

On the fine behaviors of the eigenvalues of the linearized Gel'fand problem and its applications¹

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Abstract

The purpose of this note is to overview our recent results concerning the linearized eigenvalue problem for the Gel'fand problem. The main result is a second order estimate for the first m eigenvalues of the linearized Gel'fand problem associated to solutions which blow-up at m points. From this information, we determine some qualitative properties of the first m eigenfunctions.

This is based on a joint work with Francesca Gladiali (Univ. Sassari) and Massimo Grossi (Univ. Roma "La Sapienza").

1 The Gel'fand problem

The Gel'fand problem is the following semilinear elliptic problem with exponential nonlinearity:

$$-\Delta u = \lambda e^u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $\lambda > 0$ is a real parameter. This problem appears in a wide variety of areas of mathematics such as the conformal embedding of a flat domain into a sphere, self-dual gauge field theories, equilibrium states of large number of vortices, stationary states of chemotaxis motion, and so forth. See [7, 8] for more about our motivation and further references.

Especially the asymptotic behavior of the solutions as $\lambda \downarrow 0$ was studied in detail. Let $G(x, y)$ be the Green function of $-\Delta$ in Ω with Dirichlet boundary condition. We divide the Green function into two parts as usual:

$$G(x, y) = \frac{1}{2\pi} \log |x - y|^{-1} + K(x, y), \quad (1.2)$$

$K(x, y)$ is called the regular part of $G(x, y)$ and $R(x) = K(x, x)$ is the Robin function. Using these functions we introduce a function over Ω^m , which is

¹This work is supported by Grant-in-Aid for Scientific Research (No.22540231), Japan Society for the Promotion of Science.

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known as the *Hamiltonian* function of m vortices with equal intensities in the theory of 2-dimensional incompressible non-viscous fluid:

$$H^m(x_1, \dots, x_m) := \frac{1}{2} \sum_{j=1}^m R(x_j) + \frac{1}{2} \sum_{\substack{1 \leq j, h \leq m \\ j \neq h}} G(x_j, x_h).$$

Concerning the Gel'fand problem, the following result now seems to be classical:

Theorem 1.1 ([11]). *Let $\{\lambda_n\}_{n \in \mathbb{N}}$ be a sequence of positive values such that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$ and let $u_n = u_n(x)$ be a sequence of solutions of (1.1) for $\lambda = \lambda_n$. Then there exists some $m = 0, 1, 2, \dots, +\infty$ and, along a subsequence,*

$$\lambda_n \int_{\Omega} e^{u_n} dx \rightarrow 8\pi m. \quad (1.3)$$

Moreover, the following behaviors of solutions appear in the limit $n \rightarrow \infty$:

- (i) If $m = 0$, the sequence $\{u_n\}$ converges to 0 uniformly in Ω .
- (ii) If $m = +\infty$ the entire blow-up occurs, i.e. $\inf_K u_n \rightarrow +\infty$ for any $K \Subset \Omega$.
- (iii) If $0 < m < \infty$ the solutions $\{u_n\}$ blow-up at m -points, that is, there is a set $\mathcal{S} = \{\kappa_1, \dots, \kappa_m\} \subset \Omega$ of m distinct points and a subsequence of $\{u_n\}$ such that $\|u_n\|_{L^\infty(\omega)} = O(1)$ for any $\omega \Subset \overline{\Omega} \setminus \mathcal{S}$,

$$u_n|_{\mathcal{S}} \rightarrow +\infty \quad \text{as } n \rightarrow \infty,$$

and

$$u_n(x) \longrightarrow u_\infty(x) := \sum_{j=1}^m 8\pi G(x, \kappa_j) \quad (1.4)$$

locally uniformly in $\overline{\Omega} \setminus \{\kappa_1, \dots, \kappa_m\}$. Furthermore the blow-up points $\mathcal{S} = \{\kappa_1, \dots, \kappa_m\}$ satisfy

$$\nabla H^m(\kappa_1, \dots, \kappa_m) = 0. \quad (1.5)$$

We note that a blow-up sequence of solutions for given $\mathcal{S}$ satisfying (1.5) really exists under appropriate assumptions on $\mathcal{S}$, see [1, 4, 5].

In this note we are concerned with more details about the case (iii) of Theorem 1.1. In the following we always assume that $\{u_n\}$ is a sequence of solutions to (1.1) with m blow-up points in the limit $n \rightarrow \infty$.

2 The linealized eigenvalue problem of the Gel'fand problem

Our object in this note is the following eigenvalue problem:

$$-\Delta v = \mu \lambda_n e^{u_n} v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \quad (2.1)$$

where $\{u_n\}$ is a m -points blow-up sequence of solutions to (1.1). We are able to assume that there exists a sequence of eigenvalues $\mu_n^1 \leq \mu_n^2 \leq \mu_n^3 \leq \dots$. We denote k -th eigenfunction of (2.1) corresponding to the eigenvalue μ_n^k as v_n^k .

