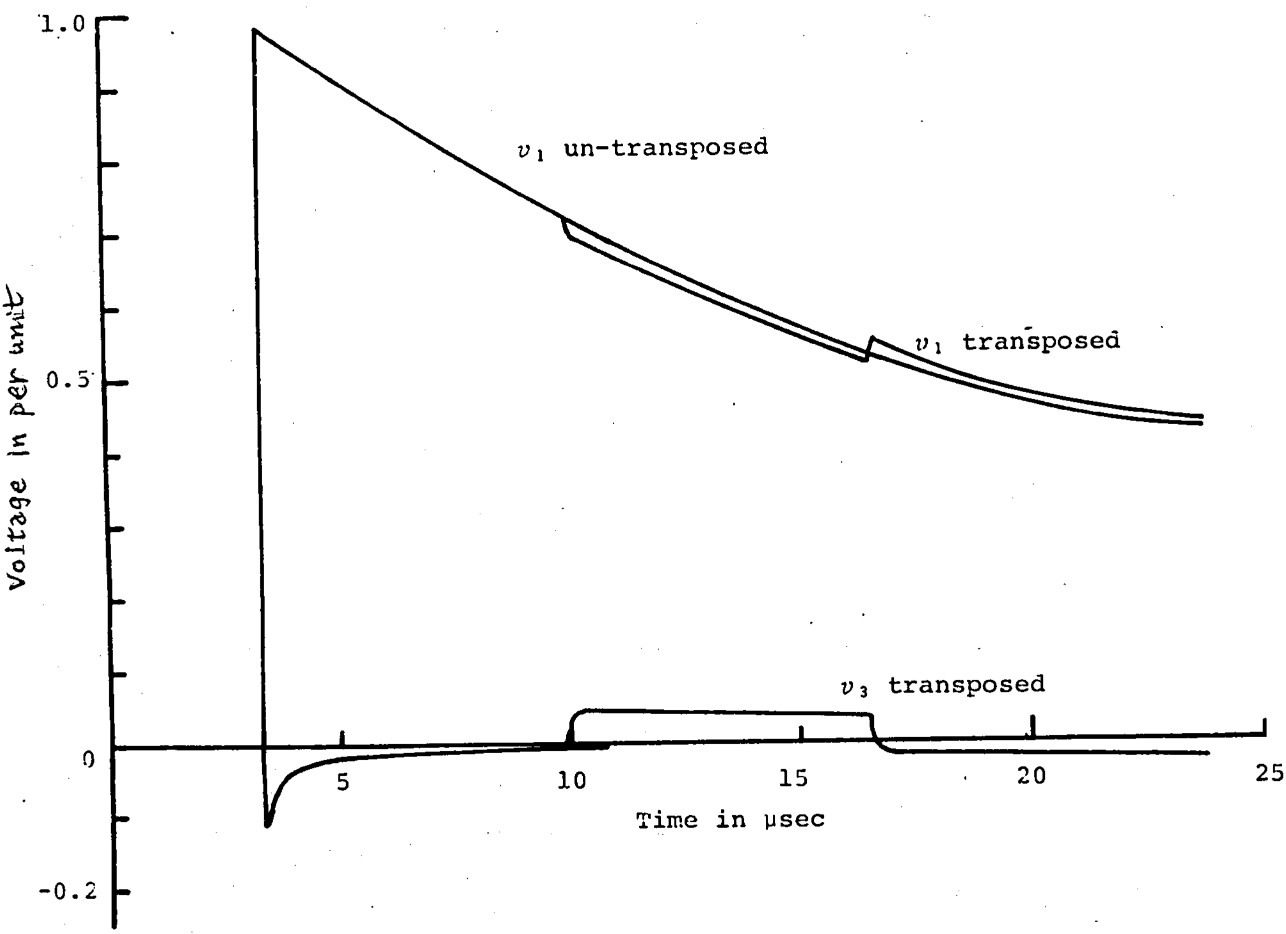


Fig 8-6-5 Numerical solution for model system

8.6.3 System with shunt load Here we shall consider a system with shunt load shown in Fig 8-6-6.

General solutions for voltage and current vectors are given the same as (8.6.2) and (8.6.3) and the boundary conditions are given by

$$\left.\begin{array}{ll} V_1=V_0 & \lambda=0 \\ V_1=V_2 \quad I_1=I_2+Y_g V_1 & \lambda=l \end{array}\right\}. \tag{8.6.6}$$

By these relations we have

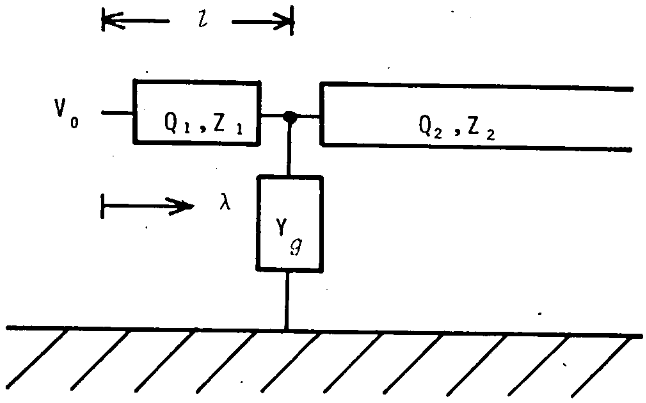


Fig 8-6-6 System with shunt load

$$\left.\begin{aligned}
C_1 &= [(Z_1^{-1}Q_1 - Z_2^{-1}Q_2 - Y_g)e^{-lQ_1} + (Z_1^{-1}Q_1 + Z_2^{-1}Q_2 + Y_g)e^{lQ_1}]^{-1} \\
&\quad (Z_1^{-1}Q_1 + Z_2^{-1}Q_2 + Y_g)e^{lQ_1}V_0 \\
C_2 &= [(Z_1^{-1}Q_1 - Z_2^{-1}Q_2 - Y_g)e^{-lQ_1} + (Z_1^{-1}Q_1 + Z_2^{-1}Q_2 + Y_g)e^{lQ_1}]^{-1} \\
&\quad (Z_1^{-1}Q_1 - Z_2^{-1}Q_2 - Y_g)e^{-lQ_1}V_0 \\
C_3 &= e^{lQ_2}e^{-lQ_1}[(Z_1^{-1}Q_1 - Z_2^{-1}Q_2 - Y_g)e^{-lQ_1} + (Z_1^{-1}Q_1 + Z_2^{-1}Q_2 \\
&\quad + Y_g)e^{lQ_1}]^{-1}(Z_1^{-1}Q_1 + Z_2^{-1}Q_2 + Y_g)e^{-lQ_1}V_0 + e^{lQ_2}e^{lQ_1} \\
&\quad [(Z_1^{-1}Q_1 - Z_2^{-1}Q_2 - Y_g)e^{-lQ_1} + (Z_1^{-1}Q_1 + Z_2^{-1}Q_2 + Y_g)e^{lQ_1}]^{-1} \\
&\quad (Z_1^{-1}Q_1 - Z_2^{-1}Q_2 - Y_g)e^{-lQ_1}V_0
\end{aligned}\right\} \qquad (8.6.7)$$

Specially for the system shown in Fig 8-6-7 we have

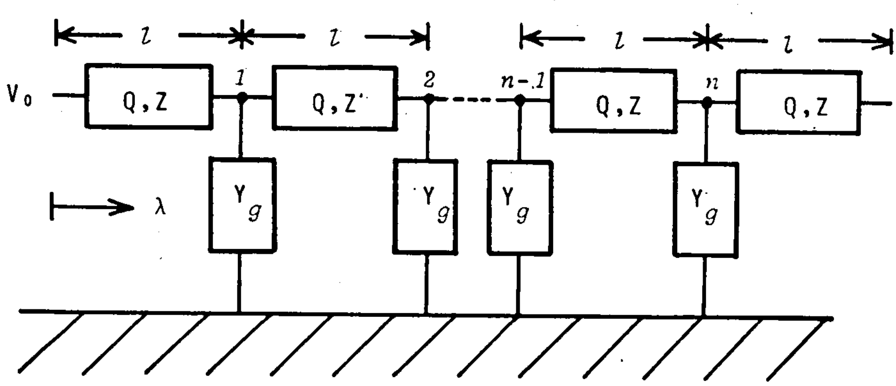


Fig 8-6-7 System with shunt loads at the points of same intervals

$$\left.\begin{aligned}&V=e^{-\lambda Q}C^{n}V_0\\&C=[(2I+Q^{-1}ZY_g)e^{lQ}-Q^{-1}ZY_ge^{-lQ}]^{-1}(2I+Q^{-1}ZY\ )e^{lQ}\\&\quad -e^{2lQ}[(2I+Q^{-1}ZY_g)e^{lQ}-Q^{-1}ZY_ge^{-lQ}]^{-1}Q^{-1}ZY_ge^{-lQ}\\&\qquad\quad \lambda\geq nl\end{aligned}\right\}\quad(8.6.8)$$

for the early phenomenon of the traveling wave.

For a numerical example we shall show the following system with an airial ground wire shown in Fig 8-6-8.

The numerical solution of the voltage vector at λ=1km apart from the excited point when No.2 and No.3 conductors are excited by the exponential wave with the time constant of 20 μsec under the following conditions

T=20μsec aT=5 K=256

is shown in Fig 8-6-9 where every value of the equivalent impedances is given by the third approximation stated in 8.2.1.

