

Keisler's theorem and cardinal invariants at uncountable cardinals

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1 Introduction

The following is an important theorem in model theory proved by Keisler and Shelah. Keisler [Kei64] proved it by assuming GCH, but Shelah [She71] removed that assumption.

Theorem 1 (Keisler–Shelah). For every (first-order) language $\mathcal{L}$ and two $\mathcal{L}$ -structures $\mathcal{A}, \mathcal{B}$, the following are equivalent:

- (1) $\mathcal{A} \equiv \mathcal{B}$ (that is, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent).
- (2) There is a nonprincipal ultrafilter $\mathcal{U}$ over an infinite set such that the ultrapowers $\mathcal{A}^{\mathcal{U}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic.

The following theorem is also known in connection with the above theorem.

Theorem 2 (Keisler, Golshani and Shelah). The following are equivalent:

- (1) The continuum hypothesis.

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- (2) For every countable language $\mathcal{L}$ and two $\mathcal{L}$ -structures $\mathcal{A}, \mathcal{B}$ of size $\leq \mathfrak{c}$, if $\mathcal{A} \equiv \mathcal{B}$ then there is a nonprincipal ultrafilter $\mathcal{U}$ over ω such that the ultrapowers $\mathcal{A}^{\mathcal{U}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic.

For this theorem, Keisler [Kei64] showed (1) $\Rightarrow$ (2) and Golshani and Shelah [GS23] (2) $\Rightarrow$ (1).

In order to analyze these theorems in detail, we introduce the following principle.

Definition 3. Let κ, μ and λ be infinite cardinals. We define a criterion $\text{KT}_{\kappa}^{\mu}(\lambda)$ by

$$\begin{aligned} \text{KT}_{\kappa}^{\mu}(\lambda) \iff & \text{for every language } \mathcal{L} \text{ of size } \leq \mu \text{ and} \\ & \text{every elementarily equivalent } \mathcal{L}\text{-structures } \mathcal{A}, \mathcal{B} \text{ of size } \leq \lambda, \\ & \text{there is a uniform ultrafilter } \mathcal{U} \text{ on } \kappa \text{ such that } \mathcal{A}^{\mathcal{U}} \simeq \mathcal{B}^{\mathcal{U}}. \end{aligned}$$

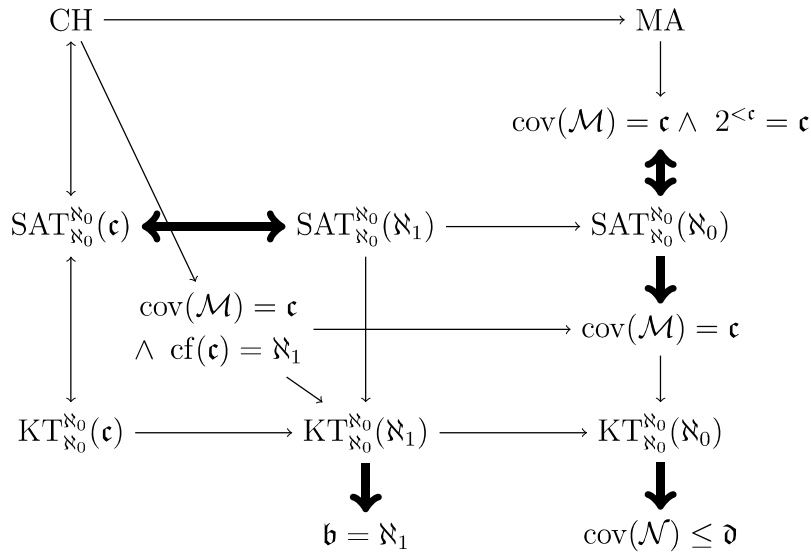
We also define a criterion $\text{SAT}_{\kappa}^{\mu}(\lambda)$ by

$$\begin{aligned} \text{SAT}_{\kappa}^{\mu}(\lambda) \iff & \text{there is a uniform ultrafilter } \mathcal{U} \text{ on } \kappa \text{ such that} \\ & \text{for every language } \mathcal{L} \text{ of size } \leq \mu \text{ and} \\ & \text{every sequence } \langle \mathcal{A}_i : i < \kappa \rangle \text{ of infinite } \mathcal{L}\text{-structures of size } \leq \lambda, \\ & \text{the ultraproduct } \left(\prod_{i \in \kappa} \mathcal{A}_i \right) / \mathcal{U} \text{ is saturated.} \end{aligned}$$

Keisler–Shelah’s theorem means that $\text{KT}_{2^{\lambda}}^{2^{\lambda}}(\lambda)$ holds for any infinite cardinal λ . Keisler’s paper also gives an example showing the following.

Fact 4 (Keisler [Kei64]). Let κ be an infinite cardinals. Then $\neg \text{KT}_{\kappa}^{\kappa^+}(\kappa^+)$ holds.

The author’s paper [Got22] showed the implications of the bold lines in the following figure.



