

New Formulation of Quantum Dynamical Entropies

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Abstract

Classical dynamical entropy is an important tool to analyse the efficiency of information transmission in communication processes.

Here we report new formulations of quantum dynamical entropy.

Introduction

Classical dynamical (or Kolmogorov - Sinai) entropy $S(T)$ for a measure preserving transformation T was defined on a message space through the measure of finite partitions of a measurable space. The classical coding theorems are the important tools to analyse communication processes, and they are formulated by the mean dynamical entropy and the mean dynamical mutual entropy. The mean entropy exhibits the amount of information per one letter for a signal sequence sending from the input source and the mean mutual entropy does the amount of information per one letter for a signal sequence transmitted from the input system to the output system.

Quantum dynamical entropy has been studied by Connes, Stormer [C.2], Emch [E.1], CNT (Connes, Narnhofer, Thirring) [C.1] and others [B.1, O.7].

Recently, quantum dynamical entropy and mutual entropy were introduced by the present author in terms of the complexity of information dynamics [O.8, M.1]. Furthermore, another formulation of the dynamical entropy through QMC was done in [A.4].

In §1, we review the formulation by CNT [C.1]. In §2, the formulation by the complexity is presented. In §3, the formulation through quantum Markov chain (QMC) [A.4] is discussed. In §4, we consider the relations among these formulations.

§1. CNT Formulation

Let $(\mathcal{A}, \theta_{\mathcal{A}}, \varphi)$ be an initial C^* -system. That is, $\mathcal{A}$ is a unital C^* -algebra, $\theta_{\mathcal{A}}$ is an automorphism of $\mathcal{A}$, and φ is an invariant state over $\mathcal{A}$ with respect to $\theta_{\mathcal{A}}$; $\varphi \circ \theta_{\mathcal{A}} = \varphi$. Let $\mathcal{N}$ be a finite dimensional C^* -subalgebra of $\mathcal{A}$. The CNT entropy [C.1] for a subalgebra $\mathfrak{N}$ is

$$H_{\varphi}(\mathcal{N}) = \sup \left\{ \sum_k \lambda_k S(\omega_k | \mathcal{N}, \varphi | \mathcal{N}); \varphi = \sum_k \lambda_k \omega_k \text{ finite decomposition of } \varphi \right\}.$$

where $\varphi|_{\mathcal{N}}$ is the restriction of a state φ to $\mathcal{N}$ and $S(\cdot, \cdot)$ is the relative entropy for C^* -algebra [A.6,U.1,O.7].

The CNT dynamical entropy with respect to $\theta_{\mathcal{A}}$ and $\mathcal{N}$ is given by

$$\tilde{H}_{\varphi}(\theta_{\mathcal{A}}, \mathcal{N}) = \limsup_{N \rightarrow \infty} \frac{1}{N} H_{\varphi}(\mathcal{N} \vee \theta \mathcal{N} \vee \dots \vee \theta^{N-1} \mathcal{N}).$$

The dynamical entropy for $\theta_{\mathcal{A}}$ is defined by

$$\tilde{H}_{\varphi}(\theta_{\mathcal{A}}) = \sup_{\mathcal{N}} \tilde{H}_{\varphi}(\theta_{\mathcal{A}}, \mathcal{N}),$$

This dynamical entropy is sometimes called the maximal average information gain with respect to $\theta_{\mathcal{A}}$.

§2. Formulation by Complexity

In this section, we first review the concept of complexity, which are the key concepts of Information Dynamics (ID for short) introduced in [O.6,O.8,O.9].

Let $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \alpha(G))$ and $(\overline{\mathcal{A}}, \overline{\mathfrak{S}}(\overline{\mathcal{A}}), \overline{\alpha}(\overline{G}))$ be an input (initial) and an output (final) C^* -systems, respectively, where $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$) is a unital C^* -algebra, $\mathfrak{S}(\mathcal{A})$ (resp. $\overline{\mathfrak{S}}(\overline{\mathcal{A}})$) is the set of all states on $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$) and $\alpha(G)$ (resp. $\overline{\alpha}(\overline{G})$) is an automorphism of $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$) indexed by a group G (resp. $\overline{G}$).

