

Nonsmooth Fractional Programming with Generalized Ratio Invexity

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Abstract: In this paper, we consider nonsmooth fractional programming problems with generalized ratio invexity. We present necessary and sufficient optimality theorems and establish duality theorems for nonsmooth fractional programming under suitable ρ -invexity assumptions.

1 Intorduction

We consider the following nonsmooth fractional programming problem :

$$\begin{aligned} \text{(NFP) Minimize} \quad & \frac{f(x)}{g(x)} \\ \text{subject to} \quad & x \in X = \{x \in \mathbb{R}^n | h_j(x) \leq 0, j = 1, \dots, m\}, \end{aligned}$$

where $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$, and $h_j : \mathbb{R} \rightarrow \mathbb{R}$, $j = 1, \dots, m$, are locally Lipschitz functions. We assume in the sequel that $f(x) \geq 0$ and $g(x) > 0$ on $\mathbb{R}^n$.

Jeyakumar [3] defined ρ -invexity for nonsmooth optimization problems, and Kuk *et al.* [6] defined the concept of V - ρ -invexity for vector valued functions, which is a generalization of the V -invex function [4]. Khan and Hanson [5] and Reddy and Mukherjee [8] applied the (generalized) ratio invexity concept for single objective fractional programming problems.

* This research of author was supported by the grant No. R01-2003-000-10825-0 from the Basic Research Program of KOSEF

Recently, Liang *et al.* [7] introduced the concept of (F, α, ρ, d) -convexity and presented optimality and duality results for a class of nonlinear fractional programming problems under generalized convexity and the properties of sublinear functional. In this paper, we present a result about the fractional objective function based on ρ -invexity assumptions. By using ρ -invexity of fractional function, we obtain necessary and sufficient optimality conditions and duality theorems for nonsmooth fractional programming problems.

2 Definitions and Generalized Invexity of Fractional Function

The real-valued function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be locally Lipschitz if for any $z \in \mathbb{R}^n$ there exists a positive constant K and a neighborhood N of z such that, for each $x, y \in N$,

$$|f(x) - f(y)| \leq K\|x - y\|,$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$. The Clarke generalized directional derivative of a locally Lipschitz function f at x in the direction $d \in \mathbb{R}^n$, denote by $f^0(x; d)$, is defined as follows:

$$f^0(x; d) = \limsup_{y \rightarrow x, t \downarrow 0} t^{-1}(f(y + td) - f(y)),$$

where y is a vector in $\mathbb{R}^n$.

The Clarke generalized subgradient of f at x is denoted by

$$\partial f(x) = \{\xi : f^0(x; d) \geq \xi d, \quad \forall d \in \mathbb{R}^n\}.$$

Definition 2.1 f is said to be regular at x if for all $d \in \mathbb{R}^n$ the one-sided directional derivative $f'(x; d)$ exists and $f'(x; d) = f^0(x; d)$.

Definition 2.2 A locally Lipschitz function $f : X_0 \rightarrow \mathbb{R}$ is said to be ρ -invex at $x_0 \in X_0$ with respect to functions η and $\theta : X_0 \times X_0 \rightarrow \mathbb{R}^n$ if there exists $\rho \in \mathbb{R}$ such that for any $x \in X_0$, and any $\xi \in \partial f(x_0)$,

$$f(x) - f(x_0) \geq \xi \eta(x, x_0) + \rho \|\theta(x, x_0)\|^2,$$

where $\theta(x, x_0) \neq 0$ if $x \neq x_0$.

When $\rho = 0$, the definition of ρ -invexity reduces to the notion of invexity in the sense of Hanson [2].

Remark. When f is of class C^1 in Definition 2.2, then the above inequality reduces to

$$f(x) - f(x_0) \geq f'_{x_0} \eta + \rho \|\theta(x, x_0)\|^2$$

where f'_{x_0} is the Frechet derivative of f at x_0 .

Theorem 2.1 If f and $-g$ are ρ -invex with respect to η and θ , and f and $-g$ are regular at x_0 , then the fractional objective function $f(x)/g(x)$ is ρ -invex with respect to $\bar{\eta}$ and $\bar{\theta}$, where

$$\bar{\eta}(x, x_0) = (g(x_0)/g(x))\eta(x, x_0), \quad \bar{\theta}(x, x_0) = (1/g(x))^{1/2}\theta(x, x_0).$$