2 Results

In this section, we discuss the principles introduced in Section 1. The case where the cardinality μ of the language and the cardinality κ of the underlying set of the ultrafilter are both $\aleph_0$ was analyzed in detail in the author's paper [Got22]. Here, the more general case is investigated. However, most of the results are naive generalisations of the arguments in [Got22].

Before proceeding to the results, we recall the basic definitions of ultrafilters.

Definition 5. Let $\mathcal{U}$ be a ultrafilter on κ . We say $\mathcal{U}$ is *regular* if there is $\mathcal{E} \subseteq \mathcal{U}$ of size κ such that for every $i < \kappa$, the set $\{E \in \mathcal{E} : i \in E\}$ is finite.

Definition 6. For ultrafilters $\mathcal{U}, \mathcal{V}$ on I, J respectively, we define

$$\mathcal{U} * \mathcal{V} = \{A \subseteq I \times J : \{i \in I : \{j \in J : (i, j) \in A\} \in \mathcal{V}\} \in \mathcal{U}\}.$$

$\mathcal{U} * \mathcal{V}$ is called the *Fubini product* of $\mathcal{U}$ and $\mathcal{V}$.

Lemma 7. Let $\kappa \leq \kappa'$ be two infinite cardinals. Then $\text{KT}_\kappa^\mu(\lambda)$ implies $\text{KT}_{\kappa'}^\mu(\lambda)$.

Proof. Fix a language $\mathcal{L}$ of size $\leq \mu$ and two elementarily equivalent $\mathcal{L}$ -structures $\mathcal{A}$ and $\mathcal{B}$ of size $\leq \lambda$. By $\text{KT}_\kappa^\mu(\lambda)$, we can take a uniform ultrafilter $\mathcal{U}$ on κ . Fix a uniform ultrafilter $\mathcal{V}$ on κ' . Then the ultrapowers of $\mathcal{A}$ and $\mathcal{B}$ by the ultrafilter $\mathcal{U} * \mathcal{V}$ are isomorphic. $\square$

Lemma 8. (1) $\text{KT}_\kappa^\mu(\lambda)$ implies there exists a regular ultrafilter witnessing $\text{KT}_\kappa^\mu(\lambda)$.

(2) If $\lambda \geq \kappa$, then every witness for $\text{SAT}_\kappa^\mu(\lambda)$ is a regular ultrafilter.

Proof. First, we show (1). Take an ultrafilter $\mathcal{U}$ on κ witnessing $\text{KT}_\kappa^\mu(\lambda)$. Take a regular ultrafilter $\mathcal{V}$ on κ . Then the product ultrafilter $\mathcal{U} * \mathcal{V}$ is regular and witnesses $\text{KT}_\kappa^\mu(\lambda)$.

Next we show (2). Take a witness $\mathcal{U}$ for $\text{SAT}_\kappa^\mu(\lambda)$. Let $M = ([\kappa]^{<\aleph_0}, \subseteq)$ and consider $M_* = M^\kappa / \mathcal{U}$. By an easy diagonal argument, we have $|M_*| \geq \kappa^+$. Define a set of formulas p with a free variable x by

$$p = \{\ulcorner \{\alpha\}_* \subseteq x^\top : \alpha < \kappa \urcorner\},$$

where $\{\alpha\}_*$ is the equivalence class of the constant sequence of $\{\alpha\}$. It can be easily checked that p is finitely satisfiable and the number of parameters of p is κ , which is smaller than $|M_*|$. Therefore, by $\text{SAT}_\kappa^\mu(\lambda)$, we can take $f: \kappa \rightarrow M$ such that $[f]$ satisfies p . This f clearly satisfies $\{i \in \kappa : \alpha \in f(i)\} \in \mathcal{U}$ for every $\alpha < \kappa$. Thus, $\mathcal{U}$ is a regular ultrafilter. $\square$

Lemma 9. $\text{SAT}_\kappa^\mu(\lambda)$ implies $\text{KT}_\kappa^\mu(\lambda)$ for every $\lambda \leq 2^\kappa$.

Proof. By regularity (Lemma 8), the ultrapowers have same cardinality. Thus uniqueness of saturated models implies this lemma. $\square$

Lemma 10. $\neg \text{SAT}_{\kappa}^{\aleph_0}(\kappa^{++})$.