A channel [O.1,O.4,O.6] is a map Λ^* from $\mathfrak{S}(\mathcal{A})$ to $\overline{\mathfrak{S}}(\overline{\mathcal{A}})$.

For a w^* -compact convex subset $\mathcal{S}$ of $\mathfrak{S}$, there exists a measure μ with the barycenter φ such that

$$\varphi = \int_{\mathcal{S}} \omega \, d\mu$$

The compound state introduced in [O.2,O.3] exhibiting the correlation between an initial state φ and its final state $\Lambda^* \varphi$ is given by

$$\mathcal{E}^* \varphi = \int_{\mathcal{S}} \omega \otimes \Lambda^* \omega \, d\mu$$

This compound state corresponds with the joint measure in classical systems.

There are two complexities in ID. One is a complexity $C_T^{\mathcal{S}}(\varphi)$ of a system itself and another is a transmitted complexity $T^{\mathcal{S}}(\varphi; \Lambda^*)$ from an initial system to a final system. These complexities should satisfy the following conditions:

(i) $\forall \varphi \in \mathcal{S} \subset \mathfrak{S}$

$$C^{\mathcal{S}}(\varphi) \geq 0, \quad T^{\mathcal{S}}(\varphi; \Lambda^*) \geq 0,$$

(ii) If there exists a bijection $j : ex\mathfrak{S} \rightarrow ex\mathfrak{S}$, the set of all extreme points in $\mathfrak{S}$, then

$$\begin{aligned} C^{j(\mathcal{S})}(j(\varphi)) &= C^{\mathcal{S}}(\varphi) \\ T^{j(\mathcal{S})}(j(\varphi); \Lambda^*) &= T^{\mathcal{S}}(\varphi; \Lambda^*) \end{aligned}$$

(iii) Let $\Psi = \varphi \otimes \psi \in \mathcal{S}_t$ and $\varphi \in \mathcal{S}$, $\psi \in \overline{\mathcal{S}}$. Then

$$C^{\mathcal{S}_t}(\Phi) = C^{\mathcal{S}}(\varphi) + C^{\overline{\mathcal{S}}}(\psi)$$

(iv) $0 \leq T^{\mathcal{S}}(\varphi; \Lambda^*) \leq C^{\mathcal{S}}(\varphi)$

(v) $T^{\mathcal{S}}(\varphi; id) = C^{\mathcal{S}}(\varphi)$

Instead of (iii) above, when for $\Phi \in \mathcal{S}_t \subset \mathfrak{S}_t = \mathfrak{S} \otimes \overline{\mathfrak{S}}$ and $\varphi \equiv \Phi|_{\mathcal{A}}$, $\psi \equiv \Phi|\overline{\mathcal{A}}$

$$C^{\mathcal{S}_t}(\Phi) \leq C^{\mathcal{S}}(\varphi) + C^{\overline{\mathcal{S}}}(\psi)$$

is satisfied, we call $\{C^{\mathcal{S}}, T^{\mathcal{S}}\}$ is a pair of strong complexities. These complexities generalize several expressions of chaos [O.10].

Let us explain the formulation of three types of entropic complexity introduced in [O.2].

Let $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \alpha(G))$, $(\overline{\mathcal{A}}, \overline{\mathfrak{S}}(\overline{\mathcal{A}}), \overline{\alpha}(\overline{G}))$ and $\mathcal{S}$ as before. Let $M_{\varphi}(\mathcal{S})$ be the set of all maximal measures μ on $\mathcal{S}$ with the fixed barycenter φ and let $F_{\varphi}(\mathcal{S})$ be the set of all measures of finite support with the fixed barycenter φ . Then we have three pairs of complexities such as