We define a diagonal matrix $D := \text{diag}[d_1, d_1, d_2, d_2, \dots, d_m, d_m]$, where d_j is a constant given by

$$d_j = \frac{1}{8} \exp \left\{ 4\pi R(\kappa_j) + 4\pi \sum_{1 \leq i \leq m, i \neq j} G(\kappa_j, \kappa_i) \right\} (> 0). \quad (2.2)$$

Previously we get the following behavior of μ_n^k :

Theorem 2.1 ([8]). *For $\lambda_n \rightarrow 0$, it holds that*

$$\begin{aligned} \mu_n^k &= -\frac{1}{2} \frac{1}{\log \lambda_n} + o\left(\frac{1}{\log \lambda_n}\right) (\rightarrow 0), \text{ for } 1 \leq k \leq m, \\ \mu_n^k &= 1 - 48\pi \eta^{(2m+1-s)} \lambda_n + o(\lambda_n) (\rightarrow 1), \text{ for } m+1 \leq k (= m+s) \leq 3m, \\ \mu_n^k &> 1, \text{ for } k \geq 3m+1 \end{aligned} \quad (2.3)$$

where η^k ($k = 1, \dots, 2m$) is the k -th eigenvalue of the matrix $D(\text{Hess}H^m)D$ at $(\kappa_1, \dots, \kappa_m)$.

We use these to calculate the Morse index of u_n for $n \gg 1$. Actually we are able to get the following estimate easily from the above behaviors of $\{\mu_n^k\}$:

$$m + \text{ind}_M\{-H^m(\kappa_1, \dots, \kappa_m)\} \leq \text{ind}_M(u_n), \quad (2.4)$$

$$\text{ind}_M^*(u_n) \leq m + \text{ind}_M^*\{-H^m(\kappa_1, \dots, \kappa_m)\}. \quad (2.5)$$

where

$$\text{ind}_M(u_n) = \#\{k \in \mathbb{N}; \mu_n^k < 1\}, \quad \text{ind}_M^*(u_n) = \#\{k \in \mathbb{N}; \mu_n^k \leq 1\}.$$

are the Morse index and the augmented Morse index of u_n , respectively.

$$\text{ind}_M\{-H^m(\kappa_1, \dots, \kappa_m)\}, \quad \text{ind}_M^*\{-H^m(\kappa_1, \dots, \kappa_m)\}$$

are the Morse index and the augmented Morse index of the $-H^m$, that is, the numbers of the negative and non-positive eigenvalues of Hessian of $-H^m$ at $(\kappa_1, \dots, \kappa_m)$, respectively. These results are a generalization of the results in [6] obtained for the case $m = 1$.

Recently we have refined the case $1 \leq k \leq m$ as follows:

Theorem 2.2 ([7]). For each $k \in \{1, \dots, m\}$, the followings hold:

(i) Let (h_{ij}) be the matrix given by

$$h_{ij} = \begin{cases} R(\kappa_i) + 2 \sum_{\substack{1 \leq l \leq m \\ l \neq i}} G(\kappa_l, \kappa_i), & \text{for } i = j, \\ -G(\kappa_i, \kappa_j), & \text{for } i \neq j, \end{cases}$$

and Λ^k be the k -th eigenvalue of (h_{ij}) , assuming $\Lambda^1 \leq \dots \leq \Lambda^m$. Then

$$\mu_n^k = -\frac{1}{2} \frac{1}{\log \lambda_n} + \left(2\pi \Lambda^k - \frac{3 \log 2 - 1}{2} \right) \frac{1}{(\log \lambda_n)^2} + o\left(\frac{1}{(\log \lambda_n)^2} \right) \quad (2.6)$$

as $n \rightarrow +\infty$.

(ii) Suppose v_n^k is normalized as

$$\|v_n^k\|_\infty = 1.$$

Then there exists a k -th eigenvector $\mathbf{c}^k = (c_1^k, \dots, c_m^k) \in [-1, 1]^m \subset \mathbb{R}^m$ ($\mathbf{c}^k \neq \mathbf{0}$) of (h_{ij}) and a subsequence satisfying

$$\frac{v_n^k}{\mu_n^k} \rightarrow \sum_{j=1}^m 8\pi c_j^k G(\cdot, \kappa_j) \quad \text{locally uniformly in } \overline{\Omega} \setminus \{\kappa_1, \dots, \kappa_m\}. \quad (2.7)$$

We note that it seems difficult to realize the matrix (h_{ij}) (and D) in the case $\#\mathcal{S} = 1$ considered in [6].