Proof. Take a witness $\mathcal{U}$ of $\text{SAT}_{\kappa}^{\aleph_0}(\kappa^{++})$. Let $\mathcal{A} = (\kappa^{++}, <)$ and $\mathcal{A}_* = \mathcal{A}^{\mathcal{U}}$. We have $|\mathcal{A}_*| \geq |\mathcal{A}| = \kappa^{++}$. Consider the following set p of formulas with one free variable x :

$$p = \{\ulcorner \alpha_* < x < (\kappa^+)_* \urcorner : \alpha < \kappa^+\}.$$

This p is finitely satisfiable and the number of parameters occurring in p is κ^+ . Thus, by $\text{SAT}_{\kappa}^{\aleph_0}(\kappa^{++})$, we can take $f: \kappa \rightarrow \kappa^+$ such that $[f]$ realizes p . Put $\beta = \sup_{\alpha < \kappa} f(\alpha)$. By $\beta_* < [f]$, we have $\{\alpha < \kappa : \beta < f(\alpha)\} \in \mathcal{U}$. This contradicts the choice of β . $\square$

Lemma 11. $\text{SAT}_{\kappa}^{\aleph_0}(\kappa^+)$ implies $2^{\kappa} = \kappa^+$.

Proof. Take a witness $\mathcal{U}$ of $\text{SAT}_{\kappa}^{\aleph_0}(\kappa^+)$ and assume $\kappa^+ < 2^{\kappa}$. Let $\mathcal{A}_* = (\kappa^+, <)^{\mathcal{U}}$. We have $|\mathcal{A}_*| = 2^{\kappa}$ since $\mathcal{U}$ is regular (Lemma 8). Consider the following set p of formulas with one free variable x :

$$p = \{\ulcorner \alpha_* < x \urcorner : \alpha < \kappa^+\}.$$

This p is finitely satisfiable and the number of parameters occurring in p is equal to κ^+ , which is smaller than 2^{κ} . Thus, by $\text{SAT}_{\kappa}^{\aleph_0}(\kappa^+)$, we can take $f: \kappa \rightarrow \kappa^+$ such that $[f]$ realizes p . Then, this f is unbounded, which contradicts that κ^+ is regular. $\square$

Lemma 12. $\neg \text{KT}_{\kappa}^{\aleph_0}(\kappa^{++})$.

Proof. This proof is based on [Tsu22]. Let $(\mathcal{M}, <)$ be a linearly ordered set with cofinality κ^{++} . We define an increasing continuous sequence $\langle A_i : i \leq \kappa^{++} \rangle$ of subsets of $\mathcal{M}$ such that:

- (1) For every $i \leq \kappa^{++}$, A_i is an elementary substructure of $\mathcal{M}$.
- (2) For every $i < \kappa^{++}$, there is $a_i \in A_{i+1}$ such that for every $b \in A_i$, we have $b < a_i$.
- (3) For every $i \leq \kappa^{++}$, we have $|A_i| \leq |i| + \aleph_0$.

We show that the pair of A_{κ^+} and $A_{\kappa^{++}}$ is a counterexample of $\text{KT}_{\kappa}(\kappa^{++})$. Let $\mathcal{U}$ be an ultrafilter on κ .

We claim that $(A_{\kappa^+})^{\mathcal{U}}$ has a cofinal increasing sequence of length κ^+ . In fact, $\langle (a_i)_* : i < \kappa^+ \rangle$ is a cofinal increasing sequence. In order to show it, take $[f] \in (A_{\kappa^+})^{\mathcal{U}}$. For each $\alpha < \kappa$, we can take $i_{\alpha} < \kappa^+$ such that $f(\alpha) \in A_{i_{\alpha}}$. Then $i = \sup_{\alpha < \kappa} i_{\alpha}$ satisfies $[f] < a_i$.

On the other hand, in $(A_{\kappa^{++}})^{\mathcal{U}}$, every κ^+ -sequence is bounded. In order to check it, take $\langle b_i : i < \kappa^+ \rangle$. We write b_i as $b_i = [f_i]$, where $f_i: \kappa \rightarrow A_{\kappa^{++}}$. Since the set $\{f_i(\alpha) : i < \kappa^+, \alpha < \kappa\}$ has size less than or equal to κ^+ , we can take $\beta < \kappa^{++}$ such that all the elements of this set belong to A_{β} . Then a_{β} is a bound of all b_i .

So we have $(A_{\kappa^+})^{\mathcal{U}} \not\preceq (A_{\kappa^{++}})^{\mathcal{U}}$. $\square$

Theorem 13. Let κ and μ be infinite cardinals satisfying $\mu \leq \kappa$. Then the following are equivalent.

- (1) $2^\kappa = \kappa^+$.
- (2) $\text{SAT}_\kappa^\mu(2^\kappa)$.
- (3) $\text{SAT}_\kappa^\mu(\kappa^+)$.
- (4) $\text{KT}_\kappa^\mu(2^\kappa)$.

Proof. Recall that there is a κ^+ -good ultrafilter U on κ . That is, for every language $\mathcal{L}$ of size $\leq \kappa$, all U -ultraproducts of $\mathcal{L}$ -structures are κ^+ -saturated. The implication $2^\kappa = \kappa^+ \implies \text{SAT}_\kappa^\mu(2^\kappa)$ follows from this fact.

The implication $\text{SAT}_\kappa^\mu(\kappa^+) \implies 2^\kappa = \kappa^+$ is just Lemma 11.