$$\begin{aligned} T^{\mathcal{S}}(\varphi; \Lambda^*) &\equiv \sup \left\{ \int_{\mathcal{S}} S(\Lambda^* \omega, \Lambda^* \varphi) d\mu; \mu \in M_{\varphi}(\mathcal{S}) \right\} \\ C_T^{\mathcal{S}}(\varphi) &\equiv T^{\mathcal{S}}(\varphi; id) \\ I^{\mathcal{S}}(\varphi; \Lambda^*) &\equiv \sup \left\{ S \left(\int_{\mathcal{S}} \omega \otimes \Lambda^* \omega d\mu, \varphi \otimes \Lambda^* \varphi \right); \mu \in M_{\varphi}(\mathcal{S}) \right\} \\ C_I^{\mathcal{S}}(\varphi) &= I^{\mathcal{S}}(\varphi; id) \\ J^{\mathcal{S}}(\varphi; \Lambda^*) &\equiv \sup \left\{ \int_{\mathcal{S}} S(\Lambda^* \omega, \Lambda^* \varphi) d\mu_f; \mu_f \in F_{\varphi}(\mathcal{S}) \right\} \\ C_J^{\mathcal{S}}(\varphi) &\equiv J^{\mathcal{S}}(\varphi; id). \end{aligned}$$

Based on the above complexities, we can formulate the quantum dynamical entropy [O.2, O.8, O.9]: Let $\theta_{\mathcal{A}}$ (resp. $\theta_{\mathcal{B}}$) be a stationary (invariant) automorphism of $\mathcal{A}$ (resp. $\mathcal{B}$); $\varphi \circ \theta_{\mathcal{A}} = \varphi$, $\psi \circ \theta_{\mathcal{B}} = \psi$, and Λ^* be a covariant channel (i.e., $\Lambda \circ \theta_{\mathcal{A}} = \theta_{\mathcal{B}} \circ \Lambda$) from $\mathfrak{S}(\mathcal{A})$ to $\mathfrak{S}(\mathcal{A})$. $\mathcal{A}_k$ (resp. $\mathcal{B}_k$) is a finite subalgebra of $\mathcal{A}$ (resp. $\mathcal{B}$). Moreover, let α_k (resp. β_k) be a completely positive unital map from $\mathcal{A}_k$ (resp. $\mathcal{B}_k$) to $\mathcal{A}$ (resp. $\mathcal{B}$) and α^M and β_{Λ}^N be

$$\begin{aligned} \alpha^M &= (\alpha_1, \alpha_2, \dots, \alpha_M), \\ \beta_{\Lambda}^N &= (\Lambda \circ \beta_1, \Lambda \circ \beta_2, \dots, \Lambda \circ \beta_N) \end{aligned}$$

Two compound states for α^M and β_{Λ}^N with respect to $\mu \in M_{\varphi}(\mathcal{S})$ are defined as

$$\begin{aligned} \Phi_{\mu}^{\mathcal{S}}(\alpha^M) &= \int_{\mathcal{S}} \bigotimes_{m=1}^M \alpha_m^* \omega d\mu, \\ \Phi_{\mu}^{\mathcal{S}}(\alpha^M \cup \beta_{\Lambda}^N) &= \int_{\mathcal{S}} \bigotimes_{m=1}^M \alpha_m^* \omega \bigotimes_{n=1}^N \beta_n^* \Lambda^* \omega d\mu. \end{aligned}$$

By using these compound states, we define three transmitted complexities [O.8]:

$$\begin{aligned}
& T_{\varphi}^S(\alpha^M, \beta_{\Lambda}^N) \\
& \equiv \sup \left\{ \int_S S \left(\bigotimes_{m=1}^M \alpha_m^* \omega \bigotimes_{n=1}^N \beta_n^* \Lambda^* \omega, \Phi_{\mu}^S(\alpha^M) \otimes \Phi_{\mu}^S(\beta_{\Lambda}^N) \right) d\mu; \mu \in M_{\varphi}(S) \right\} \\
& I_{\varphi}^S(\alpha^M, \beta_{\Lambda}^N) \equiv \sup \{ S(\Phi_{\mu}^S(\alpha^M \cup \beta_{\Lambda}^N), \Phi_{\mu}^S(\alpha^M) \otimes \Phi_{\mu}^S(\beta_{\Lambda}^N)); \mu \in M_{\varphi}(S) \} \\
& J_{\varphi}^S(\alpha^M, \beta_{\Lambda}^N) \\
& \equiv \sup \left\{ \int_S S \left(\bigotimes_{m=1}^M \alpha_m^* \omega \bigotimes_{n=1}^N \beta_n^* \Lambda^* \omega, \Phi_{\mu}^S(\alpha^M) \otimes \Phi_{\mu}^S(\beta_{\Lambda}^N) \right) d\mu_f; \mu_f \in F_{\varphi}(S) \right\}
\end{aligned}$$