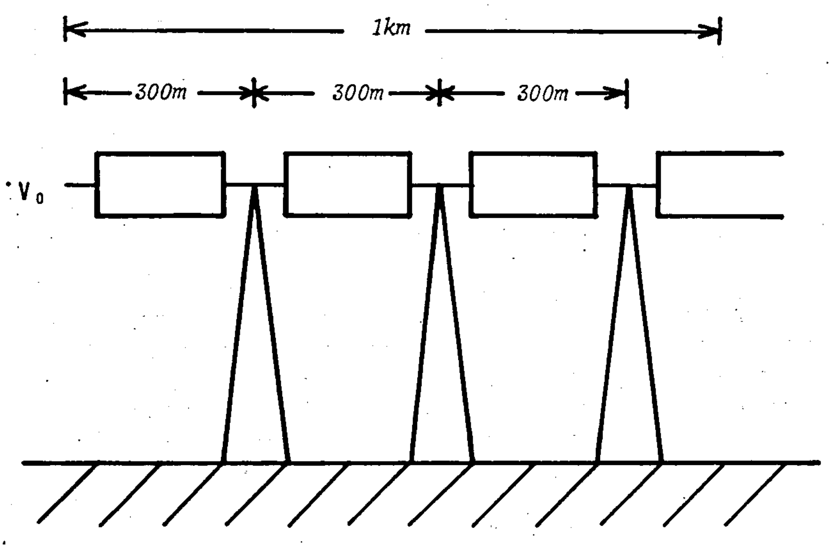


Fig 8-6-8 (a) System with airial ground wire

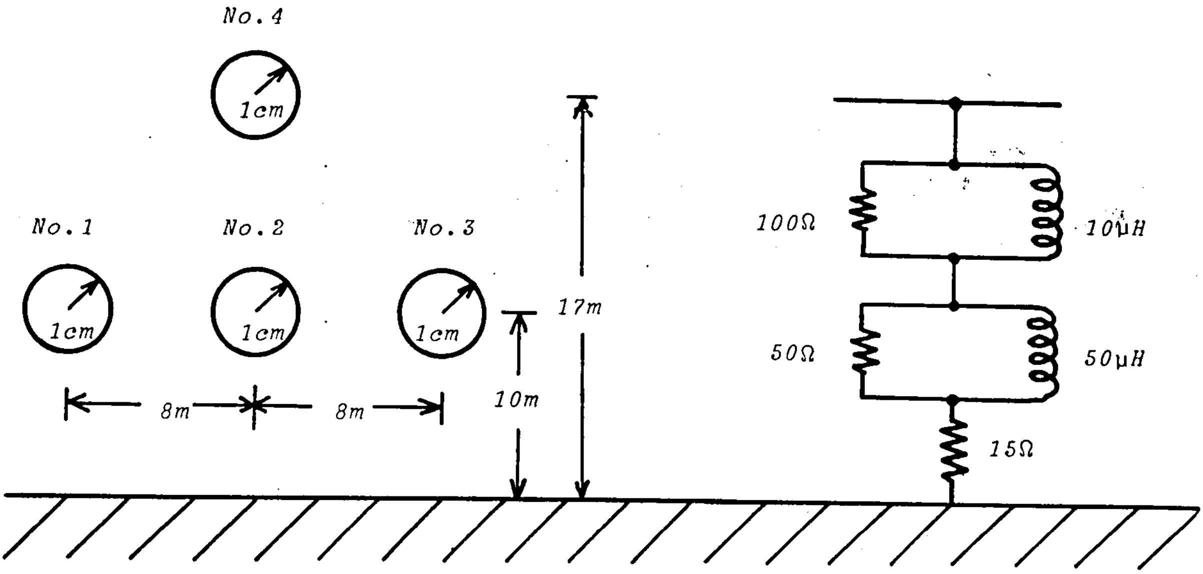


Fig 8-6-8 (b) Configuration of conductors and equivalent circuit of iron tower[35)]

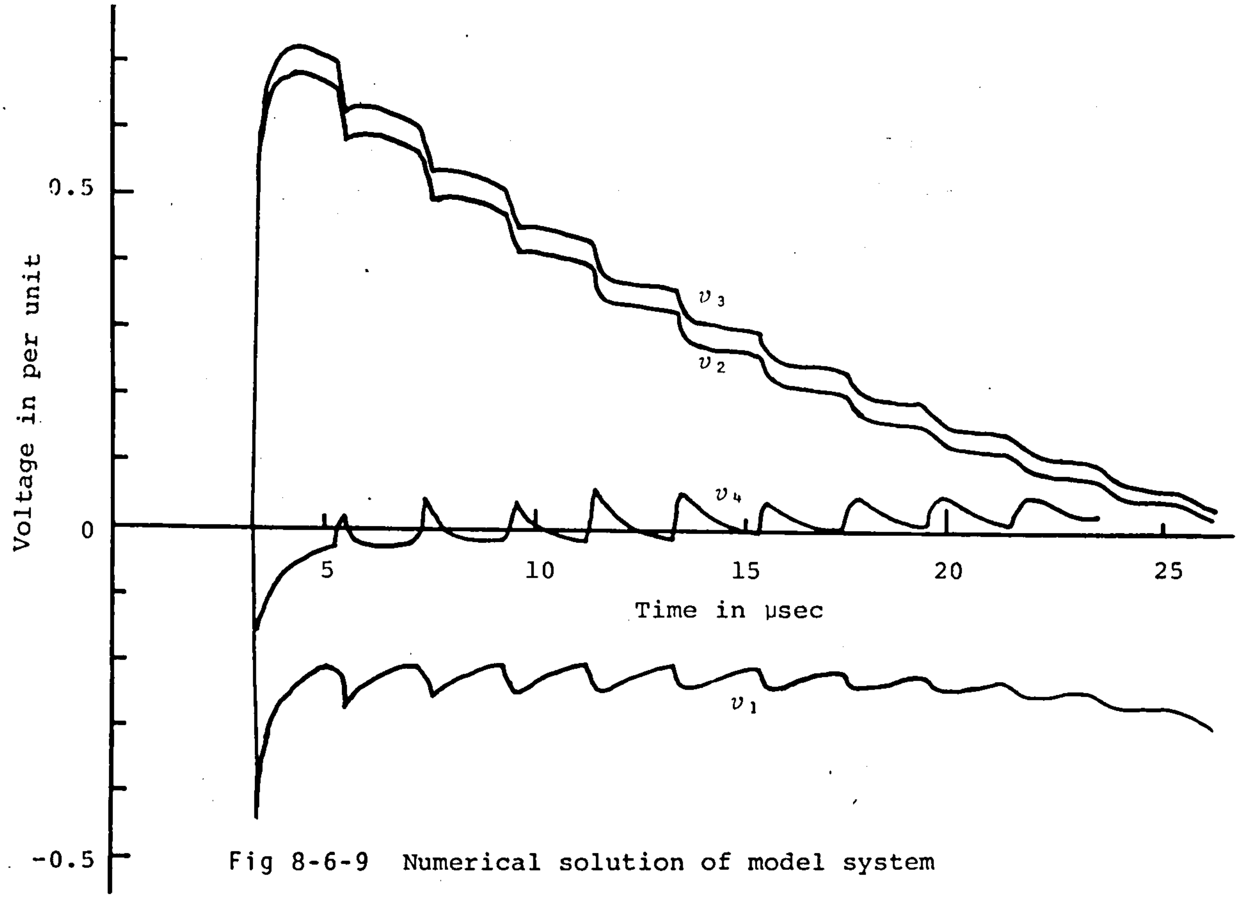


Fig 8-6-9 Numerical solution of model system

In these examples the operational solutions are fairly complicated although the model systems are not so complicated because of the non-commutativity of matrices in multiplications, therefore, in general case it is preferable to get the operational solution numerically in the same way as in 8.4.

8.7 Summary

In this chapter the numerical methods and some examples for some problems of the traveling waves on the transmission systems which can be analysed by the Laplace transform mainly related to the influences of the skin effect which had been treated only approximately were given. Every result given here is the numerical solution of the model system and has not been compa~~i~~red with the experimental or observed result.

Whether given results are good approximation or not depends only the accuracy of the mathematical models for the actual phenomena and for a given system the method stated here can be applied effectively by changing the values of some parameters.

9. Conclusion

In the study of this thesis, the results given in every chapter have been summarized in the last section of it. They are arranged collectively as follows.

(1) The operational calculus has been extended to the matrix form in order to analyse the problems of the system with many variables systematically in physical or engineering applications.

(2) A theoretical treatment of matric operational calculus has been developed by the theories of the functional analysis.

(3) The relations between the matric operational solutions and their time functions are given with respect to their treatments in the formal, analytical and numerical points of view.

(4) The relations between the operational calculus and the Laplace transform have been cleared up and a practical method of numerical inversion method of the Laplace transform by the Fourier series technique has been given.

(5) Sufficient numerical results for the properties of traveling waves on the transmission systems which had not been solved satisfactory have been obtained by ápplying those results.

The applications of the method given here are not limited to the above problems and it can be used as the powerful tool to any other fields in physical or engineering applications.

Acknowledgement

This study was done in the laboratory of the electrical networks of Department of Electrical Engineering II in the Faculty of Engineering, Kyoto University and the auther wishes to thank Professor Akira Kishima for his cordial leading and kindly encouragement.

The auther also wishes to thank Assistant Professor Takao Ozawa for his useful suggestions.

Furthermore the auther wishes to thank the members of the laboratory of the electrical networks for their useful discussions and suggestions.

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Appendix A Topological properties of operators*

Let B denote a set of continuous and bounded functions in a given interval $t\in I$, then B is a vector space and for every function $f\in B$, let us define the norm of f by

$$\|f\|=\sup_{t\in I}|f(t)| \tag{A.1}$$

then B is a Banach space.

Definition: A sequence f_n of functions of B is said to converge to f uniformly in I, if for any positive number ε, there is an index N such that

$$\|f_n-f\|<\varepsilon \tag{A.2}$$

for every $n\geq N$.

Now let C be a set of continuous functions in the interval I.

Definition: A sequence f_n of functions of C is said to converge almost uniformly to f in I, if it converges uniformly to f in every bounded and closed interval contained in I.

An equivalent definition of almost uniform convergence is as follows.

Definition: A sequence f_n of functions of C is said to converge almost uniformly to f in I, if there is a positive function $b\in B$ such that the functions $b(t)f_n(t)$ belong to B and converge uniformly to the function $b(t)f(t)$.

Now let M be a family of one to one mappings b of B into a set C which embraces B. The element assigned by the mapping b to the element f will be denoted by bf. The set of all the elements $bf(f\in B)$ will be denoted by bB and we assume that C is the union of all the sets

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* See I-1 Part 6 Chapter IV

bB $(b \in M)$.

Let Q be a set of operators.

Definition: A sequence a_n of operators of Q is said to converge to a operationally, if there exsists such an operator q that a_n/q is a sequence of functions of C tending almost uniformly to a/q.

Let us consider the mappings which consist in multiplying by an operator $c \neq 0$ and denote the mapping by the same letter c. Let O be a family of mappings of C into Q and we assume that Q is the union of all the sets cC. If the convergence in C is almost uniform, then the convergence in Q, obtained by means of the family O is the operational convergence.

Let P be a family of all the mappings cb which consist successive performances of two mappings $b \in M$ and $c \in O$, then if the convergence in B is uniform, the convergence in Q is operational.

P has the following properties.

(1) For any $a_1 \in P$ and $a_2 \in P$ there is a a_3 such that $a_1 \leq a_3$ and $a_2 \leq a_3$. The notation $a_1 \leq a_3$ means that $a_1^{-1} f_n \underset{n}{\to} a_1^{-1} f$ implies $a_3^{-1} f_n \underset{n}{\to} a_3^{-1} f$ where $a_1^{-1} f_n \underset{n}{\to} a_1^{-1} f$ means that the elements $a_1^{-1} f_n$ and $a_1^{-1} f$ exist and $a_1^{-1} f_n$ converges to the limit $a_1^{-1} f$.

(2) P contains the identical mapping of Q on Q.

Here let us define the norm of $af \in Q (a \in P$ and $f \in B)$ by

$$\|af\| = \|f\| \qquad \text{(A.3)}$$

then Q is a Banach space.

Let B be the set of continuous and bounded functions in the interval $0 \leq t < \infty$ and let P be the family of mappings defined above, we have

$$B \subset C \subset Q. \qquad \text{(A.4)}$$

The mappings b, c and $a = cb$ are injections and we can show that C and Q are not Banach spaces, but by the Miksinsky's definition of operational convergence, we can introduce the norm of functions of B in the space Q.

Appendix B Formulation of state equations by topological method[5]

B.1 Passive RLC networks Let us consider a passive RLC network with the following restrictions.

(i) It precludes a loop of voltage souces.

(ii) It precludes a cut set of current souces.

(iii) It does not contain perfectly coupled inductors.

Let N be the graph corresponding to a given network and let us choose a normal tree T and its co-tree T^c which are

$$T \cup T^c = N \qquad T \cap T^c = \phi$$

where

$$T = T_V \dotplus T_C \dotplus T_\Gamma \dotplus T_G$$
$$T^c = T_I^c \dotplus T_S^c \dotplus T_L^c \dotplus T_R^c$$

and are determined as follows.

(1) All voltage souces are designated as branches in T_V and all current souces as links in T_I^c.