The implication $\text{KT}_\kappa^\mu(2^\kappa) \implies 2^\kappa = \kappa^+$ follows from Lemma 12. $\square$

Theorem 14. Let κ be a regular cardinal. Then $\text{KT}_\kappa^{\aleph_0}(\kappa^+)$ implies $\mathfrak{b}_\kappa = \kappa^+$.

Proof. Take the same structure $\mathcal{M}$ as in Lemma 12. Consider two elementary substructures A_κ and A_{κ^+} .

Take a regular ultrafilter $\mathcal{U}$ on κ that witnesses $\text{KT}_\kappa^{\aleph_0}(\kappa^+)$. As we saw in Lemma 12, we have $\text{cf}((A_{\kappa^+})^\mathcal{U}) = \kappa^+$.

On the other hand, we have $\text{cf}(A_\kappa) = \kappa$. So it holds that $\text{cf}((A_\kappa)^\mathcal{U}) = \text{cf}(\kappa^\kappa/\mathcal{U})$.

Since the ultrafilter $\mathcal{U}$ is uniform, we have $\mathfrak{b}_\kappa \leq \text{cf}(\kappa^\kappa/\mathcal{U})$.

By $\text{KT}_\kappa^{\aleph_0}(\kappa^+)$, the two models $(A_\kappa)^\mathcal{U}$ and $(A_{\kappa^+})^\mathcal{U}$ are isomorphic. So we have $\mathfrak{b}_\kappa \leq \text{cf}(\kappa^\kappa/\mathcal{U}) = \kappa^+$. The other inequality is obvious. $\square$

Theorem 15. $\text{SAT}_\kappa^{\aleph_0}(\kappa)$ implies $2^{<2^\kappa} = 2^\kappa$.

Proof. Fix a witness $\mathcal{U}$ for $\text{SAT}_\kappa^{\aleph_0}(\kappa)$. Let $\lambda < 2^\kappa$. Define a language $\mathcal{L}$ and $\mathcal{L}$ -structure $\mathcal{A}$ by $\mathcal{L} = \{\subseteq\}$ and $\mathcal{A} = ([\kappa]^{<\omega}, \subseteq)$. We have $|\mathcal{A}| = \kappa$. Put $\mathcal{A}_* = \mathcal{A}^\kappa/\mathcal{U}$. Since $\mathcal{U}$ is regular (Lemma 8), we have $|\mathcal{A}_*| = \kappa^\kappa = 2^\kappa$. Let $\iota: \kappa^\kappa/\mathcal{U} \rightarrow \mathcal{A}_*$ be the function defined by:

$$\iota([x]) = [\langle \{x(\alpha)\} : \alpha < \kappa \rangle].$$

Fix $F \subseteq \kappa^\kappa/\mathcal{U}$ of size λ . For $X \subseteq F$, we define a set $p_X(z)$ of formulas with a free variable z by:

$$p_X(z) = \{\ulcorner \iota(y) \subseteq z \urcorner : y \in X\} \cup \{\ulcorner \iota(y) \not\subseteq z \urcorner : y \in F \setminus X\}.$$

Each $p_X(z)$ is finitely satisfiable and the number of parameters occurring in $p_X(z)$ is λ . Therefore, by $\text{SAT}_\kappa^{\aleph_0}(\kappa)$, for each $X \subseteq F$, we can take $[z_X] \in \mathcal{A}_*$ satisfying $p_X(z)$. For distinct $X, Y \subseteq F$, we have $[z_X] \neq [z_Y]$. Thus we have $2^\lambda = |\{[z_X] : X \subseteq F\}| \leq |\mathcal{A}_*| = 2^\kappa$. Since $\lambda < 2^\kappa$ was arbitrary chosen, we have $2^{<2^\kappa} = 2^\kappa$. $\square$

Theorem 16. Let κ be a regular cardinal. Let μ be a cardinal less than 2^κ . Then $\text{cov}(\mathcal{M}_\kappa) = 2^\kappa$ implies $\text{KT}_\kappa^\mu(\kappa)$.

Proof. Note that the assumption $\text{cov}(\mathcal{M}_\kappa) = 2^\kappa$ is equivalent to $\text{MA}_{<2^\kappa}(\text{Fn}_\kappa(\kappa, 2))$.

Fix an enumeration of 2^κ .

Let $\mathcal{L}$ be a language of size $\leq \mu$ and $\mathcal{A}^0$ and $\mathcal{A}^1$ are $\mathcal{L}$ -structures of size $\leq \kappa$ which are elementarily equivalent.