When $\mathcal{A}_k = \mathcal{A}_0 = \mathcal{B}_k$, $\mathcal{A} = \mathcal{B}$, $\theta_{\mathcal{A}} = \theta_{\mathcal{B}} = \theta$, $\alpha_k = \theta^{k-1} \circ \alpha = \beta_k$, the mean transmitted complexity is

$$\begin{aligned}
\tilde{T}_{\varphi}^S(\theta, \alpha, \Lambda^*) & \equiv \limsup_{N \rightarrow \infty} \frac{1}{N} T_{\varphi}^S(\alpha^N, \beta_{\Lambda}^N) \\
\tilde{T}_{\varphi}^S(\theta, \Lambda^*) & \equiv \sup_{\alpha} \tilde{T}_{\varphi}^S(\theta, \alpha, \Lambda^*)
\end{aligned}$$

Same for $\tilde{I}_{\varphi}^S, \tilde{J}_{\varphi}^S$. These quantities have the similar properties as the CNT entropy [O.8, M.1].

§3. Formulation by QMC

Another formulation of the dynamical entropy is due to quantum Markov chain [A.4].

Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, φ be a stationary faithful normal state on $\mathcal{A}$ and $\mathcal{A}_0 = M_d$ ($d \times d$ matrix algebra). Take the transition expectation $\mathcal{E}_{\gamma} : \mathcal{A}_0 \otimes \mathcal{A} \rightarrow \mathcal{A}$ of Accardi [A.1, A.2] such that

$$\mathcal{E}_{\gamma}(\tilde{A}) = \sum_i \gamma_i A_{ii} \gamma_i$$

where $\tilde{A} = \sum_{i,j} e_{ij} \otimes A_{ij} \in \mathcal{A}_0 \otimes \mathcal{A}$ and $\gamma = \{\gamma_j\}$ is a finite partition of unity $I \in \mathcal{A}$. For a state φ on $\mathcal{A}$, the quantum Markov chain $\psi \equiv \{\varphi, \mathcal{E}_{\gamma, \theta}\} \in \mathfrak{S}(\bigotimes_1^{\infty} \mathcal{A}_0)$ is defined by

$$\begin{aligned}
& \psi(j_1(A_1) \cdots j_n(A_n)) \\
& \equiv \varphi(\mathcal{E}_{\gamma, \theta}(A_1 \otimes \mathcal{E}_{\gamma, \theta}(A_2 \otimes \cdots \otimes A_{n-1} \mathcal{E}_{\gamma, \theta}(A_n \otimes I) \cdots)))
\end{aligned}$$

for each $n \in \mathbb{N}$ and each $A_1, \dots, A_n \in \mathcal{A}_0$, where $\mathcal{E}_{\gamma, \theta} = \theta \circ \mathcal{E}_{\gamma}$, $\theta \in \text{Aut}(\mathcal{A})$, $\psi|_{\bigotimes_1^n \mathcal{A}_0} \equiv \psi_n$ and j_k is the embedding from $\mathcal{A}_0$ to $\bigotimes_1^{\infty} \mathcal{A}_0$ such as $j_k(A) = I \otimes \cdots \otimes I \otimes A \otimes I \cdots$. For our $\mathcal{E}_{\gamma, \theta}$, ψ is written as

$$\begin{aligned}
\psi(j_1(A_1) \cdots j_n(A_n)) & = \psi_{[0, n]}(A_1 \otimes \cdots \otimes A_n \otimes I) \\
& = \psi_n(A_1 \otimes \cdots \otimes A_n),
\end{aligned}$$

where $\psi_{[0,n]}$ and ψ_n are the faithful normal states on $\bigotimes_1^n \mathcal{A}_0 \otimes \mathcal{A}$ and $\bigotimes_1^n \mathcal{A}_0$, respectively. When φ is defined by a trace class operator ρ such that $\varphi(\cdot) = \text{tr} \rho \cdot$, the density operators $\rho_{[0,n]}$ and ξ_n of $\psi_{[0,n]}$ and ψ_n are given by