From the conclusion (ii), we are able to show that $v_n^k \rightarrow 0$ outside the blow-up set. On the other hand, we know that $c_j^k = 0$ implies $v_n^k \rightarrow 0$ locally uniformly near κ_j , see [8, Proposition 2.11]. Therefore we introduce the following definition:

Definition 2.3. We say that an eigenfunction v_n^k concentrates at $\kappa_j \in \Omega$ if

there exists $\kappa_{j,n} \rightarrow \kappa_j$ such that

$$|v_n^k(\kappa_{j,n})| \geq C > 0 \quad \text{for } n \text{ large.} \quad (2.8)$$

As we see later, c_j^k is obtained as the limit of $v_n^k(x_{j,n})$ as $n \rightarrow \infty$ for the sequence satisfying $x_{j,n} \rightarrow \kappa_j$ and $u_n(x_{j,n}) \rightarrow \infty$. Therefore it holds that

$$v_n^k \text{ concentrates at } \kappa_j \text{ if and only if } c_j^k \neq 0, \quad (2.9)$$

that is, we are able to count the number of peaks of v_n^k from the number of non-zero components of $\mathbf{c}^k$ as an application of this work.

Remark 2.4. We note that the behavior (2.7) for *some* $\mathbf{c}^k \in [-1, 1]^m \subset \mathbb{R}^m$ ($\mathbf{c}^k \neq 0$) is obtained in the previous work [8, Proposition 2.5 and 2.13]. In this work we clarify the origin of $\mathbf{c}^k$ from the *fine* behavior of eigenvalues.

3 On the scaling argument and the behavior of eigenfunctions

In this section we sketch the proof of (2.7) and introduce the scaling argument necessary to get it.

Fix $0 < R \ll 1$ satisfying

$$B_{2R}(\kappa_i) \Subset \Omega \text{ for } i = 1, \dots, m \text{ and } B_R(\kappa_i) \cap B_R(\kappa_j) = \emptyset \text{ if } i \neq j.$$

Choose a sequence $\{x_{j,n}\}$ for each $\kappa_j \in \mathcal{S}$ satisfying

$$x_{j,n} \rightarrow \kappa_j, \quad u_n(x_{j,n}) = \max_{B_R(x_{j,n})} u_n(x) \rightarrow \infty.$$

From the Green representation formula and the behavior (1.4) of u_n , we get

$$\begin{aligned} \frac{v_n^k(x)}{\mu_n^k} &= \int_{\Omega} G(x, y) \lambda_n e^{u_n} v_n^k dy \\ &= \sum_{j=1}^m \int_{B_R(x_{j,n})} G(x, y) \lambda_n e^{u_n} v_n^k dy + O(\lambda_n). \end{aligned}$$

Since $G(x, x_{j,n})$ is smooth far from $x_{j,n}$, Taylor's theorem

$$\begin{aligned} G(x, y) &= G(x, x_{j,n}) + (y - x_{j,n}) \cdot \nabla_y G(x, x_{j,n}) + s(x, \eta, y - x_{j,n}), \\ s(x, \eta, y - x_{j,n}) &= \frac{1}{2} \sum_{1 \leq \alpha, \beta \leq 2} G_{y_\alpha y_\beta}(x, \eta) (y - x_{j,n})_\alpha (y - x_{j,n})_\beta, \\ \eta &= \eta(j, n, y) \in B_R(x_{j,n}), \end{aligned}$$

guarantees that

$$\begin{aligned}
& \int_{B_R(x_{j,n})} G(x, y) \lambda_n e^{u_n} v_n^k dy \\
&= G(x, x_{j,n}) \int_{B_R(x_{j,n})} \lambda_n e^{u_n} v_n^k dy + \nabla_y G(x, x_{j,n}) \cdot \int_{B_R(x_{j,n})} (y - x_{j,n}) \lambda_n e^{u_n} v_n^k dy \\
&\quad + \frac{1}{2} \sum_{1 \leq \alpha, \beta \leq 2} \int_{B_R(x_{j,n})} (y - x_{j,n})_\alpha (y - x_{j,n})_\beta G_{y_\alpha y_\beta}(x, \eta) \lambda_n e^{u_n} v_n^k dy \\
&=: \gamma_{j,n}^0 G(x, x_{j,n}) + \gamma_{j,n}^1 \cdot \nabla_y G(x, x_{j,n}) + \gamma_{j,n}^2.
\end{aligned}$$

So we need to see the behaviors of $\gamma_{j,n}^0$, $\gamma_{j,n}^1$, and $\gamma_{j,n}^2$ to get (2.7).