Enumerate $(\mathcal{A}^i)^\kappa$ for $i = 0, 1$ as

$$(\mathcal{A}^i)^\kappa = \{f_\alpha^i : \alpha < 2^\kappa\}.$$

By a back-and-forth method, we construct a sequence of triples $\langle (\mathcal{U}_\alpha, g_\alpha^0, g_\alpha^1) : \alpha < 2^\kappa \rangle$ satisfying:

- (1) $g_\alpha^0 \in (\mathcal{A}^0)^\kappa$,
- (2) $g_\alpha^1 \in (\mathcal{A}^1)^\kappa$,
- (3) $\mathcal{U}_\alpha$ is a filter on κ generated by $\kappa + |\alpha|$ sets,
- (4) $\langle \mathcal{U}_\alpha : \alpha < 2^\kappa \rangle$ is an increasing continuous sequence,
- (5) If $\varphi(x_0, \dots, x_{n-1})$ is an $\mathcal{L}$ -formula and $\beta_0, \dots, \beta_n \leq \alpha$, then the set

$$\{\xi \in \kappa : \mathcal{A}^0 \models \varphi(g_{\beta_0}^0(\xi), \dots, g_{\beta_{n-1}}^0(\xi)) \iff \mathcal{A}^1 \models \varphi(g_{\beta_0}^1(\xi), \dots, g_{\beta_{n-1}}^1(\xi))\}$$

belongs to $\mathcal{U}_{\alpha+1}$.

In the construction, when α is even, we put $g_\alpha^0 = f_\gamma^0$ where γ is the least ordinal $f_\gamma^0 \notin \{g_\beta^0 : \beta < \alpha\}$. And $\mathbb{P}$ is the poset of partial functions of size $< \kappa$ from κ to $\mathcal{A}^1$. This poset is forcing equivalent to $\text{Fn}_\kappa(\kappa, 2)$.

Take a generating set $\mathcal{F}$ of $\mathcal{U}_\alpha$ of size $\aleph_0 + |\alpha|$. Then by using $\text{MA}_{<2^\kappa}(\text{Fn}_\kappa(\kappa, 2))$, take a $\mathbb{P}$ -generic filter G with respect to the following family of dense sets of $\mathbb{P}$:

$$D_\xi = \{p \in \mathbb{P} : \xi \in \text{dom } p\} \text{ (for } \xi \in \kappa)$$

and

$$\begin{aligned} E_{X, \langle \varphi_\iota : \iota \in I \rangle, \langle \gamma_1^\iota, \dots, \gamma_{n_\iota}^\iota : \iota \in I \rangle} = & \{p \in \mathbb{P} : (\exists \xi \in \text{dom}(p) \cap X)(\forall \iota \in I) \\ & (\mathcal{A}^0 \models \varphi_\iota(g_{\gamma_1^\iota}^0(\xi), \dots, g_{\gamma_{n_\iota}^\iota}^0(\xi), g_\alpha^0(\xi)) \Leftrightarrow \\ & \mathcal{A}^1 \models \varphi_\iota(g_{\gamma_1^\iota}^1(\xi), \dots, g_{\gamma_{n_\iota}^\iota}^1(\xi), p(\xi))\} \end{aligned}$$

where $X \in \mathcal{F}$, $\langle \varphi_\iota : \iota \in I \rangle$ is a finite sequence of $\mathcal{L}$ -formulas and $\gamma_1^\iota, \dots, \gamma_{n_\iota}^\iota$ for $\iota \in I$ are ordinals less than α .

We now prove that $E := E_{X, \langle \varphi_\iota : \iota \in I \rangle, \langle \gamma_1^\iota, \dots, \gamma_{n_\iota}^\iota : \iota \in I \rangle}$ is dense. Let $p \in \mathbb{P}$. For each $\xi \in \kappa$ and $\iota \in I$, put

$$v(\xi, \iota) = \begin{cases} 1 & \text{if } \mathcal{A}^0 \models \varphi_\iota(g_{\gamma_1^\iota}^0(\xi), \dots, g_{\gamma_{n_\iota}^\iota}^0(\xi), g_\alpha^0(\xi)) \\ 0 & \text{otherwise.} \end{cases}$$

And for each $\xi \in \kappa$ put

$$v(\xi) = \langle v(k, \iota) : \iota \in I \rangle.$$

Then by finiteness of ${}^I 2$, for some $v_0 \in {}^I 2$, we have $\kappa \setminus v^{-1}(v_0) \notin \mathcal{U}_\alpha$.

For each $\iota \in I$, put

$$\varphi_\iota^+(x_1^\iota, \dots, x_{n_\iota}^\iota, y) \equiv \begin{cases} \varphi_\iota(x_1^\iota, \dots, x_{n_\iota}^\iota, y) & \text{if } v_0(\iota) = 1 \\ \neg \varphi_\iota(x_1^\iota, \dots, x_{n_\iota}^\iota, y) & \text{otherwise.} \end{cases}$$

Put

$$\psi \equiv \exists y \bigwedge_{\iota \in I} \varphi_\iota^+(x_1^\iota, \dots, x_{n_\iota}^\iota, y).$$

Then by the induction hypothesis (5), $Y_{\psi, \langle \gamma_1^\iota, \dots, \gamma_{n_\iota}^\iota : \iota \in I \rangle} \in \mathcal{U}_\alpha$. So take $\xi \in X \cap v^{-1}(v_0) \cap Y_{\psi, \langle \gamma_1^\iota, \dots, \gamma_{n_\iota}^\iota : \iota \in I \rangle} \setminus \text{dom}(p)$.