(2) All capacitive elements C, less the minimum required to prevent the formulation of a loop are designated as branches in T_C, those excluded S as links in T_S^c.

(3) All inductive elements L, less the minimum required to prevent the formulation of a cut set are designated as links in T_L^c, those excluded Γ as branches in T_Γ.

(4) Resistive elements G which do not form a loop are designated as branches in T_G and the remainders R as links in T_R^c.

Let $v_V, i_V, \ldots$ be voltage and current vectors for every set, then topological relations are

$$\begin{pmatrix} i_V \\ i_C \\ i_G \\ i_\Gamma \end{pmatrix} = -\begin{pmatrix} F_{VS} & F_{VR} & F_{VL} & F_{VI} \\ F_{CS} & F_{CR} & F_{CL} & F_{CI} \\ 0 & F_{GR} & F_{GL} & F_{GI} \\ 0 & 0 & F_{\Gamma L} & F_{\Gamma I} \end{pmatrix} \begin{pmatrix} i_S \\ i_R \\ i_L \\ i_I \end{pmatrix} \tag{B.1.1}$$

$$\begin{pmatrix} v_S \\ v_R \\ v_L \\ v_I \end{pmatrix} = \begin{pmatrix} F^t_{VS} & F^t_{CS} & 0 & 0 \\ F^t_{VR} & F^t_{CR} & F^t_{GR} & 0 \\ F^t_{VL} & F^t_{CL} & F^t_{GL} & F^t_{\Gamma L} \\ F^t_{VI} & F^t_{CI} & F^t_{GI} & F^t_{\Gamma I} \end{pmatrix} \begin{pmatrix} v_V \\ v_C \\ v_G \\ v_\Gamma \end{pmatrix} \tag{B.1.2}$$

and relations between voltage and current vectors in every set are

$$\begin{pmatrix} i_C \\ i_S \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} C_C & 0 \\ 0 & C_S \end{pmatrix} \begin{pmatrix} v_C \\ v_S \end{pmatrix} \tag{B.1.3}$$

$$\begin{pmatrix} v_\Gamma \\ v_L \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} L_{\Gamma\Gamma} & L_{\Gamma L} \\ L_{L\Gamma} & L_{LL} \end{pmatrix} \begin{pmatrix} i_\Gamma \\ i_L \end{pmatrix} \tag{B.1.4}$$

$$\begin{pmatrix} i_G \\ v_R \end{pmatrix} = \begin{pmatrix} G_G & 0 \\ 0 & R_R \end{pmatrix} \begin{pmatrix} v_G \\ i_R \end{pmatrix}. \tag{B.1.5}$$

Eliminating i_S, i_Γ, v_S and v_Γ in (B.1.1) and (B.1.2) by (B.1.3) and (B.1.4), we have

$$\frac{d}{dt} \begin{pmatrix} Cv_C + F_{CS} C_S F^t_{VS} v_V \\ Li_L + (F^t_{\Gamma L} L_{\Gamma\Gamma} - L_{\Gamma L}) F_{\Gamma I} i_I \end{pmatrix} = \begin{pmatrix} 0 & -F_{CR} \\ F^t_{GL} & 0 \end{pmatrix} \begin{pmatrix} v_G \\ i_R \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -F_{CL} \\ F^t_{CL} & 0 \end{pmatrix} \begin{pmatrix} v_C \\ i_L \end{pmatrix} + \begin{pmatrix} 0 & F_{CI} \\ F^t_{VL} & 0 \end{pmatrix} \begin{pmatrix} v_V \\ i_I \end{pmatrix} \tag{B.1.6}$$

where

$$C = C_C + F_{CS} C_S F^t_{CS}$$

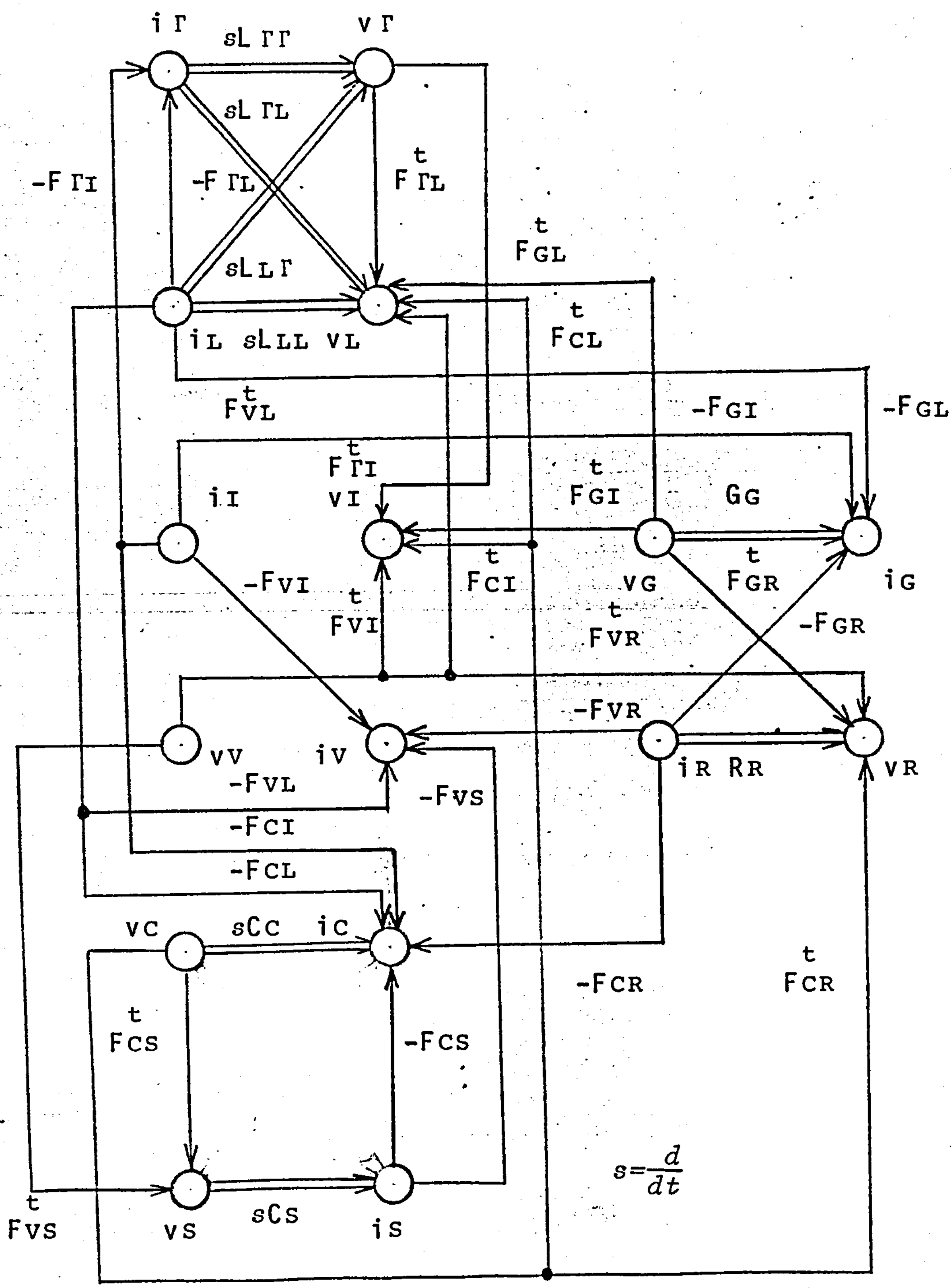

Fig B-1 Flow graph of RLC network

$$L=\begin{pmatrix}-F^t_{\Gamma L} & 1\end{pmatrix}\begin{pmatrix}L_{\Gamma\Gamma} & L_{\Gamma L}\\ L_{L\Gamma} & L_{LL}\end{pmatrix}\begin{pmatrix}F_{\Gamma L}\\ 1\end{pmatrix}.$$

Eliminating v_G, i_G, v_R and i_R in (B.1.6) by (B.1.1),(B.1.2) and (B.1.5) we have

$$\frac{d}{dt}\begin{pmatrix}Cv_C+F_{CS}C_SF^t_{VS}v_V\\ Li_L+(F^t_{\Gamma L}L_{\Gamma\Gamma}-L_{\Gamma L})F_{\Gamma I}i_I\end{pmatrix}=\begin{pmatrix}A_{CC} & A_{CL}\\ A_{LC} & A_{LL}\end{pmatrix}\begin{pmatrix}v_C\\ i_L\end{pmatrix}$$

$$+\begin{pmatrix}A_{CV} & A_{CI}\\ A_{LV} & A_{LI}\end{pmatrix}\begin{pmatrix}v_V\\ i_I\end{pmatrix} \qquad \text{(B.1.7)}$$

where

$$\begin{aligned}
A_{CC}&=-F_{CR}R^{-1}F^t_{CR}\\
A_{CL}&=F_{CR}R^{-1}F^t_{GR}G_G^{-1}-F_{CL}\\
A_{LC}&=F^t_{CL}-F^t_{GL}G^{-1}F_{GR}R_R^{-1}F^t_{CR}\\
A_{LL}&=F^t_{GL}G^{-1}F_{GL}\\
A_{CV}&=-F_{CR}R^{-1}F^t_{VR}\\
A_{CI}&=F_{CR}R^{-1}F^t_{GR}G_G^{-1}F_{GI}-F_{CI}\\
A_{LV}&=F^t_{VL}-F^t_{GL}G^{-1}F_{GR}R_R^{-1}-F^t_{VR}\\
A_{LI}&=F^t_{GL}G^{-1}F_{GI}
\end{aligned}$$

and

$$\begin{aligned}
R&=R_R+F^t_{GR}G_G^{-1}F_{GR}\\
G&=G_G+F_{GR}R_R^{-1}F^t_{GR}.
\end{aligned}$$

Let q and Φ be new vectors defined by

$$\begin{aligned}
q&=Cv_C+F_{CS}C_SF_{VS}v_V\\
\Phi&=Li_L+(F^t_{\Gamma L}L_{\Gamma\Gamma}-L_{L\Gamma})F_{\Gamma I}i_I
\end{aligned} \qquad \text{(B.1.8)}$$

the following state equation yields.

$$\frac{d}{dt}\begin{pmatrix} q \\ \Phi \end{pmatrix} = \begin{pmatrix} A_{CC}C^{-1} & A_{CL}L^{-1} \\ A_{LC}C^{-1} & A_{LL}L^{-1} \end{pmatrix} \begin{pmatrix} q \\ \Phi \end{pmatrix}$$

$$+\begin{pmatrix} A_{CV}-A_{CC}C^{-1}F_{CS}C_S F^t_{VS} & A_{CI}-A_{CL}L^{-1}(F^t_{\Gamma L}L_{\Gamma\Gamma}-L_{\Gamma L})F_{\Gamma I} \\ A_{LV}-A_{LC}C^{-1}F_{CS}C_C F^t_{VS} & A_{LI}-A_{LL}L^{-1}(F^t_{\Gamma L}L_{\Gamma\Gamma}-L_{\Gamma L})F_{\Gamma I} \end{pmatrix}$$

$$\cdot\begin{pmatrix} v_V \\ i_I \end{pmatrix} \qquad (B.1.9)$$

If a given network has following two restrictions

(iv) it precludes a loop of voltage souces and capacitors

(v) it precludes a cut set of current souces and inductors

then

$$F_{\Gamma I}=0 \qquad F_{VS}=0$$

and state equation becomes

$$\frac{d}{dt}\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \begin{pmatrix} C^{-1}A_{CC} & C^{-1}A_{CL} \\ L^{-1}A_{LC} & L^{-1}A_{LL} \end{pmatrix} \begin{pmatrix} v_C \\ i_L \end{pmatrix} + \begin{pmatrix} C^{-1}A_{CV} & C^{-1}A_{CI} \\ L^{-1}A_{LV} & L^{-1}A_{LI} \end{pmatrix} \begin{pmatrix} v_V \\ i_I \end{pmatrix}. \qquad (B.1.10)$$

B.2 Active RLC networks The general method to formulate the state equation topologically for a given network composed by RLC elements, ideal transformers,gyrators,active elements and independent souces is not perfectly given. Therefore we shall consider a simple example.