To this purpose we rescale the solution u_n and eigenfunction v_n^k around $x_{j,n}$. Let $\delta_{j,n}$ be a parameter determined by the relation

$$\lambda_n e^{u_n(x_{j,n})} \delta_{j,n}^2 = 1 \quad (3.1)$$

and set

$$\begin{aligned}
\tilde{u}_{j,n}(\tilde{x}) &:= u_n(\delta_{j,n} \tilde{x} + x_{j,n}) - u_n(x_{j,n}) \quad \text{in } B_{\frac{R}{\delta_{j,n}}}(0), \\
\tilde{v}_{j,n}^k(\tilde{x}) &:= v_n^k(\delta_{j,n} \tilde{x} + x_{j,n}) \quad \text{in } B_{\frac{R}{\delta_{j,n}}}(0).
\end{aligned}$$

Then it holds that

$$-\Delta \tilde{u}_{j,n} = e^{\tilde{u}_{j,n}} \quad \text{in } B_{\frac{R}{\delta_{j,n}}}(0), \quad \tilde{u}_{j,n} \leq \tilde{u}_{j,n}(0) = 0 \quad \text{in } B_{\frac{R}{\delta_{j,n}}}(0) \quad (3.2)$$

and

$$-\Delta \tilde{v}_{j,n}^k = \mu_n^k e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k, \quad \text{in } \tilde{x} \in B_{\frac{R}{\delta_{j,n}}}(0), \quad \|\tilde{v}_{j,n}^k\|_{L^\infty(B_{\frac{R}{\delta_{j,n}}}(0))} \leq 1. \quad (3.3)$$

We note that there exists $d_j > 0$ such that

$$\delta_{j,n} = d_j \lambda_n^{\frac{1}{2}} + o(\lambda_n^{\frac{1}{2}}) (\longrightarrow 0) \quad (3.4)$$

from Y.Y.Li's estimate ([10], see also [9, Corollary 4.3]). We also note that d_j is known to be given by (2.2), see [7, Proposition 3.4].

Now assuming

$$\mu_n^k \longrightarrow \mu_\infty^k \in \mathbb{R},$$

we reach the following problem at the limit $n \longrightarrow \infty$:

$$-\Delta U = e^U \quad \text{in } \mathbb{R}^2, \quad U(\tilde{x}) \leq U(0) \quad (3.5)$$

and

$$-\Delta V = \mu_\infty^k e^U V, \quad \|V\|_{L^\infty(\mathbb{R}^2)} \leq 1. \quad (3.6)$$

From Chen-Li's result [3], we see

$$U(\tilde{x}) = \log \frac{1}{\left(1 + \frac{|\tilde{x}|^2}{8}\right)^2}$$

and we get

$$\tilde{u}_{j,n}(\tilde{x}) \rightarrow U(\tilde{x}) \quad \text{in } C_{loc}^{2,\alpha}(\mathbb{R}^2).$$

On the other hand, we can show

$$\tilde{v}_{j,n}^k \rightarrow V_j^k \quad \text{in } C_{loc}^{2,\alpha}(\mathbb{R}^2)$$

holds for a subsequence, where V_j^k is a solution of (3.6) ([8, Proposition 2.2]). Since we know that there exists $j \in \{1, \dots, m\}$ such that $V_j^k \not\equiv 0$ ([8, Proposition 2.11]), we see that μ_∞^k is an eigenvalue of the linearized eigenvalue problem (3.6) for (3.5). These eigenvalues (and the eigenfunctions) are studied in [6] and we know

$$\mu_\infty^k = \frac{l(l+1)}{2} \quad \text{for some } l = 0, 1, 2, \dots$$

In this note, we are interested in the case $\mu_\infty^k = 0$ and for this case the eigenfunction is known to

$$V_j^k \equiv \text{const. } (= c_j^k) \in [-1, 1].$$

Consequently we are able to confirm that

$$\gamma_{j,n}^0 = \int_{B_R(x_{j,n})} \lambda_n e^{u_n} v_n^k = \int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \rightarrow c_j^k \int_{\mathbb{R}^2} e^U = 8\pi c_j^k.$$

Similarly we see

$$\begin{aligned} \gamma_{j,n}^1 &= \int_{B_R(x_{j,n})} (y - x_{j,n}) \lambda_n e^{u_n} v_n^k = \delta_{j,n} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \tilde{y} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \\ &= \delta_{j,n} \left\{ \int_{\mathbb{R}^2} \tilde{y} e^{\tilde{U}} \tilde{V}_j^k + o(1) \right\} = o(\delta_{j,n}) = o\left(\lambda_n^{\frac{1}{2}}\right) \end{aligned}$$

and, taking $\varepsilon \in (0, 1)$,

$$\begin{aligned}
& \left| \int_{B_R(x_{j,n})} G_{y_\alpha y_\beta}(x, \eta) (y - x_{j,n})_\alpha (y - x_{j,n})_\beta \lambda_n e^{u_n} v_n^k dy \right| \\
& \leq CR^\varepsilon \int_{B_R(x_{j,n})} |y - x_{j,n}|^{2-\varepsilon} \lambda_n e^{u_n} dy \\
& = CR^\varepsilon \delta_{j,n}^{2-\varepsilon} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} |\tilde{y}|^{2-\varepsilon} e^{\tilde{u}_{j,n}} d\tilde{y} = O(\delta_{j,n}^{2-\varepsilon}) = o\left(\lambda_n^{\frac{1}{2}}\right).
\end{aligned}$$

For simplify the presentation, we omitted here some additional argument necessary to the process $n \rightarrow \infty$, see [8, Proposition 2.5 and 2.13] for details.

