

SDEのEuler-丸山型近似解に対するSample Path Large Deviations

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Introduction : Itô's SDE

Let $\{B(t), 0 \leq t \leq 1\}$ be an r -dimensional standard Brownian motion. Consider Itô's stochastic differential equation (SDE) for a d -dimensional continuous process $\{X(t), 0 \leq t \leq 1\}$ ($d \geq 1$):

$$(1) \quad \begin{cases} dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt, 0 \leq t \leq 1 \\ X(0) = X_0. \end{cases}$$

Suppose that $\sigma(t, x)$ and $b(t, x)$ satisfy the Lipschitz condition. Then there exists a unique solution of (1). (See e.g., Ikeda-Watanabe (1981).)

1. The Euler-Maruyama Algorithm

Maruyama (1955) showed the existence of the unique solution of (1) using an Euler type approximation solution $Z_n := \{Z_n(t), 0 \leq t \leq 1\}$ defined by

$$(2) \quad Z_n(t) := X_0 + \int_0^t \sigma_n(u)dB(u) + \int_0^t b_n(u)du, \quad 0 \leq t \leq 1,$$

where

$$\sigma_n(t) := \sigma\left(\frac{k-1}{n}, x_{k-1}\right), \quad k/n \leq t \leq (k+1)/n, \quad k=0, \dots, n-1,$$

$$b_n(t) := b\left(\frac{k-1}{n}, x_{k-1}\right), \quad k/n \leq t \leq (k+1)/n, \quad k=0, \dots, n-1,$$

$$x_k := X_0 + \sum_{j=1}^k \sigma\left(\frac{j-1}{n}, x_{j-1}\right)\eta_j + \sum_{j=1}^k b\left(\frac{j-1}{n}, x_{j-1}\right)/n, \quad k=0, 1, \dots, n,$$

$$\eta_k := B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right), \quad k=1, \dots, n.$$

Theorem A. (Maruyama (1955)) Suppose that $\sigma(t,x)$ and $b(t,x)$ satisfy the Lipschitz condition, i.e.

$$|\sigma(t,x) - \sigma(s,y)|^2 + |b(t,x) - b(s,y)|^2 \leq K_1(|x-y|^2 + |t-s|^2),$$

where K_1 is a positive constant independent of x, y, t and s . Then for any $t \geq 0$

$$\lim_{n \rightarrow \infty} E(|X(t) - Z_n(t)|^2) = 0.$$

The above scheme for the construction of $Z_n(t)$ has no practical value since it needs the complete knowledge about whole trajectory over the interval $[0,1]$ of the Brownian motion, which is also to be simulated. Therefore we find it necessary to introduce a stochastic process $X_n := \{X_n(t), 0 \leq t \leq 1\}$ in $D(0,1)$ by a slight modification of Z_n

$$\begin{cases} X_n(t) := x_k, & k/n \leq t < (k+1)/n, \quad k=0, \dots, n-1 \\ X_n(1) := x_n, \end{cases}$$

here $\{\eta_k\}$ are i.i.d. random variables constructed from pseudo-random numbers with the r -dimensional normal distribution $N(0,1/n)$.

As for the error estimation for Z_n , Ghiman-Skorohod (1979), Shimizu (1984) showed the rate of convergence of them to the real solution X of (1) in L^p -mean for some $p \geq 2$. On the other hand Kanagawa (1988) considered the error of X_n as follows.

Theorem 1. Kanagawa(1988) Suppose that for any $0 \leq s, t \leq 1$ and $x, y \in \mathbb{R}^d$

$$(4) \quad |\sigma(t,x) - \sigma(s,y)|^2 + |b(t,x) - b(s,y)|^2 \leq K_1(|x-y|^2 + |t-s|^2),$$

$$(5) \quad |\sigma(t,x)|^2 + |b(s,y)|^2 \leq K_2,$$

where K_1 and K_2 are some positive constants independent of s, t, x and y . Then for any $p \geq 2$ and for some $\varepsilon > p/2$

$$(6) \quad E\left(\max_{0 \leq t \leq 1} |X(t) - X_n(t)|^p\right) = o\left(n^{-p/2}(\log n)^\varepsilon\right) \quad \text{as } n \rightarrow \infty,$$

$$(7) \quad E\left(\max_{0 \leq t \leq 1} |X(t) - Z_n(t)|^p\right) = O\left(n^{-p/2}\right) \quad \text{as } n \rightarrow \infty.$$