Here let us consider a network composed by RLC elements, dependent and independent souces with the following restrictions.

(1) It precludes a loop of dependent and independent voltage souces and branches of voltages which control dependent souces.

(2) It precludes a cut set of dependent and independent current souces and links of currents which control dependent souces.

(3) Dependent souces are controled by currents of links in T^C_R

where value of resistors are equal to 0 and by voltages of branches in T_G where value of conductors are equal to 0.

Then dependent voltage souces are designated as branches in T_R^c and dependent current souces as links in T_G.

In this case topological relations are the same as (B.1.1) and (B.1.2) where $F_{VS}=0, F_{\Gamma I}=0$ and relations between voltage and current vectors in capacitive and inductive sets are the same as (B.1.3)and (B.1.4), but in resistive set

$$\begin{pmatrix} i_G \\ v_R \end{pmatrix} = \begin{pmatrix} H_{GG} & H_{GR} \\ H_{RG} & H_{GG} \end{pmatrix} \begin{pmatrix} v_G \\ i_R \end{pmatrix}. \qquad \text{(B.2.1)}$$

Eliminating i_S, i_Γ, v_S and v_Γ the same equation as (B.1.6) holds where $F_{VS}=0, F_{\Gamma I}=0$. To eliminate v_G, i_G, v_R and i_R, it is necessary that the matrix

$$\begin{pmatrix} H_{GG} & H_{GR}+F_{GR} \\ H_{RG}-F_{GR}^t & H_{RR} \end{pmatrix}$$

is non-singular. Here let us assume that above condition is satisfied and denote

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} H_{GG} & H_{GR}+F_{GR} \\ H_{RG}-F_{GR}^t & H_{RR} \end{pmatrix}^{-1} \qquad \text{(B.2.2)}$$

then the same equation as (B.1.7) holds where $F_{VS}=0, F_{\Gamma I}=0$ and

$$A_{CC}=-F_{CR}H_{22}F_{CR}^t$$
$$A_{CL}=F_{CR}H_{21}F_{GL}-F_{CI}$$
$$A_{LC}=F_{CL}^t+F_{GL}^tH_{12}F_{CR}^t$$
$$A_{LL}=-F_{GL}^tH_{11}F_{GL}$$
$$A_{CV}=-F_{CR}H_{22}F_{VI}^t$$
$$A_{CI}=F_{CR}H_{21}F_{GI}-F_{CI}$$

$$A_{LV}=F_{VL}^{t}+F_{GL}^{t}H_{12}F_{VI}^{t}$$
$$A_{LI}=-F_{GL}^{t}H_{11}F_{GI}$$

and state equation is the same as (B.1.10).

Appendix C Derivatives and integrals of operational functions*

C.1 Derivatives of operational functions For a given function of two variables $f(\lambda,t)$ defined for $t\geq 0$ and for some value of λ, we shall write

$$f(\lambda)=\{f(\lambda,t)\} \tag{C.1.1}$$

and call $f(\lambda)$ as operational function.

An operational function $f(\lambda)$ will be termed continuous in a finite interval I if it can be represented in that interval as a product of a certain operator q and a operational function $f_1(\lambda)=\{f_1(\lambda,t)\}$ such that the function of two variables $f_1(\lambda,t)$ is continuous in the domain $D(\lambda\in I, 0\leq t<\infty)$

$$f(\lambda)=q\{f_1(\lambda,t)\}. \tag{C.1.2}$$

Above definition can be extended to infinite interval I if $f(\lambda)$ is continuous in every finite interval contained in I. For such an operational function $f(\lambda)$, we can define its norm as shown in Appendix A and define its derivative as the limit

$$f'(\lambda)=\lim_{\alpha\to 0}\frac{f(\lambda+\alpha)-f(\lambda)}{\alpha} \tag{C.1.3}$$

where the convergence is operational. Namely if $f(\lambda)$ is represented by (C.1.2) and the function $\{f_1(\lambda,t)\}$ has derivative with respect to λ, then we have

$$f'(\lambda)=q\{\frac{\partial}{\partial\lambda}f_1(\lambda,t)\}. \tag{C.1.4}$$

Derivatives of operational functions have the following properties

(1) $f(\lambda)=k$ then $f'(\lambda)=0$

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* See I-1 Part 6 Chapter IV

(2) $[f(\lambda)\pm g(\lambda)]'=f'(\lambda)\pm g'(\lambda)$

(3) $[f(\lambda)g(\lambda)]'=f'(\lambda)g(\lambda)+f(\lambda)g'(\lambda)$.

(4) $[cf(\lambda)]'=cf'(\lambda)$

(5) $\left[\frac{f(\lambda)}{g(\lambda)}\right]'=\frac{f'(\lambda)g(\lambda)-f(\lambda)g'(\lambda)}{g(\lambda)^2}$

(6) $F(\lambda)=f[\phi(\lambda)]$ then $F'(\lambda)=f'[\phi(\lambda)]\phi'(\lambda)$.

We can define higher order derivative of operational function $f(\lambda)$ in the same way. Namely if $f(\lambda)$ can be represented as a product of a certain operator q_n by

$$f(\lambda)=q_n\{f_n(\lambda,t)\} \tag{C.1.5}$$

and the operational function $\{f_n(\lambda,t)\}$ has n-th derivative with respect to λ, then we have

$$f^{(n)}(\lambda)=q_n\{\frac{\partial^n}{\partial\lambda^n}f_n(\lambda,t)\}. \tag{C.1.6}$$

C.2 Integrals of operational functions Suppose that an operational function $f(\lambda)$ assumes a step function of a Banach space in a given interval $[\alpha,\beta]$ and let m_i denote a Lebesgue mesure of the set in which $f(\lambda)$ assumes the value f_i, then

$$\int_\alpha^\beta f(\lambda)\,d\lambda=\sum_{i=1}^n f_i m_i . \tag{C.2.1}$$

Now let $f_n(\lambda)$ be a sequence of step functions in $[\alpha,\beta]$ which converges almost everywhere to a function $f(\lambda)$ such that

$$|f_n(\lambda)|\le\phi(\lambda)$$

where $\phi(\lambda)$ is a real positive function integrable in $[\alpha,\beta]$ in the sence of Lebesgue, then the function $f(\lambda)$ is called integrable and written as follows.

$$\int_\alpha^\beta f(\lambda)\,d\lambda=\lim_{n\to\infty}\int_\alpha^\beta f_n(\lambda)\,d\lambda \tag{C.2.2}$$

Integrals of operational functions have the following properties.

(1) $\int_{\alpha}^{\alpha} f(\lambda)\,d\lambda=0$

(2) $\int_{\alpha}^{\beta} f(\lambda)\,d\lambda=-\int_{\beta}^{\alpha} f(\lambda)\,d\lambda$

(3) $\int_{\alpha}^{\beta} f(\lambda)\,d\lambda=\int_{\alpha}^{\gamma} f(\lambda)\,d\lambda+\int_{\gamma}^{\beta} f(\lambda)\,d\lambda$

(4) $\int_{\alpha}^{\beta} cf(\lambda)\,d\lambda=c\int_{\alpha}^{\beta} f(\lambda)\,d\lambda$

(5) $\int_{\alpha}^{\beta} [f(\lambda)\pm g(\lambda)]\,d\lambda=\int_{\alpha}^{\beta} f(\lambda)\,d\lambda\pm\int_{\alpha}^{\beta} g(\lambda)\,d\lambda$

(6) $\int_{\alpha}^{\beta} f'(\lambda)g(\lambda)\,d\lambda=f(\beta)g(\beta)-f(\alpha)g(\alpha)-\int_{\alpha}^{\beta} f(\lambda)g'(\lambda)\,d\lambda$

(7) $\int_{\alpha}^{\beta} f[\phi(\lambda)]\phi'(\lambda)\,d\lambda=\int_{\phi(\alpha)}^{\phi(\beta)} f(\lambda)\,d\lambda$

Appendix D Higher order partial differential equations*

Let us consider the following partial differential equation in a given domain $D(\lambda\in I, 0\le t<\infty)$.

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\alpha_{\mu\nu}\frac{\partial^{\mu+\nu}}{\partial\lambda^{\mu}\partial t^{\nu}}\{x(\lambda,t)\}=\{\phi(\lambda,t)\} \tag{D.1}$$

For a given initial condition Φ we must take account of the following problems.

(1) Whether a solution $\{x(\lambda,t)\}$ exists or not.

(2) If a solution exists, whether it is unique or not.

(3) Whether continuity of $\{x(\lambda,t)\}$ with respect to $\{\phi(\lambda,t)\}$ and Φ holds or not.

(4) Whether the solution $\{x(\lambda,t)\}$ exists globally or not.

To get the unique solution satisfying the above questions we must impose fairly severe restrictions upon the relations among the given equation, external function ϕ, initial condition Φ and domain D.

Now let us show some known results summarily given from the standpoint of the operational calculus.

The equation of (D.1) can be reduced to the following operational form.

$$\left.\begin{aligned}
&a_m x^{(m)}+a_{m-1}x^{(m-1)}+\dots+a_0x=f(\lambda)\\
&a_\mu=\alpha_{\mu n}s^n+\alpha_{\mu,n-1}s^{n-1}+\dots+\alpha_{\mu 0}\qquad \mu=0,1,\dots,m\\
&f(\lambda)=\phi(\lambda)+\sum_{k=0}^{n-1}s^{n-k-1}\sum_{\mu=0}^{m}\sum_{\nu=0}^{k}\alpha_{\mu,n-k+\nu}\frac{\partial^{\mu+\nu}}{\partial\lambda^{\mu}\partial t^{\nu}}x(\lambda,0)
\end{aligned}\right\} \tag{D.2}$$

To get the operational solution of the above equation it is necessary to know the function $f(\lambda)$ and to do so it is sufficient to know the forms of the following functions in a given interval.

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* See I-1 Part 4 Chapters I - III and I-4 Chapter IV

$$\left.\begin{array}{l}\sum_{\mu=0}^{m}\sum_{\nu=0}^{k}\alpha_{\mu,n-k+\nu}x_{\lambda^{\mu}t^{\nu}}(\lambda,0)=g_k(\lambda)\\ k=0,1,\ldots,n-1\end{array}\right\} \qquad (D.3)$$

In some cases above conditions may be replaced by the following Cauchy's conditions.

$$\left.\begin{array}{l}x_{t^k}(\lambda,0)=h_k(\lambda)\\ k=0,1,\ldots,n-1\end{array}\right\} \qquad (D.4)$$

For a certain partial differential equation if the conditions (D.3) and (D.4) are not equivalent, the equation is said to be restrictive.