4 On the improvement of the behavior of the eigenvalues

The sharp formula (2.6) is from the determination of the following constant L :

$$\frac{1}{\mu_n^k} = -2 \log \lambda_n + L + o(1). \quad (4.1)$$

Indeed, this leads

$$\begin{aligned}
\mu_n^k &= \frac{1}{-2 \log \lambda_n + L + o(1)} = -\frac{1}{2 \log \lambda_n} \cdot \frac{1}{1 - \frac{L+o(1)}{2 \log \lambda_n}} \\
&= -\frac{1}{2 \log \lambda_n} \left\{ 1 + \frac{L+o(1)}{2 \log \lambda_n} + o\left(\frac{1}{\log \lambda_n}\right) \right\} \\
&= -\frac{1}{2 \log \lambda_n} - \frac{L}{4} \cdot \frac{1}{(\log \lambda_n)^2} + o\left(\frac{1}{(\log \lambda_n)^2}\right).
\end{aligned}$$

It is easy to see that we get (2.3) from this if L is not specified. In the following we sketch how to get

$$L = -8\pi\Lambda^k + 2(3 \log 2 - 1).$$

The constant L in (4.1) comes from two formula:

$$u_n(x_{j,n}) = -2 \log \lambda_n - 2 \log d_j + o(1), \quad (4.2)$$

where d_j is the number given in (2.2), and

$$\left\{ \frac{1}{\mu_n^k} - u_n(x_{j,n}) \right\} \gamma_{j,n}^0 + 16\pi c_j^k = -(8\pi)^2 \sum_{i=1}^m g_{ji} c_i^k + o(1), \quad (4.3)$$

where

$$g_{ji} = \begin{cases} \sum_{\substack{1 \leq h \leq m \\ h \neq j}} G(\kappa_j, \kappa_h), & \text{for } j = i, \\ -G(\kappa_j, \kappa_i), & \text{for } j \neq i. \end{cases}$$

Concerning this point, previously we used

$$u_n(x_{j,n}) = -2 \log \lambda_n + O(1)$$

and

$$\left\{ \frac{1}{\mu_n^k} - u_n(x_{j,n}) \right\} \gamma_{j,n}^0 = O(1).$$

Here we eliminate $u_n(x_{j,n})$ from these and get

$$\left\{ \frac{1}{\mu_n^k} + 2 \log \lambda_n + O(1) \right\} \gamma_{j,n}^0 = O(1).$$

We know that $\gamma_{j,n}^0 \rightarrow 8\pi c_j^k$ and there exists at least one $j = 1, \dots, m$ such that $c_j^k \neq 0$ since $\mathbf{c}^k = (c_1^k, \dots, c_m^k) \neq \mathbf{0}$. Therefore we get

$$\frac{1}{\mu_n^k} = -2 \log \lambda_n + O(1).$$

Similarly, eliminating $u_n(x_{j,n})$ from (4.2) and (4.3), we get

$$\left\{ \frac{1}{\mu_n^k} + 2 \log \lambda_n + 2 \log d_j + o(1) \right\} \gamma_{j,n}^0 + 16\pi c_j^k = -(8\pi)^2 \sum_{i=1}^m g_{ji} c_i^k + o(1),$$

that is,

$$\begin{aligned} \frac{1}{\mu_n^k} &= -2 \log \lambda_n - 2 - 2 \log d_j - \frac{8\pi \sum_{i=1}^m g_{ji} c_i^k}{c_j^k} + o(1) \\ &= -2 \log \lambda_n + 2(3 \log 2 - 1) - \frac{8\pi \sum_{i=1}^m h_{ji} c_i^k}{c_j^k} + o(1). \end{aligned}$$

Here we assume $c_j^k \neq 0$ for simplicity. Since this formula exists for each $j = 1, \dots, m$. We are able to get

$$\frac{\sum_{i=1}^m h_{ji} c_i^k}{c_j^k} = \frac{\sum_{i=1}^m h_{li} c_i^k}{c_l^k}$$

for $j \neq l$. This means that there exists a constant Λ^k such that

$$\sum_{i=1}^m h_{ji} c_i^k = \Lambda^k c_j^k$$

for every $j = 1, \dots, m$, that is, Λ^k is an eigenvalue of the matrix (h_{ji}) , see [7] for details. Obviously $\mathbf{c}^k = (c_1^k, \dots, c_m^k)$ is an eigenvector of (h_{ji}) .

