

正定値行列の幾何構造について

On geometric structure of positive definite matrices

大阪教育大学・教養学科・情報科学 藤井 淳一

Jun Ichi Fujii

Departments of Arts and Sciences (Information Science)
Osaka Kyoiku University

In this note, from the viewpoint of Corach-Porta-Recht [3, 4], we discuss a Riemannian geometry for the $n \times n$ positive definite matrices $\mathcal{C}(n)$ by Bhatia-Holbrook [2], say the *CPRBH geometry*: The principal fiber bundle is the regular matrices $\mathcal{G} = \mathcal{G}(n)$ with the unitary group $\mathcal{U}(n)$ as the structure one and the projection $\pi(X) = XX^*$. The fiber at $A \in \mathcal{C}(n)$ is $\pi^{-1}(A) = \sqrt{A}\mathcal{U}(n)$ and the Riemannian metric $g_A(X, Y) = \text{tr}(A^{-1}XA^{-1}Y)$ at $A \in \mathcal{C}(n)$. It was shown in [4] that the path of the geometric means

$$A \#_t B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}}$$

is the geodesic from A at $t = 0$ to B at $t = 1$.

The manifold $\mathcal{C}(n)$ is a homogeneous space $\mathcal{G}(n)/\mathcal{U}(n)$ with the involution $\sigma(T) = (T^*)^{-1}$ for $T \in \mathcal{G}(n)$. The differential $d\sigma(Z) = -Z^*$ for $Z \in \mathcal{T}(\mathcal{G}(n)) = \mathcal{M}_n$ is the Cartan involution with the Cartan decomposition as a Lie group and a Lie algebra;

$$\mathcal{G}(n) = \mathcal{U}(n)\mathcal{C}(n), \quad \mathfrak{gl}(n) = \mathfrak{u}(n) \oplus \mathcal{TC}(n) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$$

where the Lie algebra $\mathfrak{u}(n)$ is the skew-hermitian matrices and the tangent bundle $\mathcal{TC}(n)$ is the hermitian ones. In fact, $d\sigma$ is the Cartan involution since

$$-B(X, d\sigma(X)) = \text{tr} X \text{ad} X^* = 2n \text{tr} X X^* - 2 \text{tr} X \text{tr} X^* \geq 0$$

where B is the Killing form.

It is related to the connection in $\mathcal{G}$: The vertical space in the tangent space $\mathcal{T}\pi^{-1}(A)$ is $\sqrt{A}\mathcal{U}\mathfrak{u}(n)$ and the horizontal one is $\sqrt{A}\mathcal{U}\mathcal{TC}(n)$ for some unitary U . In fact, for an invertible matrix G , the orthogonal decomposition at $T = \sqrt{A}U$ is

$$G = \frac{T(T^{-1}G - G^*(T^*)^{-1})}{2} + \frac{T(T^{-1}G + G^*(T^*)^{-1})}{2}$$

Thereby the horizontal lift Γ of γ should satisfy that $\Gamma^{-1}\dot{\Gamma}$ is hermitian, i.e., the horizontal condition is

$$\dot{\Gamma}\Gamma^* = \Gamma\dot{\Gamma}^*.$$

Moreover, as Pálfi [12] pointed, $\mathcal{C}(n)$ is a symmetric space with the symmetry s_A at $A \in \mathcal{C}(n)$ satisfying $s_A(B) = AB^{-1}A$. The Cartan decomposition shows that a symmetric space $\mathcal{U}(n) = \mathcal{U}(n) \times \mathcal{U}(n)/\Delta\mathcal{U}(n)$ is the real form and its dual symmetric space $\mathcal{U}(n)_{\mathbb{C}}/\mathcal{U}(n)$ is $\mathcal{C}(n)$ itself where $\Delta\mathcal{U}(n)$ is the diagonal subspace and $\mathcal{U}(n)_{\mathbb{C}}$ is the complexification of $\mathcal{U}(n)$. This shows that it is not compact and the sectional curvature is non-positive, that is $\mathcal{C}(n)$ is a $CAT(0)$ -space. Let γ and δ be geodesics. If