In such a case the conditions on the λ-axis must be given in the form (D.3).

For the criterion to decide whether the given equation is restrictive or not the following theorem is given.*

<< *The equation is restrictive if no partial derivative whose order with respect to t is the highest in the given equation is a mixed derivative.* >>

In those ways we can solve the operational solution $x(\lambda)$ from the operational differential equation in the same way as in the ordinary differential equation.

Then among the roots of the characteristic equation of the homogeneous equation

$$a_m w^m + a_{m-1} w^{m-1} + \ldots + a_0 = 0 \qquad (D.5)$$

significant roots are only the logarithmic roots and if there exist p logarithmic roots, we can determine the operational solution $x(\lambda)$ by giving p conditions on the λ-axis.

For the criterion to decide whether the roots of the characteristic equation are logarithmic or not the following theorem is given.**

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* ** See I-1 Part 4 Chapter III

<< *If the operational equation* (D.2) *has been deduced from the partial differential equation with constant coefficients* (D.1), *then the characteristic equation has m roots and all these roots can be represented in the form of an infinite convergent series with numerical coefficients*

$$w=\beta_k s^{k/q}+\dots+\beta_1 s^{1/q}+\beta_0+\sum_{\kappa=0}^{\infty}\gamma_\kappa l^{\kappa/q} \qquad \text{(D.6)}$$

where k is a certain non-negative integer and q is a natural number not greater than m and if $k/q>1$ *and* $\beta_k\neq 0$ *or if* $k/q=1$ *and* β_k *is not real then the operator w is not logarithmic. In the opposit case w is always logarithmic.* >>

Appendix E Algorithm of Fast Fourier Transform[18]

When a function $y(t)$ is given, the Fourier transform and its inversion formulas are defined as follows.

$$Y(\omega)=\int_{-\infty}^{\infty} y(t)\, e^{-i\omega t} dt \tag{E.1}$$

$$y(t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} Y(\omega)\, e^{i\omega t} d\omega \tag{E.2}$$

Now we shall consider the sampled function $y^*(t)$ defined as

$$y^*(t)=\sum_{k=0}^{K-1} y(t)\,\delta(t-kT). \tag{E.3}$$

The Fourier transform $Y^*(\omega)$ is given by

$$Y^*(\omega)=\sum_{k=0}^{K-1} y(kT)\, e^{-i\omega kT} \tag{E.4}$$

and if frequencies are chosen as integer multiples n of $2\pi/KT$ it becomes

$$Y^*\left(\frac{2n\pi}{KT}\right)=\sum_{k=0}^{K-1} y(kT)\, e^{-i2\pi nk/K} \tag{E.5}$$

and by this equation we can get the frequency spectra of $y(t)$ from its sampled values.

If K Fourier coefficients ($n=0,1,\ldots,K-1$) are computed, the matrix representation of (E.5) becomes

$$\mathrm{Y}^*=\mathrm{W}\mathrm{y} \tag{E.6}$$

where

$$\mathrm{Y}^*=\begin{pmatrix} Y^*(0) \\ Y^*(2\pi/KT) \\ \vdots \\ Y^*(\overline{K-1}2\pi/KT) \end{pmatrix} \quad \mathrm{y}=\begin{pmatrix} y(0) \\ y(T) \\ \vdots \\ y(\overline{K-1}T) \end{pmatrix} \quad \mathrm{W}=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{K-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & w^{K-1} & w^{2(K-1)} & \cdots & w^{(K-1)^2} \end{pmatrix}$$

$$w=e^{-i2\pi/K}.$$

Because this equation is the product of matrices, the number of multiplications increases rapidly as K increases.

However, when K is chosen suitably and the Fast Fourier Transform based on the periodicity of powers of w is used, the matrix W may be solved into the products of sparse matrices which significantly reduce the number of multiplications required.

Here we shall present the method. If r is any factor of K , that is $K=rK'$, then for i,j element of the matrix W we have

$$W_{ij}=w^{(i-1)(j-1)}.$$

Denote by W_j j-th col~~eum~~umn vector of W, that is

$$\mathrm{W}_j=(W_{1j}\ W_{2j}\ \ldots.\ W_{kj})^t$$

then from the relations

$$\left.\begin{array}{l}
W_{i+r,j}=w^{(i+r-1)(j-1)}=w^{r(j-1)}W_{ij} \\
W_{i+2r,j}=w^{2r(j-1)}W_{ij} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
W_{i+(K'-1)r,j}=w^{(K'-1)r(j-1)}W_{ij} \\
\quad i=1,2,\ldots.,r
\end{array}\right\}$$

and from the periodicity of $w^r=e^{-i2\pi/K'}$ that is

$$w^{r(j-1)}=w^{r(j-1)\mathrm{MOD}K'}$$

we have

$$\mathrm{W}_j^t=\left\{\begin{array}{ll}
(\mathrm{r}_j\ \mathrm{r}_j\ \ldots\ldots\ldots\ldots\ \mathrm{r}_j) & (j-1)\mathrm{MOD}K'=0 \\
(\mathrm{r}_j\ w^r\mathrm{r}_j\ \ldots\ldots\ldots\ w^{(K'-1)r}\mathrm{r}_j) & (j-1)\mathrm{MOD}K'=1 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
(\mathrm{r}_j\ w^{(K'-1)r}\mathrm{r}_j\ \ldots.\ w^{(K'-1)^2r}\mathrm{r}_j) & (j-1)\mathrm{MOD}K'=K'-1
\end{array}\right\} \qquad (\mathrm{E}.7)$$

where

$$\mathrm{r}_j=(1\ w^{j-1}\ w^{2(j-1)}\ \ldots.\ w^{(r-1)(j-1)})$$

and W can be represented as the product of two matrices as follows

$$W=\begin{pmatrix} r_1 & 0_r \ldots\ldots 0_r \\ 0_r & r_2 \ldots\ldots 0_r \\ \ldots\ldots & \ldots\ldots \\ 0_r & 0_r \ldots\ldots r_{K'} \\ r_{K'+1} & 0_r \ldots\ldots 0_r \\ \ldots\ldots & \ldots\ldots \\ 0_r & 0_r \ldots\ldots r_K \end{pmatrix} \begin{pmatrix} U_r & U_r & \ldots\ldots\ldots\ldots U_r \\ U_r & w^r U_r & \ldots\ldots\ldots\ldots w^{(K'-1)r} U_r \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ U_r & w^{(K'-1)r} U_r & \ldots\ldots w^{(K'-1)^2 r} U_r \end{pmatrix} \qquad \text{(E.8)}$$

where 0_r is the row vector of order r whose elements are all zero and U_r is the unit matrix of the order r.

The second matrix of the right side of (E.8) is of the same form as W and if K' has any factor it can be solved into the products of more simple matrices in the same way as stated above.

Specially if $K=2^N$, W is solved into the products of N sparse matrices and for each of them the number of non-zero elements is $2K$ (every row has 1, any power of w and $K-2$ 0s), therefore by using this method the number of multiplications is reduced from K^2 to NK and additions from $K(K-1)$ to NK compairing with the direct method of (E.6) and the execution time in computation can be reduced extremely.

The processes of factoring the matrices are regular MOD operations and we can perform them by digital computers, therefore, this method is very attractive for numerical calculations.

In Fig E-1 flow chart of the above processes is given where W is solved into

$$W=W_1 W_2 \ldots\ldots W_N$$

and

$$Z_1=y$$

$$Z_2=W_N Z_1$$

$$\ldots\ldots$$

$$Z_N=W_2 Z_{N-1}$$

$$Z_{N+1}=W_1 Z_N=Y^*.$$

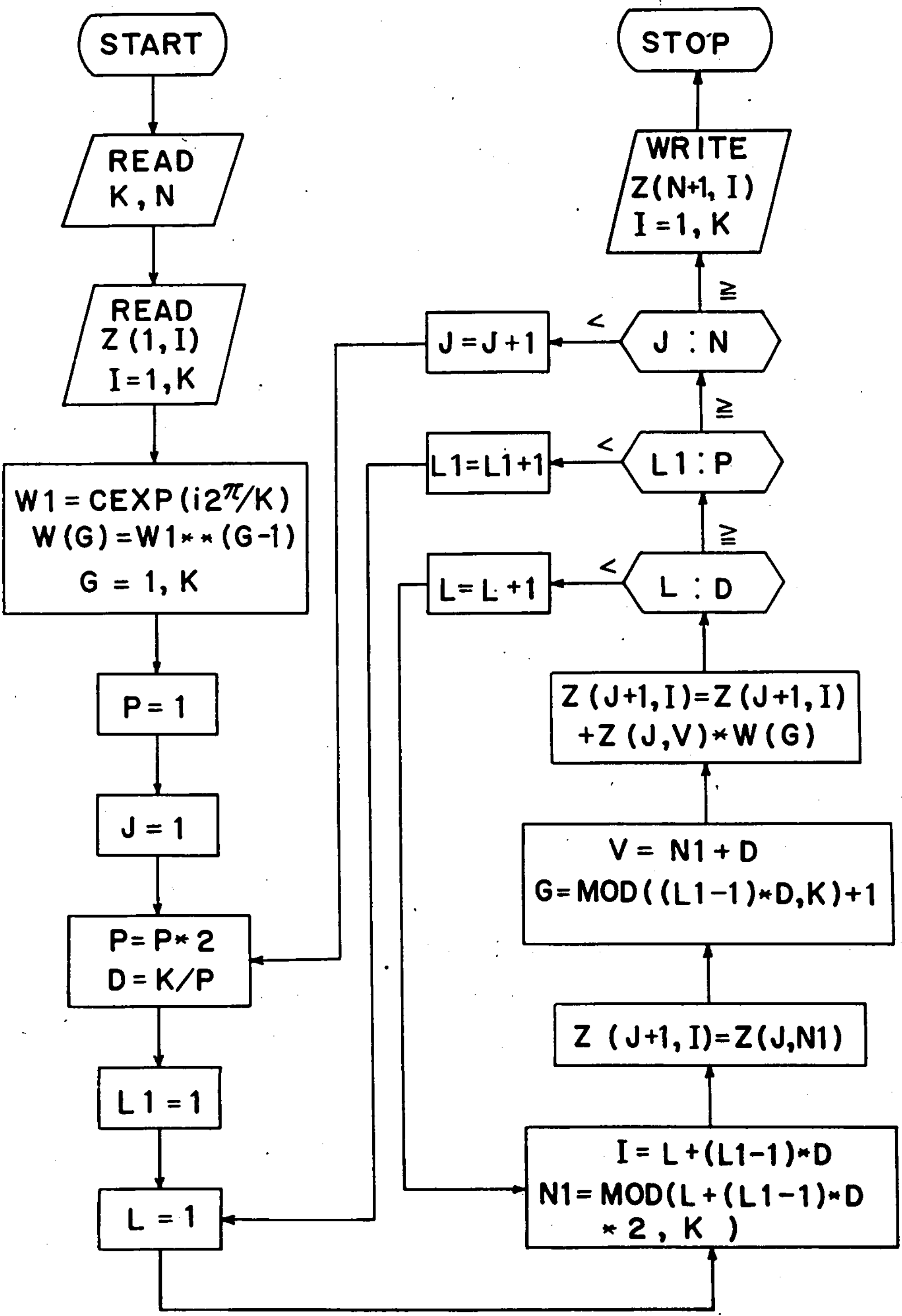


Fig E-1 Flow chart of Fast Fourier Transform

Appendix F Equivalent impedance of skin effect[I-10]

F.1 Skin effect of conductor For the high frequency current, the current density in the conductor is not constant and the current flows almost in the neighborhood of the surface.