$$\begin{aligned}\rho_{[0,n]} &= \sum_{i_1} \cdots \sum_{i_n} e_{i_1 i_1} \otimes \cdots \otimes e_{i_n i_n} \otimes \theta^n(\gamma_{i_n}) \cdots \gamma_{i_1} \rho \gamma_{i_1} \cdots \theta^n(\gamma_{i_n}) \\ \xi_n &\equiv \sum_{i_1} \cdots \sum_{i_n} \text{tr}_{\mathcal{A}}(\theta^n(\gamma_{i_n}) \cdots \gamma_{i_1} \rho \gamma_{i_1} \cdots \theta^n(\gamma_{i_n})) e_{i_1 i_1} \otimes \cdots \otimes e_{i_n i_n}\end{aligned}$$

Take

$$P_{i_n \cdots i_1} = \text{tr}_{\mathcal{A}}(\theta^n(\gamma_{i_n}) \cdots \gamma_{i_1} \rho \gamma_{i_1} \cdots \theta^n(\gamma_{i_n}))$$

The mean dynamical entropy [A.4] through QMC is

$$\begin{aligned}\tilde{S}_\varphi(\theta; \gamma) &\equiv \lim_{n \rightarrow \infty} \frac{1}{n} S_n(\gamma, \theta) \\ &= - \lim_{n \rightarrow \infty} \frac{1}{n} \text{tr} \xi_n \log \xi_n \\ &= - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i_1, \dots, i_n} P_{i_n \cdots i_1} \log P_{i_n \cdots i_1},\end{aligned}$$

When $P_{i_n \cdots i_1}$ satisfies the Markov property, the above equation becomes

$$\tilde{S}_\varphi(\theta; \gamma) = - \sum_{i_1, i_2} P(i_2 | i_1) P(i_1) \log P(i_2 | i_1).$$

The dynamical entropy through QMC with respect to θ and a subalgebra $\mathcal{A}_1$ of $\mathcal{A}$ is

$$\tilde{S}_\varphi(\theta; \mathcal{A}_1) \equiv \sup \{ \tilde{S}_\varphi(\theta; \gamma); \gamma \subset \mathcal{A}_1 \}.$$

§4. Relations Among Three Formulations

In this section, we discuss the relations among the above three formulations. The $\mathcal{S}$ -mixing entropy in GQS (general quantum systems) introduced in [O.5] is

$$S^\mathcal{S}(\varphi) = \inf \{ H(\mu) ; \mu \in M_\varphi(\mathcal{S}) \},$$

where $H(\mu)$ is given by

$$H(\mu) = \sup \left\{ - \sum_{A_k \in \tilde{A}} \mu(A_k) \log \mu(A_k) : \tilde{A} \in P(\mathcal{S}) \right\}$$

where $P(\mathcal{S})$ is the set of all partitions of $\mathcal{S}$.

The following theorem [O.8, M.1] shows the relation between the CNT formulation and the formulation by complexity.

Theorem 4.1 Under the above settings, we have the following relations:

- (1) $0 \leq I^S(\varphi; \Lambda^*) \leq T^S(\varphi; \Lambda^*) \leq J^S(\varphi; \Lambda^*)$
- (2) $C_I^S(\varphi) = C_T^S(\varphi) = C_J^S(\varphi) = S^S(\varphi) = H_\varphi(\mathcal{A})$
- (3) $\mathcal{A} = \tilde{\mathcal{A}} = B(\mathcal{H})$, for any density operator ρ

$$0 \leq I^S(\rho; \Lambda^*) = T^S(\rho; \Lambda^*) \leq J^S(\rho; \Lambda^*)$$

Since there exists a model showing $S^{I(\alpha)}(\varphi) > H_\varphi(\mathcal{A}_\alpha)$, $S^S(\varphi)$ distinguishes states more sharply than $H_\varphi(\mathfrak{N})$, where $\mathcal{A}_\alpha = \{A \in \mathcal{A}; \alpha(A) = A\}$.

Moreover we have [O.9].