Since $M^0 \models \psi(\langle g_{\gamma_1^\iota}^0(\xi), \dots, g_{\gamma_{n_\iota}^\iota}^0(\xi) : \iota \in I \rangle)$, we have $M^1 \models \psi(\langle g_{\gamma_1^\iota}^1(\xi), \dots, g_{\gamma_{n_\iota}^\iota}^1(\xi) : \iota \in I \rangle)$.

By the definition of ψ , we can take $y \in M^1$ such that $M^1 \models \varphi_\iota^+(g_{\gamma_1^\iota}^1(\xi), \dots, g_{\gamma_{n_\iota}^\iota}^1(\xi), y)$ for every $\iota \in I$. We now put $q = p \cup \{(\xi, y)\}$. This witnesses denseness of E .

Then we put $g_\alpha^1 = \bigcup G$ and letting $\mathcal{U}_{\alpha+1}$ contain $\mathcal{U}_\alpha$ and the sets in (5) and have either the α -th element of the enumeration of 2^κ or its complement.

When α is odd, do the same construction above except for swapping 0 and 1.

Then the construction guarantees that $\mathcal{U} = \bigcup_{\alpha < 2^\kappa} \mathcal{U}_\alpha$ is an ultrafilter and that the function

$$\langle ([g_\alpha^0]u, [g_\alpha^1]u) : \alpha < 2^\kappa \rangle$$

is an isomorphism from $(\mathcal{A}^0)^\mathcal{U}$ to $(\mathcal{A}^1)^\mathcal{U}$. $\square$

Fact 17 ([Vlu23, Theorem 4.3]). Let κ be an inaccessible cardinal. Then $\text{cov}(\mathcal{M}_\kappa) \geq \lambda$ holds iff for every $X \subseteq \kappa^\kappa$ of size $< \lambda$ there is $S \in \prod_{i < \kappa} [\kappa]^{\leq |i|+1}$ such that for all $x \in X$ we have $\{i < \kappa : x(i) \in S(i)\}$ is cofinal in κ .

Fact 17 does not seem to generalize to anything other than inaccessible cardinals. In fact, it is known that when κ is a successor cardinal, the cardinal invariant determined by slaloms as claimed above is equal to $\mathfrak{d}_\kappa$.

Theorem 18. Let κ be an inaccessible cardinal. Then $\text{SAT}_\kappa^{\aleph_0}(\kappa)$ implies $\text{cov}(\mathcal{M}_\kappa) = 2^\kappa$.

Proof. Let $\mathcal{U}$ be a regular ultrafilter on κ witnessing $\text{SAT}_\kappa^{\aleph_0}(\kappa)$. Let $X \subseteq \kappa^\kappa$ of size $< 2^\kappa$. Define a language $\mathcal{L}$ by $\mathcal{L} = \{\subseteq\}$. For $i < \kappa$, define a $\mathcal{L}$ -structure $\mathcal{A}_i$ by $\mathcal{A}_i = ([\kappa]^{<|i|}, \subseteq)$. Since κ is inaccessible, we have $|\mathcal{A}_i| = \kappa$. For $x \in \kappa^\kappa$, we define $S_x = \langle \{x(i)\} : i < \kappa \rangle$. Put $\mathcal{A}_* = \prod_{i < \kappa} \mathcal{A}_i / \mathcal{U}$. Consider a set of formulas $p(S)$ defined by

$$p(S) = \{\ulcorner [S_x] \subseteq S^\urcorner : x \in X\}.$$

Then $p(S)$ is finitely satisfiable and the number of parameters occurring in $p(S)$ is $< 2^\kappa$. Thus, by $\text{SAT}_\kappa^{\aleph_0}(\kappa)$, we can take $[S] \in \mathcal{A}_*$ realizing $p(S)$. Then we have

$$(\forall x \in X)(\{i < \kappa : x(i) \in S(i)\} \in \mathcal{U}).$$

But since our ultrafilter $\mathcal{U}$ is uniform, we have

$$(\forall x \in X)(\{i < \kappa : x(i) \in S(i)\} \text{ is cofinal}).$$

So by Fact 17, we showed $\text{cov}(\mathcal{M}_\kappa) = 2^\kappa$. $\square$

Theorem 19. Let κ be a regular cardinal. Then $\text{cov}(\mathcal{M}_\kappa) = 2^{<2^\kappa} = 2^\kappa$ implies $\text{SAT}_\kappa^\kappa(\kappa)$.

Proof. Let $\langle b_\alpha : \alpha < 2^\kappa \rangle$ be an enumeration of κ^κ .