To make clear this phenomenon mathematically, we must solve the Maxwell's equations.

Let us assume that the cross section of the conductor is circular and take the cylindrical coordinate system shown in Fig F-1-1, then from the physical point of view the electric field has only λ-component E_λ and determined by the following equation.

$$\nabla^2 E_\lambda = \sigma\mu \frac{dE_\lambda}{dt} \qquad \text{(F.1.1)}$$

Because of the axial symmetry, E_λ is a function of variable r only and suppose that E_λ is equal to 0 at $t=0$, then the above equation becomes

$$\frac{d^2E_\lambda}{dr^2}+\frac{1}{r}\frac{dE_\lambda}{dr} - \sigma\mu s E_\lambda = 0. \qquad \text{(F.1.2)}$$

From the physical meaning E_λ is regular at $t=0$.

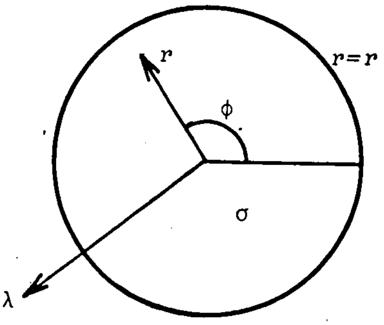


Fig F-1-1 Coordinate system for circular conductor.

Let us consider the following infinite series

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{i\lambda\sqrt{s}}{2}\right)^{2n}\frac{1}{(n!)^2}=1+\frac{\lambda^2}{2^2}s-\frac{\lambda^4}{2^2 4^2}s^2+\dots. \qquad \text{(F.1.3)}$$

then from the theorem of Ryll-Nardzewski*

<< *An infinite series*

$$\alpha_0+\alpha_1\lambda s+\alpha_2\lambda^2 s^2+\dots. \qquad \text{(F.1.4)}$$

---

* See I-1 Part 5 Chapter V

converges for every complex number λ if

$$\limsup_{n\to\infty} n^{\delta n}|\alpha_n|<\infty \tag{F.1.5}$$

for some $\delta>1$. >>

and the relation

$$\lim_{n\to\infty}\frac{n^{\delta n}}{2^{2n}(n!)^2}=\lim_{n\to\infty}\frac{n^{\delta n}e^{2n}}{2^{2n}2\pi n^{2n+1}}=\lim_{n\to\infty}\frac{n^{\delta n}e^{2n}}{(2n)^{2n+1}\pi}=0$$

for $1<\delta<2$ the infinite series of (F.1.3) is convergent. Furthermore, we can differentiate it term by term.

Let us denote it by $I_0(\lambda\sqrt{s})$, then we can treat it formally the same as the ordinary first kind modified Bessel function of order 0 and we can write the solution of the electric field as

$$E\lambda=AI_0(\sqrt{\sigma\mu s}r). \tag{F.1.6}$$

The current density is $J\lambda=\sigma E\lambda$ and the integral on the cross section is equal to the total current I such that

$$I=\int_0^r J\lambda 2\pi r dr=2\pi A\sqrt{\frac{\sigma}{\mu s}}rI_1(\sqrt{\sigma\mu s}r) \tag{F.1.7}$$

and we have the following relation at $r=r$

$$-\frac{dV}{d\lambda}=E\lambda=Z_c(s)I \tag{F.1.8}$$

where

$$Z_c(s)=\frac{1}{2\pi}\sqrt{\frac{\mu s}{\sigma}}\;\frac{I_0(\sqrt{\sigma\mu s}r)}{rI_1(\sqrt{\sigma\mu s}r)}\;. \tag{F.1.9}$$

F.2 Skin effect of ground for airial line Here we shall consider the following coordinate system shown in Fig F-2-1.

Assume that the initial values of the electric and magnetic field are equal to 0 and that the current in the ground flows parallel to the λ-axis, then the electric field has x and y components and we have the following set of the differential equations from the Maxwell's equations.

$$\left.\begin{aligned} &\frac{\partial^2 E\lambda}{\partial x^2}+\frac{\partial^2 E\lambda}{\partial y^2}+\frac{\partial^2 E\lambda}{\partial \lambda^2}=k^2 E\lambda \\ &k^2=\mu s(\sigma+\varepsilon s) \\ &H_x=-\frac{1}{\mu s}\frac{\partial E\lambda}{\partial y} \\ &H_y=\frac{1}{\mu s}\frac{\partial E\lambda}{\partial x} \end{aligned}\right\} \quad \text{(F.2.1)}$$

For every component in the ground we shall mark suffix 1 and in the free space 0.

From the physical point of view we may note.

$$E_{1\lambda}=E_{1\lambda}(x,y)e^{-\gamma_1\lambda} \quad \text{(F.2.2)}$$

then we can assume $\gamma_1^2<<k_1^2$ and we have

$$\frac{\partial^2 E_{1\lambda}}{\partial x^2}+\frac{\partial^2 E_{1\lambda}}{\partial y^2}=k^2 E_{1\lambda}\,. \quad \text{(F.2.3)}$$

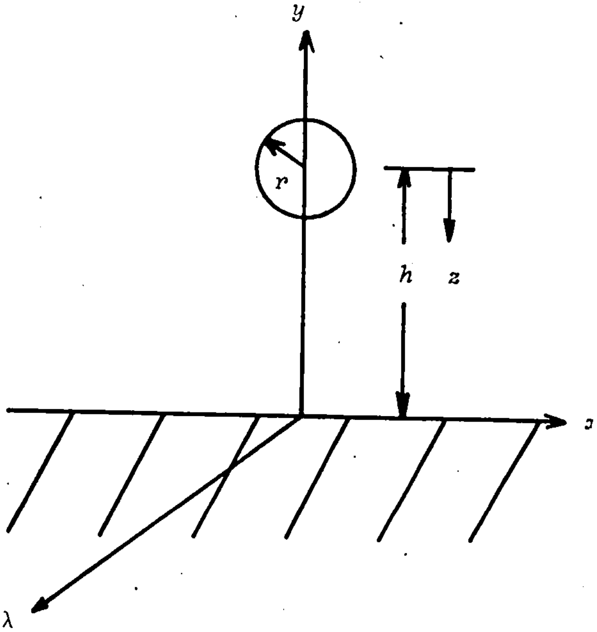


Fig F-2-1 Coordinate system for airial line

Now let us assume the solution as

$$E_{1\lambda}(x,y)=X_1(x)Y_1(y) \quad \text{(F.2.4)}$$

then we have

$$\frac{d^2X_1}{dx^2}+u^2X_1=0 \quad \text{(F.2.5)}$$

$$\frac{d^2Y_1}{dy^2}-(k_1^2+u^2)Y=0. \quad \text{(F.2.6)}$$

From the physical meaning we have

$$\left.\begin{aligned} &E_{1\lambda}(x,y)=E_{1\lambda}(-x,y) \\ &\lim_{y\to-\infty}E_{1\lambda}(x,y)=0 \end{aligned}\right\} \quad \text{(F.2.7)}$$

therefore, the boundary conditions are given by

$$X_1(x)=X_1(-x) \quad \text{(F.2.8)}$$

$$\lim_{y\to-\infty}Y_1(y)=0 \quad \text{(F.2.9)}$$

and we have

$$X_1=\cos ux \tag{F.2.10}$$

$$Y_1=e^{\sqrt{k_1^2+u^2}y}. \tag{F.2.11}$$

Consequently we have

$$E_{1\lambda}=-\int_0^\infty F(u)\cos ux e^{\sqrt{u^2+k_1^2}y}du \qquad y\le 0 \tag{F.2.12}$$

$$H_{1x}=\frac{1}{\mu_1 s}\int_0^\infty \sqrt{u^2+k_1^2}F(u)\cos ux e^{\sqrt{u^2+k_1^2}y}du \quad y\le 0 \tag{F.2.13}$$

$$H_{1y}=\frac{1}{\mu_1 s}\int_0^\infty uF(u)\sin ux e^{\sqrt{u^2+k_1^2}y}du \quad y\le 0. \tag{F.2.14}$$

The magnetic field in the free space caused by the current in the ground is given by the following equations.

$$\left.\begin{aligned}
&\frac{\partial^2 H'_{0x}}{\partial x^2}+\frac{\partial^2 H'_{0x}}{\partial y^2}=0\\
&\frac{\partial^2 H'_{0y}}{\partial x^2}+\frac{\partial^2 H'_{0y}}{\partial y^2}=0\\
&\frac{\partial H'_{0x}}{\partial x}+\frac{\partial H'_{0y}}{\partial y}=0
\end{aligned}\right\} \tag{F.2.15}$$

From the physical meaning we have

$$\left.\begin{aligned}
&H'_{0x}(x,y)=H'_{0x}(-x,y)\\
&H'_{0y}(x,y)=-H'_{0y}(-x,y)\\
&\lim_{y\to\infty}H'_{0x}(x,y)=0\\
&\lim_{y\to\infty}H'_{0y}(x,y)=0
\end{aligned}\right\} \tag{F.2.16}$$

and consequently we have the following solutions.

$$H'_{0x}=\int_0^\infty G(u)\cos ux e^{-uy}du \qquad y>0 \tag{F.2.17}$$

$$H'_{0y}=-\int_0^\infty G(u)\sin ux e^{-uy}du \qquad y>0\ . \tag{F.2.18}$$

The magnetic field in the free space caused by the current in the conductor is given by

$$H''_{0x}=\frac{I}{2\pi}\,\frac{z}{x^2+z^2}=\frac{I}{2\pi}\int_0^\infty \cos ux e^{-zu}du \tag{F.2.19}$$

$$H''_{0y}=\frac{I}{2\pi}\ \frac{x}{x^2+z^2}=\frac{I}{2\pi}\int_0^\infty \sin ux e^{-zu}du. \quad (F.2.20)$$

By the continuity of the magnetic field we have

$$\left.\begin{aligned}[H'_{0x}+H''_{0x}]_{y=0}&=[H_{1x}]_{y=0}\\ [H'_{0y}+H''_{0y}]_{y=0}&=[H_{1y}]_{y=0}\end{aligned}\right\} \quad (F.2.21)$$

and we have

$$F(u)=\mu_1 s\frac{I}{\pi}\ \frac{1}{u+\sqrt{u^2+k_1^2}}e^{-hu} \quad (F.2.22)$$

$$G(u)=\frac{I}{2\pi}\frac{\sqrt{u^2+k_1^2}-u}{\sqrt{u^2+k_1^2}+u}e^{-hu}. \quad (F.2.23)$$

Consequently we have

$$H_{1x}=\ \frac{I}{\pi}\int_0^\infty \frac{\sqrt{u^2+k_1^2}}{u+\sqrt{u^2+k_1^2}}\cos ux e^{-hu+\sqrt{u^2+k_1^2}y}du \quad (F.2.24)$$

$$H_{1y}=\frac{I}{\pi}\int_0^\infty \frac{u}{u+\sqrt{u^2+k_1^2}}\sin ux e^{-hu+\sqrt{u^2+k_1^2}y}du \quad (F.2.25)$$

$$E_{1\lambda}=-\mu_1 s\frac{I}{\pi}\int_0^\infty \frac{1}{u+\sqrt{u^2+k_1^2}}\cos ux e^{-hu+\sqrt{u^2+k_1^2}y}du \quad (F.2.26)$$

$$H_{0x}=\frac{I}{2\pi}\left\{\int_0^\infty \cos ux e^{-|h-y|u}du+\int_0^\infty \cos ux e^{-(y+h)u}\frac{\sqrt{u^2+k_1^2}-u}{\sqrt{u^2+k_1^2}+u}du\right\} \quad (F.2.27)$$

$$H_{0y}=\frac{I}{2\pi}\left\{\int_0^\infty \sin ux e^{-|h-y|u}du-\int_0^\infty \sin ux e^{-(y+h)u}\frac{\sqrt{u^2+k_1^2}-u}{\sqrt{u^2+k_1^2}+u}du\right\}. \quad (F.2.28)$$

From the relation

$$\frac{dE_{0\lambda}}{dy}=-\mu_0 sH_{0x}$$

we have

$$E_{0\lambda}(x,y)=E_{0\lambda}(x,0)-\mu_0 s\int_0^y H_{0x}(x,y)dy=E_{1\lambda}(x,0)-\mu_0 s\int_0^y H_{0x}(x,y)dy \quad (F.2.29)$$

and consequently we have

$$E_{0\lambda}(x,y)=-\mu_0 s\frac{I}{4\pi}\log\frac{x^2+(h+y)^2}{x^2+(h-y)^2}-\mu_0 s\frac{I}{\pi}\int_0^\infty \frac{\cos ux}{u+\sqrt{u^2+k_1^2}}e^{-(h+y)u}du \quad (F.2.30)$$

where we assume $\mu_1=\mu_0$.