Furthermore, in the case when the approximate solutions are constructed from r -dimensional i.i.d. random variables $\{\xi_k\}$ which do not always obey the normal distribution, Kangawa (1989) showed the rate of convergence of $E\left(\max_{0 \leq t \leq 1} |X(t) - Y_n(t)|^p\right)$ to zero, where $Y_n := \{Y_n(t), 0 \leq t \leq 1\}$ is a stochastic process in $D(0,1)$ defined by for $k=0,1,\dots,n$

$$(8) \quad \begin{cases} Y_n(t) := y_k, & k/n \leq t < (k+1)/n, \quad k=0, \dots, n-1 \\ Y_n(1) := y_n, \end{cases}$$

where

$$y_k := X_0 + \sum_{j=1}^k \sigma\left(\frac{j-1}{n}, y_{j-1}\right) \xi_j / \sqrt{n} + \sum_{j=1}^k b\left(\frac{j-1}{n}, y_{j-1}\right) / n.$$

Theorem 2. Kanagawa(1989) Let $\{\xi_k, k \geq 1\}$ be r -dimensional i.i.d. random variables with

$$(9) \quad E(\xi_1) = 0, E(|\xi_1|^2) = 1, E(|\xi_1|^{2+\delta}) < \infty \text{ for some } 0 < \delta \leq 1.$$

Assume $\sigma(t,x)$ and $b(t,x)$ satisfy (4) and (5). Then we can redefine $\{X(t), 0 \leq t \leq 1\}$ and $\{Y_n(t), 0 \leq t \leq 1\}$ on a common probability space such that for any $p \geq 2$ and $\varepsilon > (2+\delta)^2/2(3+\delta)$,

$$(10) \quad E\left(\max_{0 \leq t \leq 1} |X(t) - Y_n(t)|^p\right) = o\left(n^{-p\delta/2(2+\delta)} (\log n)^\varepsilon\right) \text{ as } n \rightarrow \infty,$$

where the power of n cannot be improved by better one.

Furthermore, under the Cramér's condition instead of $E(|\xi_1|^{2+\delta}) < \infty$, we have the next result.

Theorem 3. Kanagawa(1995) Let $\{\xi_k, k \geq 1\}$ be r -dimensional i.i.d. random variables with $E(\xi_1) = 0, E(|\xi_1|^2) = 1$. Suppose that $E(e^{4|\xi_1|}) \leq \infty$ in a neighborhood of $t=0$. Assume $\sigma(t,x)$ and $b(t,x)$ satisfy (4) and (5). Then we can redefine $\{X(t), 0 \leq t \leq 1\}$ and $\{Y_n(t), 0 \leq t \leq 1\}$ on a common probability space such that for any $p \geq 2, \varepsilon > p/2$ and for sufficiently large n

$$(11) \quad E\left(\max_{0 \leq t \leq 1} |X(t) - Y_n(t)|^p\right) = o\left(n^{-p/4} (\log n)^\varepsilon\right) \text{ as } n \rightarrow \infty.$$

2. Sample Path Large Deviations

We can apply Schiler's Brownian motion sample path large deviations to Euler-Maruyama approximate solutions $X_n := \{X_n(t), 0 \leq t \leq 1\}$ for SDE's.

Theorem 4. Consider the following SDE,

$$\begin{cases} dX(t) = dB(t) + b(t, X(t))dt, 0 \leq t \leq 1 \\ X(0) = X_0. \end{cases}$$

Suppose that $b(t, x)$ satisfies the Lipschitz condition (4). Let $\{\xi_k, k \geq 1\}$ be r -dimensional i.i.d. random variables with $E(\xi_1) = 0, E(|\xi_1|^2) = 1$. Suppose that $E(e^{4|\xi_1|}) \leq \infty$ in a neighborhood of $t = 0$. Put

$$\Lambda(\lambda) = \log E(e^{\lambda \xi_1}), \quad \Lambda^*(x) = \sup_{\lambda} E(\lambda x - \Lambda(\lambda)),$$

$$I(\phi) := \begin{cases} \int_0^1 \Lambda^* \left(\dot{\phi}(t) - \frac{1}{\sqrt{n}} b(\sqrt{n}\phi(t)) \right) dt, & \text{if } \phi \in AC, \phi(0) = 0 \\ 0, & \text{otherwise} \end{cases}$$

Then we have for any closed $F \in \mathcal{C}[0, 1]$

$$(12) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log P\{\sqrt{n}X_n \in F\} = O\left(-\inf_{x \in F} I(x)\right) \quad \text{as } n \rightarrow \infty.$$

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