Finally we are able to conclude that Λ^k is the k -th eigenvalue of (h_{ji}) because $\mu_n^1 \leq \dots \leq \mu_n^k$.

4.1 Derivation of (4.2)

The formula (4.2) was essentially proved by C. C. Chen and C.-S. Lin [2, Estimate D] from the Green's representation formula:

$$\begin{aligned} u_n(x_{j,n}) &= \int_{\Omega} G(x_{j,n}, y) \lambda_n e^{u_n(y)} dy \\ &= \frac{1}{2\pi} \int_{B_R(x_{j,n})} \log |x_{j,n} - y|^{-1} \lambda_n e^{u_n(y)} dy \\ &\quad + \int_{B_R(x_{j,n})} K(x_{j,n}, y) \lambda_n e^{u_n(y)} dy \\ &\quad + \sum_{\substack{1 \leq i \leq m \\ i \neq j}} \int_{B_R(x_{i,n})} G(x_{j,n}, y) \lambda_n e^{u_n(y)} dy \\ &\quad + \int_{\Omega \setminus \bigcup_{i=1}^m B_R(x_{i,n})} G(x_{j,n}, y) \lambda_n e^{u_n(y)} dy \\ &= -\frac{\sigma_{j,n}}{2\pi} \log \delta_{j,n} + \frac{1}{2\pi} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \log |\tilde{y}|^{-1} e^{\tilde{u}_{j,n}(\tilde{y})} d\tilde{y} \\ &\quad + 8\pi \left\{ R(x_{j,n}) + \sum_{\substack{1 \leq i \leq m \\ i \neq j}} G(x_{j,n}, x_{i,n}) \right\} + o(1), \end{aligned}$$

where

$$\sigma_{j,n} := \lambda_n \int_{B_R(x_{j,n})} e^{u_n} dx.$$

We are able to confirm that

$$\frac{1}{2\pi} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \log |\tilde{y}|^{-1} e^{\tilde{u}_{j,n}(\tilde{y})} d\tilde{y} \rightarrow \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |\tilde{y}|^{-1} e^{U(\tilde{y})} d\tilde{y} = -6 \log 2 \quad (4.4)$$

and

$$8\pi \left\{ R(x_{j,n}) + \sum_{\substack{1 \leq i \leq m \\ i \neq j}} G(x_{j,n}, x_{i,n}) \right\} \longrightarrow 2 \log(8d_j). \quad (4.5)$$

On the other hand, we know $\sigma_{j,n} \longrightarrow 8\pi$ from Theorem 1.1. Therefore we get the following formula from the relation (3.1):

$$u_n(x_{j,n}) = -\frac{\sigma_{j,n}}{\sigma_{j,n} - 4\pi} \log \lambda_n - 2 \log d_j + o(1) \quad (4.6)$$

$$= -2 \log \lambda_n + \frac{\sigma_{j,n} - 8\pi}{\sigma_{j,n} - 4\pi} \log \lambda_n - 2 \log d_j + o(1). \quad (4.7)$$

To get (4.2), we need to know more precise behavior of $\sigma_{j,n}$ along $\lambda_n \longrightarrow 0$. To this purpose the following one obtained in [2, (3.56)] is sufficient:

$$\sigma_{j,n} = 8\pi + o(\lambda_n). \quad (4.8)$$

We note that a weaker version

$$\sigma_{j,n} = 8\pi + o\left(\lambda_n^{\frac{1}{2}}\right),$$

which is also sufficient for our purpose, can be obtained rather easily, see [13].

4.2 Derivation of (4.3)

To get the formula (4.3) we use the Green's theorem for u_n and $\frac{v_n^k}{\mu_n^k}$ around $B_R(x_{j,n})$:

$$\int_{\partial B_R(x_{j,n})} \left\{ \frac{\partial u_n}{\partial \nu} \frac{v_n^k}{\mu_n^k} - u_n \frac{\partial}{\partial \nu} \left(\frac{v_n^k}{\mu_n^k} \right) \right\} d\sigma = \int_{B_R(x_{j,n})} \left\{ \Delta u_n \frac{v_n^k}{\mu_n^k} - u_n \Delta \frac{v_n^k}{\mu_n^k} \right\} dx \quad (4.9)$$

The choice of u_n and $\frac{v_n^k}{\mu_n^k}$ seems to be a kind of trick. Indeed we know the behaviors of u_n and $\frac{v_n^k}{\mu_n^k}$ far from $\mathcal{S} = \{\kappa_1, \dots, \kappa_m\}$, see (1.4) and (2.7). Therefore the left-hand side of (4.9) has limit in the process $n \longrightarrow \infty$.