By the relation

$$\frac{dV}{d\lambda}=E_{0\lambda}=-[LsI+Z_g(s)I] \tag{F.2.31}$$

we have

$$L=\frac{\mu_0}{4\pi}\log\frac{x^2+(h+y)^2}{x^2+(h-y)^2} \tag{F.2.32}$$

$$Z_g(s)=\frac{\mu_0 s}{\pi}\int_0^\infty\frac{\cos ux}{u+\sqrt{u^2+k_1^2}}e^{-(h+y)u}du \tag{F.2.33}$$

where L is determined only by the configuration of the conductor and $Z_g(s)$ is caused by the skin effect of the ground.

In the ground we can assume

$$k_1^2=(\sigma_1+\varepsilon_1 s)\mu_0 s\simeq\sigma_1\mu_0 s^2 \tag{F.2.34}$$

and we have

$$Z_g(s)=\frac{\mu_0}{\pi}\int_0^\infty(\sqrt{\xi^2+s}-\xi)\cos\sqrt{\sigma_1\mu_0}x\xi e^{-\sqrt{\sigma_1\mu_0}(h+y)\xi}d\xi. \tag{F.2.35}$$

Here appear many integrals of the operational functions and they are generalized to the following form.

$$\int_0^\infty[\alpha(s)u^2+\beta(s)u][\gamma(s)\cos ux+\delta(s)\sin ux]e^{-\kappa(s)uy}du$$

For every operational function $\alpha(s),\beta(s),\gamma(s),\delta(s)$ and $\kappa(s)$ we can give the norm of the continuous and bounded function and denote it by the same letter, then for the integral of the numerical function we have

$$\left|\int_0^\infty(\alpha u^2+\beta u)(\gamma\cos ux+\delta\sin ux)e^{-\kappa uy}du\right|<\infty$$

and it is shown that every integral exists.

F.3 Skin effect of two-layer ground for airial line Here we shall consider the following coordinate system shown if Fig F-3-1 and mark suffix 0 for every component in the free space, 1 in the upper layer and 2 in the lower layer, then in the same way as the uniform ground we have

$$E_{2\lambda}=-\int_0^\infty F(u)\cos ux e^{\sqrt{u^2+k_1^2}y}du \qquad y\leq -d \tag{F.3.1}$$

$$H_{2x}=\frac{1}{\mu_2 s}\int_0^\infty\sqrt{u^2+k_1^2}F(u)\cos ux e^{\sqrt{u^2+k_1^2}y}du \qquad y\leq -d \tag{F.3.2}$$

$$H_{2y}=\frac{1}{\mu_2 s}\int_0^\infty uF(u)\sin ux\, e^{\sqrt{u^2+k_2^2}y}\, du \qquad y\le -d \qquad (F.3.3)$$

$$E_{1\lambda}=-\int_0^\infty [G_1(u)e^{\sqrt{u^2+k_1^2}y}+G_2(u)e^{-\sqrt{u^2+k_1^2}y}]\cos ux\, du \qquad -d<y\le 0 \qquad (F.3.4)$$

$$H_{1x}=\frac{1}{\mu_1 s}\int_0^\infty \sqrt{u^2+k_1^2}[G_1(u)e^{\sqrt{u^2+k_1^2}y}+G_2(u)e^{-\sqrt{u^2+k_1^2}y}]\cos ux\, du \qquad -d<y\le 0 \qquad (F.3.5)$$

$$H_{1y}=\frac{1}{\mu_1 s}\int_0^\infty u[G_1(u)e^{\sqrt{u^2+k_1^2}y}-G_2(u)e^{-\sqrt{u^2+k_1^2}y}]\sin ux\, du \qquad -d<y\le 0 \qquad (F.3.6)$$

$$H_{0x}=\int_0^\infty [H(u)e^{-uy}+\frac{I}{2\pi}e^{-zu}]\cos ux\, du \qquad y\ge 0 \qquad (F.3.7)$$

$$H_{0y}=\int_0^\infty [-H(u)e^{-hu}+\frac{I}{2\pi}e^{-zu}]\sin ux\, du \qquad y\ge 0. \qquad (F.3.8)$$

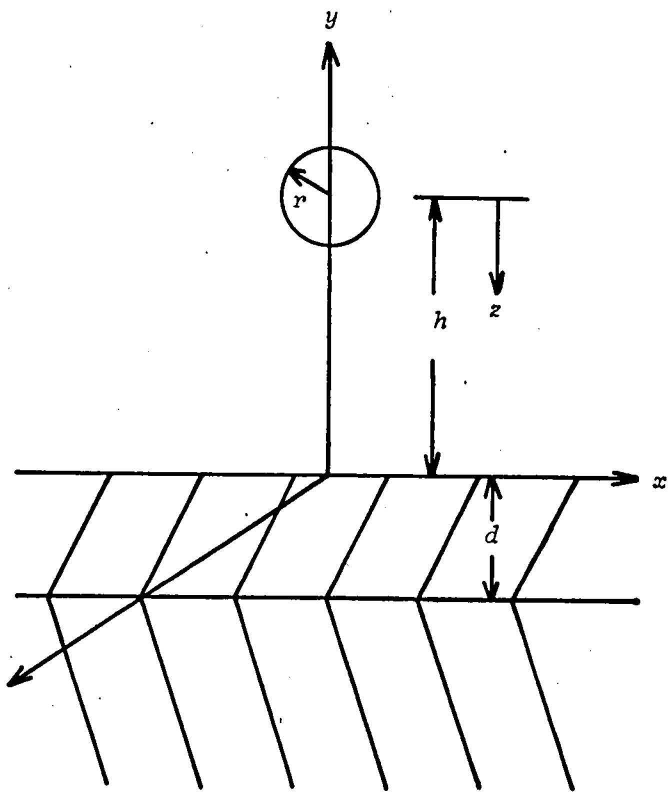


Fig F-3-1 Coordinate system of two-layer ground for airial line

By the continuity of the magnetic field at the boundary surfaces we have

$$H(u)=\frac{I}{2\pi}e^{-hu}\frac{[\sqrt{(u^2+k_1^2)(u^2+k_2^2)}-u\sqrt{u^2+k_1^2}]\cosh\sqrt{u^2+k_1^2}d+[(u^2+k_1^2)-u\sqrt{u^2+k_2^2}]\sinh\sqrt{u^2+k_1^2}d}{[\sqrt{(u^2+k_1^2)(u^2+k_2^2)}+u\sqrt{u^2+k_1^2}]\cosh\sqrt{u^2+k_1^2}d+[(u^2+k_1^2)+u\sqrt{u^2+k_2^2}]\sinh\sqrt{u^2+k_1^2}d} \qquad (F3.3.9)$$

$$G_1(u)=\frac{\mu_1 sI}{2\pi}e^{-hu}\frac{(\sqrt{u^2+k_1^2}+\sqrt{u^2+k_2^2})e^{\sqrt{u^2+k_1^2}d}}{''} \qquad (F.3.10)$$

$$G_2(u)=\frac{\mu_1 sI}{2\pi}e^{-hu}\frac{(\sqrt{u^2+k_1^2}-\sqrt{u^2+k_2^2})e^{-\sqrt{u^2+k_1^2}d}}{''} \qquad (F.3.11)$$

$$F(u)=\frac{\mu_2 sI}{\pi}e^{-hu}\frac{\sqrt{u^2+k_1^2}e^{\sqrt{u^2+k_1^2}d}}{''}. \qquad (F.3.12)$$

Assume that $\mu_1=\mu_0$, then we have

$$E_0\lambda=-\mu_0 s\frac{I}{4\pi}\log\frac{x^2+(h+y)^2}{x^2+(h-y)^2}-\frac{\mu_0 sI}{\pi}\int_0^\infty M(u)\cos ux e^{-(h+y)u}du \qquad y\geq 0 \qquad \text{(F.3.13)}$$

and

$$Z_g(s)=\frac{\mu_0 s}{\pi}\int_0^\infty M(u)\cos ux e^{-(h+y)u}du \qquad \text{(F.3.14)}$$

where

$$M(u)=\frac{\sqrt{u^2+k_2^2}\cosh\sqrt{u^2+k_2^2}d+\sqrt{u^2+k_2^2}\sinh\sqrt{u^2+k_2^2}d}{''} . \qquad \text{(F.3.15)}$$

In the ground we can assume

$$k_1^2\simeq\mu_0\sigma_1 s^2$$

$$k_2^2\simeq\mu_0\sigma_2 s^2$$

then we have

$$Z_g(s)=\frac{\mu_0 s}{\pi}\int_0^\infty M(\xi)e^{-\sqrt{\sigma_1\mu_0}(h+y)\xi}d\xi \qquad \text{(F.3.16)}$$

$$M(\xi)=\frac{\sqrt{\xi^2+s}+\sqrt{\xi^2+r's}+(\sqrt{\xi^2+s}-\sqrt{\xi^2+r's})e^{-2\sqrt{\sigma_1\mu_0}d\sqrt{\xi^2+s}}}{(\sqrt{\xi^2+s}+\sqrt{\xi^2+r's})(\sqrt{\xi^2+s}+\xi)+(\sqrt{\xi^2+s}-\sqrt{\xi^2+r's})(\xi-\sqrt{\xi^2+s})e^{-2\sqrt{\sigma_1\mu_0}d\sqrt{\xi^2+s}}}$$

(F.3.17)

where $r'=\sigma_2/\sigma_1$.

It is shown easily as in the uniform ground that every integral appeares here also exists.