- (1) When $\mathcal{A}_n, \mathcal{A}$ are the abelian C*-algebras and α_k is an embedding map, then

$$T^\mathfrak{S}(\mu; \alpha^M) = S_\mu^{\text{classical}} \left(\bigvee_{m=1}^M \tilde{A}_m \right)$$

$$I^\mathfrak{S}(\mu; \alpha^M, \beta^N) = I_\mu^{\text{classical}} \left(\bigvee_{n=1}^M \tilde{A}_n, \bigvee_{n=1}^N \tilde{B}_n \right)$$

are satisfied for any finite partitions $\tilde{A}_n, \tilde{B}_n$ on the probability space $(\Omega = \text{spec}(\mathcal{A}), \mathcal{F}, \mu)$.

- (2) When Λ is the restriction of $\mathcal{A}$ to a subalgebra $\mathcal{M}$ of $\mathcal{A}$; $\Lambda = |\mathcal{M}$ and

$$\mathcal{N} \subset \mathcal{A}_0, \mathcal{A} = \bigotimes_{i=1}^N \mathcal{A}_0, \theta \in \text{Aut}(\mathcal{A});$$

$$\alpha^N \equiv (\alpha, \theta \circ \alpha, \dots; \theta^{N-1} \circ \alpha);$$

$$\alpha = \beta; \mathcal{A}_0 \rightarrow \mathcal{A} \text{ embedding};$$

$$\mathcal{N}_N \equiv \bigotimes_{i=1}^N \mathcal{N},$$

we have

$$H_\varphi(\mathcal{M}) = J^\mathfrak{S}(\varphi; |\mathcal{M}|) = J_\varphi^\mathfrak{S}(\text{id}; |\mathcal{M}|),$$

$$\tilde{H}_\varphi(\theta; \mathcal{N}) = \tilde{J}_\varphi^\mathfrak{S}(\theta; \mathcal{N}) = \limsup_{N \rightarrow \infty} \frac{1}{N} J_\varphi^\mathfrak{S}(\alpha^N; |\mathcal{N}_N|).$$

We show the relation between the formulation by complexity and the formulation by QMC.

When φ is defined by a trace class operator ρ such that $\varphi(\cdot) = \text{tr} \rho \cdot$, we define a map $\mathcal{E}_{(n)}^*$ from $\mathfrak{S}(\mathcal{A})$ to $\mathfrak{S}((\bigotimes_{i=1}^n \mathcal{A}_0) \otimes \mathcal{A})$ by

$$\mathcal{E}_{(n)}^*(\varphi)(A) = \text{tr} \sum_{i_1} \cdots \sum_{i_n} e_{i_1 i_1} \otimes \cdots \otimes e_{i_n i_n} \otimes \theta^n(\gamma_{i_n}) \cdots \gamma_{i_1} \rho \gamma_{i_1} \cdots \theta^n(\gamma_{i_n}) A$$

for any $A \in (\bigotimes_1^n \mathcal{A}_0) \otimes \mathcal{A}$. Take a map $E_{(n)}$ from $\mathfrak{S}((\bigotimes_1^n \mathcal{A}_0) \otimes \mathcal{A})$ to $\mathfrak{S}(\bigotimes_1^n \mathcal{A}_0)$ such that

$$(E_{(n)}\omega)(Q) = \omega(Q \otimes I), \quad \forall Q \in \bigotimes_1^n \mathcal{A}_0$$

Then a channel $\Gamma_{(n)}^*$ from $\mathfrak{S}(\mathcal{A})$ to $\mathfrak{S}(\bigotimes_1^n \mathcal{A}_0)$ is given by

$$\Gamma_{(n)}^* \equiv E_{(n)} \circ \mathcal{E}_{(n)}^*$$

so that $\Gamma_{(n)}^*(\varphi)(A) = \text{tr} \xi_n A$ for any $A \in \bigotimes_1^n \mathcal{A}_0$ and

$$\tilde{S}_\varphi(\theta; \gamma) = \lim_{n \rightarrow \infty} \frac{1}{n} C_I^\mathfrak{S}(\Gamma_{(n)}^* \varphi) = \tilde{C}_I(\varphi; \theta, \gamma).$$

In any case, the formulation by the entropic complexities contains other formulations, moreover it opens other possibility to classify the dynamical systems more fine [A.5].

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