$$d(\gamma(1/2), \delta(1/2)) \leq \frac{d(\gamma(1), \delta(1))}{2}$$

always holds, then it is said that *Busemann curvatures are non-positive*. If

$$d^2(Z, \gamma(t)) \leq (1-t)(d^2(Z, \gamma(0)) + td^2(Z, \gamma(1))) - t(1-t)d^2(\gamma(0), \gamma(1))$$

always holds, it is said that *Alexandrov curvatures are non-positive*. This inequality is called *Courbure négative one* or *semi-parallelogram law* for the case $t = 1/2$ ([1]). In the Riemannian case, they are equivalent to nonpositivity of sectional curvature [10]. Moreover, $\mathcal{C}(n)$ is a (simply connected) complete space, it is called *Hadamard manifold*. Then it is known that $F(t) = d(\gamma(t), \delta(t))$ is convex.

Since every symmetric space is geodesically complete (hence we also have that it is complete as a metric space), the extended curve

$$\gamma(t) = A \natural_t B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}}$$

for $t \in (-\infty, \infty)$ is the geodesic including $A \#_t B$. Then we have the parallel translate along the geodesic is given by

Theorem. *One of the horizontal lift of the geodesic $\gamma(t) = A \natural_t B$ is*

$$\Gamma(t) = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{t}{2}}$$

and the parallel translate P_t^s from $\gamma(s)$ to $\gamma(t)$ along γ in the tangent bundle $T\mathcal{C}(n)$ is given by

$$P_t^s X = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{t-s}{2}} A^{-\frac{1}{2}} X A^{-\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{t-s}{2}} A^{\frac{1}{2}}.$$

Proof. By

$$\pi(\Gamma(t)) = \Gamma(t)\Gamma(t)^* = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}} = \gamma(t),$$

Γ is a lift of γ . The horizontality follows from the fact that

$$2\Gamma(t)^{-1}\dot{\Gamma}(t) = 2\dot{\Gamma}(t)^*\Gamma^*(t)^{-1} = \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)$$

is hermitian. The parallel translate of X from s to t is

$$\begin{aligned} P_t^s X &= \Gamma(t)\Gamma(s)^{-1}X(\Gamma(s)^{-1})^*\Gamma(t)^* \\ &= A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{t-s}{2}} A^{-\frac{1}{2}} X A^{-\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{t-s}{2}} A^{\frac{1}{2}}. \quad \square \end{aligned}$$

Consider the triangle closed path $I \xrightarrow{A^t} A \xrightarrow{A\#_t B} B \xrightarrow{B^{1-t}} I$. Then the parallel translate of X is $V^* X V$ for

$$V = A^{\frac{1}{2}} A^{-\frac{1}{2}} C^{\frac{1}{2}} A^{\frac{1}{2}} B^{-\frac{1}{2}} = A^{\frac{1}{2}} A^{-\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}} A^{\frac{1}{2}} B^{-\frac{1}{2}} = \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}} A^{\frac{1}{2}} B^{-\frac{1}{2}}.$$

Thus, $V^* V = I$ and $\det V = (\det A)^0 (\det B)^0 = 1$, so that $V \in \mathcal{SU}(n)$. Approximating any loop by a polygon of geodesics, we have:

Corollary. *The holonomy group of $\mathcal{C}(n)$ is included by $\mathcal{SU}(n)$.*

Remark. In virtue of the Ambrose-Singer theorem, Pálfi [12] showed that they coincide via the Lie algebra $\mathfrak{su}(n)$, which might be already known.