Let $\mathcal{L}^+ = \mathcal{L} \cup \{c_\alpha : \alpha < \mathfrak{c}\}$ where the c_α 's are new constant symbols and let $\text{Fml}(\mathcal{L}^+)$ be the set of all $\mathcal{L}^+$ formulas with one free variable.

Let $\langle (\mathcal{L}_\xi, T_\xi, \mathcal{B}_\xi, \Delta_\xi) : \xi < 2^\kappa \rangle$ be an enumeration of tuples $(\mathcal{L}, T, \mathcal{B}, \Delta)$ such that $\mathcal{L}$ is a language of size $\leq \kappa$, $T : \kappa \rightarrow \kappa + 1$, $\mathcal{B} = \langle \mathcal{A}_i : i < \kappa \rangle$ is a κ -sequence of $\mathcal{L}$ -structures with i -th universe $T(i)$ and Δ is a subset of $\text{Fml}(\mathcal{L}^+)$ with $|\Delta| < 2^\kappa$. Here we used $(2^\kappa)^{<2^\kappa} = 2^\kappa$. Ensure each $(\mathcal{L}, T, \mathcal{B}, \Delta)$ occurs cofinally in this sequence.

For $\mathcal{B}_\xi = \langle \mathcal{A}_i^\xi : i < \kappa \rangle$, we put

$$\mathcal{B}_\xi(i) = \langle \mathcal{A}_i^\xi, b_0(i) \upharpoonright T_\xi(i), b_1(i) \upharpoonright T_\xi(i), \dots \rangle,$$

which is a $\mathcal{L}^+$ -structure. Here $\alpha \upharpoonright \beta = \begin{cases} \alpha & \text{if } \alpha < \beta \\ 0 & \text{otherwise} \end{cases}$ for α and β are ordinals.

Let $\langle X_\xi : \xi < 2^\kappa \rangle$ be an enumeration of $\mathcal{P}(\kappa)$. We construct a sequence of filters $\langle F_\xi : \xi < 2^\kappa \rangle$ satisfying following conditions:

- (1) F_0 is the filter generated by a regularizing set for κ .
- (2) $F_\xi \subseteq F_{\xi+1}$ and $F_\xi = \bigcup_{\alpha < \xi} F_\alpha$ for a limit ξ .
- (3) $X_\xi \in F_{\xi+1}$ or $\kappa \setminus X_\xi \in F_{\xi+1}$.
- (4) F_ξ is generated by $< 2^\kappa$ members.
- (5) If

$$(\forall \Gamma \in [\Delta_\xi]^{<\aleph_0})(\{i < \kappa : \Gamma \text{ is satisfiable in } \mathcal{B}_\xi(i)\} \in F_\xi) \quad (*)$$

Then there is $f \in \prod_{i < \kappa} T_\xi(i)$ such that for every $\varphi \in \Delta_\xi$ we have $\{i < \kappa : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}_\xi(i)\} \in F_{\xi+1}$.

Suppose that F_ξ is constructed and $(*)$ holds. Let

$$\mathbb{P} = \{p : p \text{ is a partial function of size } < \kappa \text{ from } \kappa \text{ to } \kappa\}$$

This forcing notion $\mathbb{P}$ is forcing equivalent to the forcing adding a κ -Cohen function.

Fix a generating set F'_ξ of F_ξ of size $< 2^\kappa$. For each $A \in F'_\xi$ and $\varphi_1, \dots, \varphi_n \in \Delta_\xi$, we put

$$E_{A, \varphi_1, \dots, \varphi_n} = \{p \in \mathbb{P} : (\exists i \in \text{dom}(p) \cap A)(p(i) \text{ is element of } T_\xi(i) \text{ and satisfies } \varphi_1, \dots, \varphi_n \text{ in } \mathcal{B}_\xi(i))\}$$

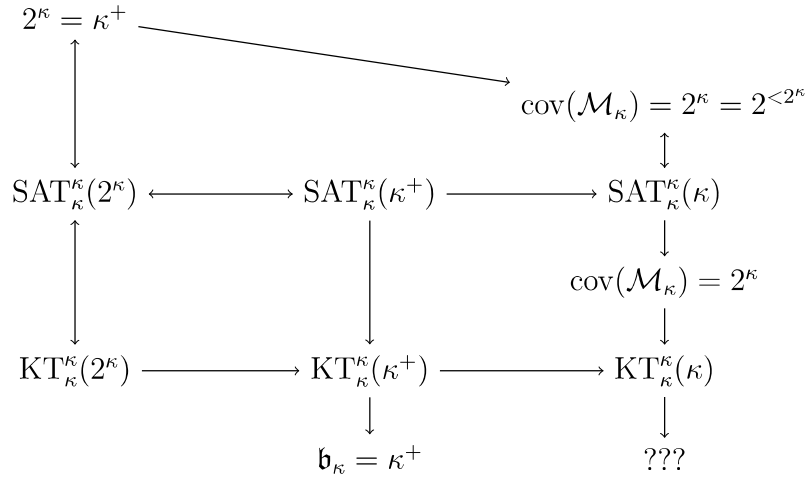
By assumption $(*)$, these $E_{A, \varphi_1, \dots, \varphi_n}$'s are dense subsets in $\mathbb{P}$.