In fact, we have

$$\begin{aligned} & \int_{\partial B_R(x_{j,n})} \left\{ \frac{\partial u_n}{\partial \nu} \frac{v_n^k}{\mu_n^k} - u_n \frac{\partial}{\partial \nu} \left(\frac{v_n^k}{\mu_n^k} \right) \right\} d\sigma \\ & \rightarrow (8\pi)^2 \sum_{h=1}^m \sum_{i=1}^m c_i^k \int_{\partial B_R(\kappa_j)} \left\{ \frac{\partial}{\partial \nu} G(x, \kappa_h) G(x, \kappa_i) - G(x, \kappa_h) \frac{\partial}{\partial \nu} G(x, \kappa_i) \right\} d\sigma. \end{aligned} \quad (4.10)$$

It is easy to see that

$$\begin{aligned} & \int_{\partial B_R(\kappa_j)} \left\{ \frac{\partial}{\partial \nu} G(x, \kappa_h) G(x, \kappa_i) - G(x, \kappa_h) \frac{\partial}{\partial \nu} G(x, \kappa_i) \right\} d\sigma \\ &= \begin{cases} 0, & h = i \\ -G(\kappa_j, \kappa_i) \delta_h^j + G(\kappa_j, \kappa_h) \delta_j^i, & h \neq i, \end{cases} \end{aligned}$$

where $\delta_a^b = 1$ if $a = b$ and $\delta_a^b = 0$ else.

Therefore, from (4.10) we have

$$\begin{aligned} & \int_{\partial B_R(x_{j,n})} \left\{ \frac{\partial u_n}{\partial \nu} \frac{v_n^k}{\mu_n^k} - u_n \frac{\partial}{\partial \nu} \left(\frac{v_n^k}{\mu_n^k} \right) \right\} d\sigma \\ &= (8\pi)^2 \sum_{h=1}^m \sum_{\substack{1 \leq i \leq m \\ i \neq h}} c_i^k \{ -G(\kappa_j, \kappa_i) \delta_h^j + G(\kappa_j, \kappa_h) \delta_j^i \} + o(1) \\ &= (8\pi)^2 \sum_{i=1}^m g_{ji} c_i^k + o(1). \end{aligned}$$

On the other hand, we are able to apply the scaling argument to the right-hand side of (4.9). Indeed the following holds:

$$\begin{aligned} & \int_{B_R(x_{j,n})} \left\{ \Delta u_n \frac{v_n^k}{\mu_n^k} - u_n \Delta \frac{v_n^k}{\mu_n^k} \right\} dx \\ &= -\frac{1}{\mu_n^k} \lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k dx + \lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k u_n dx \\ &= -\frac{1}{\mu_n^k} \lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k dx + u_n(x_{j,n}) \lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k dx \\ &\quad + \int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \tilde{u}_{j,n} d\tilde{x} \\ &= -\left(\frac{1}{\mu_n^k} - u_n(x_{j,n}) \right) \gamma_{j,n}^0 + \int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \tilde{u}_{j,n} d\tilde{x}. \end{aligned}$$

Here

$$\int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \tilde{u}_{j,n} d\tilde{x} \longrightarrow \int_{\mathbb{R}^2} e^U V_j^k U d\tilde{x} = -16\pi c_j^k.$$

Consequently we get (4.3).

5 Examples of $\mathbf{c}^k = (c_1^k, \dots, c_m^k) \neq 0$

Let us fix an integer $m \geq 2$ and Ω be an annulus $\{x \in \mathbb{R}^2; (0 <) a < |x| < 1\}$. In [12] there was constructed a m -mode solution u_n to (1.1), i.e. a solution which is invariant with respect to a rotation of $\frac{2\pi}{m}$ in $\mathbb{R}^2$,

$$u(r, \theta) = u\left(r, \left(\theta + \frac{2\pi}{m}\right)\right).$$

This solution blows-up at m points $\kappa_1 = \dots = \kappa_m$ which are located on a circle concentric with the annulus and are vertices of a regular polygon with m sides. So we can assume that $\kappa_1 = (r_0, 0)$, $\kappa_2 = r_0 \left(\cos \frac{2\pi}{m}, \sin \frac{2\pi}{m}\right), \dots, \kappa_m = r_0 \left(\cos \frac{2(m-1)\pi}{m}, \sin \frac{2(m-1)\pi}{m}\right)$ for some $r_0 \in (a, 1)$.