In this geometry, the tangent vector at $\gamma(t)$ is given by (cf. [9])

$$S_t(A|B) = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}},$$

in particular, the tangent one at $t = 0$ is the relative operator entropy [5, 6]:

$$S(A|B) = S_0(A|B) = A^{\frac{1}{2}} \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}.$$

For the above lift Γ , the horizontal condition is now

$$2\Gamma(t)^{-1}\dot{\Gamma}(t) = 2\dot{\Gamma}(t)^*\Gamma^*(t)^{-1} = A^{-\frac{1}{2}}S(A|B)A^{-\frac{1}{2}}.$$

Recently E.Kamei pointed in a seminar talk that the tangent vector at r

$$S_r(A|B) = (A\sharp_r B)(A\sharp_t B)^{-1}S_t(A|B).$$

shows the parallel translate of the tangent vector $S_t(A|B)$ to $S_r(A|B)$. In fact, for $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$, we have

$$\begin{aligned} \Gamma(r)\Gamma(t)^{-1}S_t(A|B)\Gamma(t)^{-1}\Gamma(r) &= A^{\frac{1}{2}}C^{\frac{r-t}{2}}A^{-\frac{1}{2}}S_t(A|B)A^{-\frac{1}{2}}C^{\frac{r-t}{2}}A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}C^{\frac{r-t}{2}}C^t(\log C)C^{\frac{r-t}{2}}A^{\frac{1}{2}} = A^{\frac{1}{2}}C^r \log C A^{\frac{1}{2}} = S_r(A|B). \end{aligned}$$

In Hadamard manifolds, the *parallel* geodesics are defined by the boundedness;

$$d(\gamma(t), \delta(t)) < \exists M$$

for all $t \in \mathbb{R}$ (it is also called *asymptotic*). But the parallel translates for the parallel vectors along parallel geodesics are not always parallel. So, considering flat geometry in $\mathcal{C}(n)$, we need Γ -commutativity ([2]): A , B and C are Γ -*commute* if matrices $C^{-\frac{1}{2}}AC^{-\frac{1}{2}}$, $C^{-\frac{1}{2}}BC^{-\frac{1}{2}}$ commute. It is equivalent to the commutativity of matrices

$$X^{-\frac{1}{2}}AX^{-\frac{1}{2}}, X^{-\frac{1}{2}}BX^{-\frac{1}{2}}, X^{-\frac{1}{2}}CX^{-\frac{1}{2}}$$

for some X .

参考文献

- [1] R.Bhatia, "Positive Definite Matrices", Princeton Univ. Press, 2007.
- [2] R.Bhatia and J.A.R.Holbrook, Riemannian geometry and matrix geometric means, Linear Algebra. Appl. **423** (2006), 594–618.
- [3] G.Corach, H.Porta and L.Recht, Geodesics and operator means in the space of positive operators. Internat. J. Math. **4** (1993), 193–202.
- [4] G.Corach and A.L.Maestripieri, Differential and metrical structure of positive operators, Positivity **3** (1999), 297–315.
- [5] J.I.Fujii and E.Kamei, Relative operator entropy in noncommutative information theory, Math. Japon. **34** (1989), 341–348.
- [6] J.I.Fujii and E.Kamei, Uhlmann's interpolational method for operator means, Math. Japon. **34** (1989), 541–547.
- [7] J.I.Fujii and E.Kamei, Interpolational paths and their derivatives, Math. Japon. **39** (1993), 557–560.
- [8] J.I.Fujii, The Hiai-Petz geodesic for strongly convex norm is the unique shortest path, Sci. Math. Japon., **71**(2010), 19–26.
- [9] T.Furuta, Parametric extensions of Shannon inequality and its reverse one in Hilbert space operators, Linear Alg. Appl., **381**(2004), 219–235.
- [10] J.Jost, "Nonpositive Curvature: Geometric and Analytic Aspects", Springer, 1997.
- [11] F.Kubo and T.Ando, Means of positive linear operators, Math. Ann. **246** (1980), 205–224.
- [12] M.Pálfia, Semigroups of operator means and generalized Karcher equations, Preprint.