So using $\text{MA}_{<2^\kappa}(\mathbb{P})$, we have a filter G of $\mathbb{P}$ that intersects all $E_{A, \varphi_1, \dots, \varphi_n}$'s. Put $f(i) = (\bigcup G)(i) \upharpoonright T_\xi(i)$. Then we can extend our filter F_ξ to $F_{\xi+1}$ such that for every $\phi \in \Delta_\xi$ $\{i < \kappa : f(i) \text{ satisfies } \phi \text{ in } \mathcal{B}_\xi(i)\} \in F_{\xi+1}$. Moreover we can extend this filter satisfying $X_\xi \in F_{\xi+1}$ or $\kappa \setminus X_\xi \in F_{\xi+1}$. This finishes the construction.

In order to check that the resulting ultrafilter $F = \bigcup_{\xi < 2^\kappa} F_\xi$ witnesses $\text{SAT}_\kappa^\kappa(\kappa)$, let $\mathcal{L}$ be a language of size $\leq \kappa$ and $\mathcal{B} = \langle \mathcal{A}_i : i \in \kappa \rangle$ be a sequence of $\mathcal{L}$ -structures. We may assume that, for each $i < \kappa$, the universe of $\mathcal{A}_i$ is an ordinal. Let $T(i) =$ the universe of $\mathcal{A}_i$. Let Δ be a subset of $\text{Fml}(\mathcal{L}^+)$ with $|\Delta| < 2^\kappa$. Assume that for all $\Gamma \subseteq \Delta$ finite, $X_\Gamma := \{i \in \kappa : \Gamma \text{ is satisfiable in } \mathcal{B}(i)\} \in F$. By the regularity of 2^κ which follows from the cardinal arithmetical assumption of the theorem, we have $\alpha < 2^\kappa$ such that for all $\Gamma \subseteq \Delta$ finite, $X_\Gamma \in F_\alpha$. Let $\xi \geq \alpha$ be satisfying $(\mathcal{L}_\xi, T_\xi, \mathcal{B}_\xi, \Delta_\xi) = (\mathcal{L}, T, \mathcal{B}, \Delta)$. Then by (5), there is a $f \in \prod_{i < \kappa} T(i)$ such that for all $\varphi \in \Delta$, $\{i \in \kappa : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}(i)\} \in F$. Thus $\prod_{i \in \kappa} \mathcal{A}_i / F$ is saturated. $\square$

3 Discussion

From the results of Section 2, the following figure can be drawn for an inaccessible cardinal κ .



In light of this, the following two questions naturally arise.

Question 20. (1) Can we eliminate the inaccessibility assumption from the result which states $\text{SAT}_\kappa^{\aleph_0}(\kappa)$ implies $\text{cov}(\mathcal{M}_\kappa) = 2^\kappa$?

(2) Can we prove the consistency of $\neg \text{KT}_\kappa^\kappa(\kappa)$?

For the second question the answer Yes is obtained when $\kappa = \aleph_0$ ([She92]). [Got22] improves on that result, showing $\text{KT}_{\aleph_0}^{\aleph_0}(\aleph_0) \Rightarrow \text{cov}(\mathcal{N}) \leq \mathfrak{d}$. By generalizing the proof to an inaccessible cardinal, we obtain the following.

Theorem 21. Let κ be an inaccessible cardinal. Then $\text{KT}_\kappa^{\aleph_0}(\kappa)$ implies $\mathfrak{v}_\kappa^\forall \leq \mathfrak{d}_\kappa$. $\square$

Here, for a cardinal κ and $c, h \in \kappa^\kappa$, letting $\prod c = \prod_{\alpha < \kappa} c(\alpha)$ and $S(c, h) = \prod_{\alpha < \kappa} [c(\alpha)]^{<h(\alpha)}$, we define

$$\mathfrak{v}_{\kappa, c, h}^\forall = \min\{|X| : X \subseteq \prod c, (\forall \varphi \in S(c, h))(\exists x \in X) \\ (\forall \alpha < \kappa)(\exists \beta \in [\alpha, \kappa))(x(\alpha) \notin \varphi(\alpha))\}.$$

Also, we define $\mathfrak{v}_\kappa^\forall = \min\{\mathfrak{v}_{\kappa, c, h}^\forall : c, h \in \kappa^\kappa, \text{ and } h \text{ diverges to } \infty\}$.

However, for an inaccessible cardinal κ , the consistency of $\mathfrak{d}_\kappa < \mathfrak{v}_\kappa^\forall$ is not currently known. The situation differs from cardinal invariants at ω in that forcing notions such as random forcing are not known for higher cardinals, nor are good generalizations of properties such as ω^ω -bounding proper forcing.

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