F.4 Skin effect of ground for underground line Here we shall consider the coordinate system shown in Fig F-4-1 and mark the suffix 0 for every component in the free space and 1 in the ground, then we have

$$E_0\lambda=-\int_0^\infty G(u)\cos ux e^{-uy}du \quad y\geq 0 \qquad \text{(F.4.1)}$$

$$\left.\begin{aligned} E_1'\lambda&=-\int_0^\infty F(u)\cos ux e^{-\sqrt{u^2+k_1^2}y}du \\ k_1^2&=(\sigma_1+\varepsilon_1 s)\mu_1 s \qquad y\geq 0 \end{aligned}\right\} . \qquad \text{(F.4.2)}$$

The electric field in the ground caused by the current in the conductor is given by the following differential equation

$$\frac{d^2E_{1\lambda}^{\prime\prime}}{d\rho^2}+\frac{1}{\rho}\frac{dE_{1\lambda}^{\prime\prime}}{d\rho}-k_1^2E_{1\lambda}^{\prime\prime}=0 \quad (F.4.3)$$

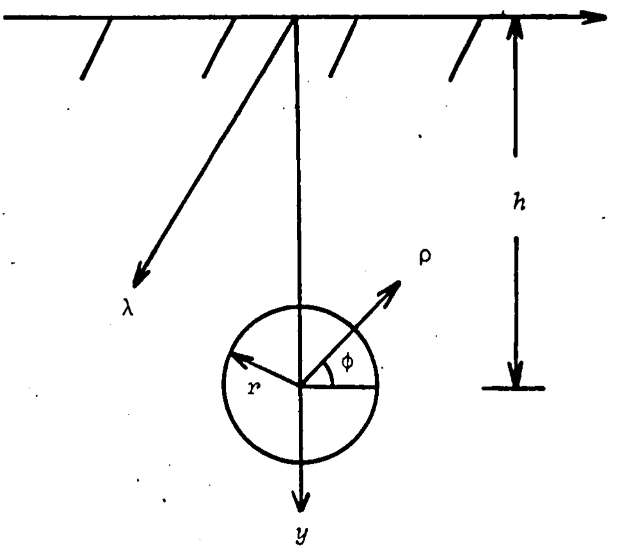
and from the physical meaning it must satisfy the following condition

$$\lim_{\rho\to\infty}E_{1\lambda}^{\prime\prime}(\rho)=0. \quad (F.4.4)$$

The solution of (F.4.3) satisfying the condition (F.4.4) is given by

$$E_{1\lambda}^{\prime\prime}=-AK_0(k_1\rho) \quad (F.4.5)$$

where K_0 means the second kind modified Bessel function of the order 0 and the magnetic field caused by $E_{1\lambda}^{\prime\prime}$ has only ϕ-component and is given by



Coordinate system for underground line

$$H_{1\phi}^{\prime\prime}=\frac{1}{\mu_1 s}Ak_1K_1(k_1\rho) \quad (F.4.6)$$

and satisfies the following condition at the surface of the conductor

$$\frac{1}{\mu_1 s}Ak_1K_1(k_1r)=\frac{I}{2\pi r}. \quad (F.4.7)$$

Consequently we have

$$E_{1\lambda}^{\prime\prime}=-\frac{\mu_1 s}{2\pi r k_1}\frac{K_0(k_1\rho)}{K_1(k_1r)} \quad (F.4.8)$$

and by the relation

$$K_0(k_1\rho)=K_0[k_1\sqrt{x^2+(y-h)^2}]=\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{-|y-h|\sqrt{u^2+k_1^2}-ixu}}{\sqrt{u^2+k_1^2}}du$$
$$=\int_0^{\infty}\frac{e^{-|y-h|\sqrt{u^2+k_1^2}}}{\sqrt{u^2+k_1^2}}\cos ux\,du \quad (F.4.9)$$

we have

$$E_{1\lambda}^{\prime\prime}=-\frac{\mu_1 sI}{2\pi r k_1K_1(k_1r)}\int_0^{\infty}\frac{e^{-|y-h|\sqrt{u^2+k_1^2}}}{\sqrt{u^2+k_1^2}}\cos ux\,du. \quad (F.4.10)$$

Every solution of the magnetic fields is given by

$$H_{0x}=-\frac{1}{\mu_0 s}\int_0^\infty uG(u)\cos ux e^{uy}du \qquad y>0 \qquad (F.4.11)$$

$$H_{0y}=\frac{1}{\mu_0 s}\int_0^\infty uG(u)\sin ux e^{uy}du \qquad y>0 \qquad (F.4.12)$$

$$H'_{1x}=\frac{1}{\mu_1 s}\int_0^\infty \sqrt{u^2+k_1^2}F(u)\cos ux e^{-\sqrt{u^2+k_1^2}y}du \qquad y\geq 0 \qquad (F.4.13)$$

$$H'_{1y}=\frac{1}{\mu_1 s}\int_0^\infty uF(u)\sin ux e^{-\sqrt{u^2+k_1^2}y}du \qquad y\geq 0 \qquad (F.4.14)$$

$$H''_{1x}=-\frac{I}{2\pi k_1 r K_1(k_1 r)}\int_0^\infty e^{-|y-h|\sqrt{u^2+k_1^2}}\cos ux du \qquad y\geq 0 \qquad (F.4.15)$$

$$H''_{1y}=\frac{I}{2\pi k_1 r K_1(k_1 r)}\int_0^\infty \frac{ue^{-|y-h|\sqrt{u^2+k_1^2}}}{\sqrt{u^2+k_1^2}}\sin ux du \qquad y\geq 0. \qquad (F.4.16)$$

Assume that $\mu_1=\mu_0$, then by the relations

$$[H_{0x}]_{y=0}=[H'_{1x}+H''_{1x}]_{y=0}$$

$$[H_{0y}]_{y=0}=[H'_{1y}+H''_{2y}]_{y=0}$$

we have

$$F(u)=\frac{\mu_0 sI}{2\pi k_1 r K_1(k_1 r)}\left(\frac{2}{\sqrt{u^2+k_1^2}+u}-\frac{1}{\sqrt{u^2+k_1^2}}\right)e^{-h\sqrt{u^2+k_1^2}} \qquad (F.4.17)$$

$$G(u)=\frac{\mu_0 sI}{2\pi k_1 r K_1(k_1 r)}\ \frac{2}{\sqrt{u^2+k_1^2}+u}e^{-h\sqrt{u^2+k_1^2}} \qquad (F.4.18)$$

and

$$E_{1\lambda}=E'_{1\lambda}+E''_{1\lambda}$$

$$=-\frac{\mu_0 sI}{2\pi k_1 r K_1(k_1 r)}\{K_0[k_1\sqrt{x^2+(y-h)^2}]-K_0[k_1\sqrt{x^2+(y-h)^2}]$$

$$+2\int_0^\infty \frac{e^{-(y+h)\sqrt{u^2+k_1^2}}}{\sqrt{u^2+k_1^2}+u}\cos ux du\}. \qquad (F.4.19)$$

By the relations

$$\lim_{k_1 r\to 0} k_1 r K_1(k_1 r)=1 \qquad (F.4.20)$$

$$\lim_{k_1\to 0}\{K_0[k_1\sqrt{x^2+(y-h)^2}]-K_0[k_1\sqrt{x^2+(y-h)^2}]\}=\log\frac{\sqrt{x^2+(y+h)^2}}{\sqrt{x^2+(y-h)^2}} \qquad (F.4.21)$$

we have approximately the following equivalent impeadance.

$$Z(s)=\frac{\mu_0 s}{4\pi}\log\frac{x^2+(y+h)^2}{x^2+(y-h)^2}+\frac{\mu_0 s}{\pi}\int_0^\infty \frac{e^{-(y+h)\sqrt{u^2+k_1^2}}}{\sqrt{u^2+k_1^2}+u}\cos ux du \qquad (F.4.22)$$

The first term is determined by the configuration of the conductor and is equal to Ls and we have

$$Z_g(s)=\frac{\mu_0 s}{\pi}\int_0^\infty \frac{e^{-(y+h)\sqrt{u^2+k_1^2}}}{\sqrt{u^2+k_1^2}+u}\cos ux\,du. \qquad \text{(F.4.23)}$$

It is easy to show that every integral and infinite series of the operational functions exists.

F.5 Skin effect of co-axial cable Here we shall consider the following coordinate system shown in Fig F-5-1 and mark suffix 1 for the components in the inner conductor and 2 in the outer conductor, then the equation is given the same as (F.1.2) and for the inner conductor we have

$$Z_{c1}(s)=\frac{\mu_1 s}{2\pi\sqrt{\sigma_1\mu_1 s}r}\frac{\mathrm{I}_0(\sqrt{\sigma_1\mu_1 s}r)}{\mathrm{I}_1(\sqrt{\sigma_1\mu_1 s}r)}. \qquad \text{(F.5.1)}$$

For the outer conductor we have

$$E_{2\lambda}=A\mathrm{I}_0(\sqrt{\sigma_2\mu_2 s}r)+B\mathrm{K}_0(\sqrt{\sigma_2\mu_2 s}r) \qquad \text{(F.1.2)}$$

and the boundary conditions are given by

$$[E_{2\lambda}]_{r=D}=0 \qquad \text{(F.5.3)}$$

$$\int_R^D \sigma_2 E_{2\lambda}2\pi r\,dr=I \qquad \text{(F.5.4)}$$

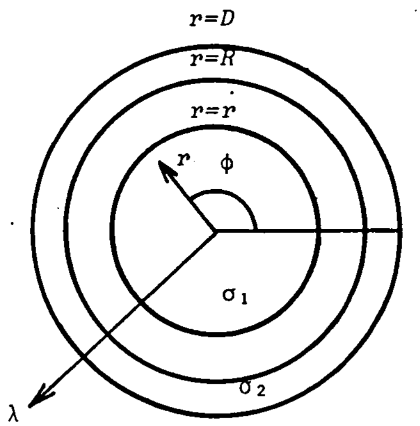


Fig F-5-1 Coordinate system for co-axial cable

then at $r=R$ we have

$$E_{2\lambda}=Z_{c2}(s)I \qquad \text{(F.5.5)}$$

where

$$\left.\begin{aligned} Z_{c2}(s)&=\frac{\mu_2 s}{2\pi}\frac{\mathrm{K}_0(\sqrt{\sigma_2\mu_2 s}D)\mathrm{I}_0(\sqrt{\sigma_2\mu_2 s}R)-\mathrm{I}_0(\sqrt{\sigma_2\mu_2 s}D)\mathrm{K}_0(\sqrt{\sigma_2\mu_2 s}R)}{\mathrm{K}_0(\sqrt{\sigma_2\mu_2 s}D)\mathrm{C}_1-\mathrm{I}_0(\sqrt{\sigma_2\mu_2 s}D)\mathrm{C}_2} \\ \mathrm{C}_1&=\sqrt{\sigma_2\mu_2 s}D\mathrm{I}_1(\sqrt{\sigma_2\mu_2 s}D)-\sqrt{\sigma_2\mu_2 s}R\mathrm{I}_1(\sqrt{\sigma_2\mu_2 s}R) \\ \mathrm{C}_2&=-\sqrt{\sigma_2\mu_2 s}D\mathrm{K}_1(\sqrt{\sigma_2\mu_2 s}D)+\sqrt{\sigma_2\mu_2 s}R\mathrm{K}_1(\sqrt{\sigma_2\mu_2 s}R) \end{aligned}\right\}. \qquad \text{(F.5.6)}$$

The total equivalent impedance of the co-axial cable is given by

$$Z_c(s)=Z_{c1}(s)+Z_{c2}(s). \qquad \text{(F.5.7)}$$