Since $G(x, \kappa_1)$ is symmetric with respect to the x_1 -axis, we get $G(\kappa_j, \kappa_1) = G(\kappa_{m-j+2}, \kappa_1)$, $j = 2, \dots, m$. Similarly the value $G(\kappa_i, \kappa_j)$ depends only on the distance between κ_i and κ_j . Since Ω is an annulus, the Robin function $R(x)$ is radial, so that $R(\kappa_1) = \dots = R(\kappa_m) = R$.

Here we set $G_i := G(\kappa_i, \kappa_1)$ and $R_l := R + 4 \sum_{h=2}^l G_h$ for simplicity. Then the matrix h_{ij} becomes as follows:
for $m = 2l$ ($l = 1, 2, \dots$),

$$(h_{ij}) = \begin{pmatrix} R_l + 2G_{l+1} & -G_2 & -G_3 & \dots & -G_{l+1} & \dots & -G_2 \\ -G_2 & R_l + 2G_{l+1} & -G_2 & \dots & \dots & \dots & -G_3 \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ -G_2 & -G_3 & \dots & \dots & \dots & \dots & R_l + 2G_{l+1} \end{pmatrix},$$

and for $m = 2l + 1$ ($l = 1, 2, \dots$),

$$(h_{ij}) = \begin{pmatrix} R_l & -G_2 & -G_3 & \dots & -G_l & -G_l & \dots & -G_2 \\ -G_2 & R_l & -G_2 & \dots & \dots & \dots & \dots & -G_3 \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -G_2 & -G_3 & \dots & \dots & \dots & \dots & -G_2 & R_l \end{pmatrix},$$

A straightforward computation shows the following facts:

- $m=3$

$$\lambda_1 = R + 2G_2, \mathbf{c}_1 = (1, 1, 1).$$

$$\lambda_2 = \lambda_3 = R + 5G_2, \mathbf{c}_2 = (1, -1, 0), \mathbf{c}_3 = (1, 0, -1).$$

- $m=4$

$$\lambda_1 = R + 2G_2 + G_3, \mathbf{c}_1 = (1, 1, 1, 1).$$

$$\lambda_2 = \lambda_3 = R + 4G_2 + 3G_3, \mathbf{c}_2 = (1, 0, -1, 0), \mathbf{c}_3 = (0, 1, 0, -1).$$

$$\lambda_4 = R + 6G_2 + G_3, \mathbf{c}_4 = (1, -1, 1, -1).$$

- $m=5$

$$\lambda_1 = R + 2G_2 + 2G_3, \mathbf{c}_1 = (1, 1, 1, 1, 1),$$

$$\lambda_2 = \lambda_3 = R + \frac{9-\sqrt{5}}{2}G_2 + \frac{9+\sqrt{5}}{2}G_3,$$

$$\mathbf{c}_2 = (1, \frac{-1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, -1, 0), \mathbf{c}_3 = (1, -1, \frac{-1-\sqrt{5}}{2}, 0, \frac{1+\sqrt{5}}{2}),$$

$$\lambda_4 = \lambda_5 = R + \frac{9+\sqrt{5}}{2}G_2 + \frac{9-\sqrt{5}}{2}G_3.$$

$$\mathbf{c}_4 = (1, \frac{-1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, -1, 0), \mathbf{c}_5 = (1, -1, \frac{-1+\sqrt{5}}{2}, 0, \frac{1-\sqrt{5}}{2}).$$

- $m=6$

$$\lambda_1 = R + 2G_2 + 2G_3 + G_4, \mathbf{c}_1 = (1, 1, 1, 1, 1, 1),$$

$$\lambda_2 = \lambda_3 = R + 3G_2 + 5G_3 + 3G_4,$$

$$\mathbf{c}_2 = (1, 0, -1, -1, 0, 1), \mathbf{c}_3 = (1, 1, 0, -1, -1, 0)$$

$$\lambda_4 = \lambda_5 = R + 5G_2 + 5G_3 + G_4,$$

$$\mathbf{c}_4 = (1, -1, 0, -1, 1, 0), \mathbf{c}_5 = (1, 0, 1, -1, 0, 1)$$

$$\lambda_6 = R + 6G_2 + 2G_3 + 3G_4,$$

$$\mathbf{c}_6 = (1, -1, 1, -1, 1, -1)$$

- ...

In general, it is easy to see that the first eigenvalue of (h_{ij}) is $\Lambda^1 = R + 2\sum_{h=2}^l G_h + G_{l+1}$ for $m = 2l$ and $R + 2\sum_{h=2}^l G_h$ for $m = 2l + 1$ which is simple. It is easy to see that the eigenspace corresponding to Λ^1 is spanned by $\mathbf{c}^1 = (1, 1, \dots, 1)$.

Unfortunately we are not yet able to get further information on the multiplicity of the eigenvalues even for these cases and we will leave this for future work, see also [7, Remark 5.1].

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