Proof. Let $x, x_0 \in X_0$. By the ρ -invexity of f and $-g$, we have

$$\begin{aligned} & f(x)/g(x) - f(x_0)/g(x_0) \\ &= (f(x) - f(x_0))/g(x) - f(x_0)(g(x) - g(x_0))/g(x)g(x_0) \\ &\geq (1/g(x))\xi\eta(x, x_0) + \rho\|(1/g(x))^{1/2}\theta(x, x_0)\|^2 \\ &\quad + (f(x_0)/(g(x)g(x_0)))(-\zeta\eta(x, x_0) + \rho\|\theta(x, x_0)\|^2), \end{aligned}$$

for any $x \in X_0$, any $\xi \in \partial f(x_0)$ and any $\zeta \in \partial g(x_0)$. Since $f(x) \geq 0$ and $g(x) > 0$,

$$\begin{aligned} & f(x)/g(x) - f(x_0)/g(x_0) \\ &\geq (g(x_0)/g(x))((\xi/g(x_0))\eta(x, x_0) + (-f(x_0)\zeta/(g^2(x_0))\eta(x, x_0)) \\ &\quad + \rho\|(1/g(x))^{1/2}(1 + (f(x_0)/g(x_0)))^{1/2}\theta(x, x_0)\|^2. \end{aligned}$$

Since f and $-g$ are regular at x_0 , we obtain, for any $\delta \in \partial(f(x_0)/g(x_0))$,

$$\begin{aligned} & f(x)/g(x) - f(x_0)/g(x_0) \\ &\geq (g(x_0)/g(x))\delta\eta(x, x_0) + \rho\|(1/g(x))^{1/2}(1 + (f(x_0)/g(x_0)))^{1/2}\theta(x, x_0)\|^2. \end{aligned}$$

Considering that

$$1 + f(x_0)/g(x_0) \geq 1,$$

we have

$$\begin{aligned} & f(x)/g(x) - f(x_0)/g(x_0) \\ & \geq (g(x_0)/g(x))\delta\eta(x, x_0) + \rho\|(1/g(x))^{1/2}\theta(x, x_0)\|^2. \end{aligned}$$

Therefore, the function $f(x)/g(x)$ is ρ -invex, where

$$\begin{aligned} \bar{\eta}(x, x_0) &= (g(x_0)/g(x))\eta(x, x_0), \\ \bar{\theta}(x, x_0) &= (1/g(x))^{1/2}\theta(x, x_0). \end{aligned}$$

3 Optimality Conditions

The Cottle constraint qualification

The Cottle constraint qualification is satisfied at x_0 if either $h_j(x_0) < 0$ for all $j = 1, \dots, m$ or $0 \notin \text{conv}\{\partial h_j(x_0) : h_j(x_0) = 0\}$, where $\text{conv}S$ denotes the convex hull of a set S .

By Theorem 6.1.1 in [1], we can present the following Fritz John necessary conditions.

Theorem 3.1 (Fritz John Necessary Conditions). *If $x_0 \in X$ is an optimal solution of (NFP), then there exist λ and r_j , $j = 1, 2, \dots, m$, such that*

$$0 \in \lambda \partial \left(\frac{f(x_0)}{g(x_0)} \right) + \sum_{j=1}^m r_j \partial h_j(x_0),$$

$$\sum_{j=1}^m r_j h_j(x_0) = 0,$$

$$(\lambda, r_1, \dots, r_m) \geq 0 \text{ and } (\lambda, r_1, \dots, r_m) \neq 0.$$

Assuming the Cottle constraint qualification, we obtain the Karush-Kuhn-Tucker necessary conditions.

Theorem 3.2 (Karush-Kuhn-Tucker Necessary Conditions). Assume that $x_0 \in X$ is an optimal solution for (NFP) at which the Cottle constraint qualification is satisfied. Then there exist $\mu_j \geq 0, j = 1, 2, \dots, m$, such that

$$0 \in \partial\left(\frac{f(x_0)}{g(x_0)}\right) + \sum_{j=1}^m \mu_j \partial h_j(x_0),$$

$$\sum_{j=1}^m \mu_j h_j(x_0) = 0,$$

$$(\mu_1, \dots, \mu_m) \geq 0.$$

Theorem 3.3 (Karush-Kuhn-Tucker Sufficient Conditions). Let (x_0, μ) satisfy the Karush-Kuhn-Tucker conditions as follows:

$$0 \in \partial\left(\frac{f(x_0)}{g(x_0)}\right) + \sum_{j=1}^m \mu_j \partial h_j(x_0),$$

$$\sum_{j=1}^m \mu_j h_j(x_0) = 0,$$

$$(\mu_1, \dots, \mu_m) \geq 0.$$

Assume that f and $-g$ are ρ -invex at x_0 with respect to η and θ , and f and $-g$ are regular at x_0 , and h_j is ρ'_j -invex at x_0 with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \sum_{j=1}^m \mu_j \rho'_j \geq 0$.

Then x_0 is an optimal solution of (NFP).

Proof. Let $x_0, x \in X$ and (x_0, μ) satisfy the Karush-Kuhn-Tucker conditions. Then there exist $\delta \in \partial(f(x_0)/g(x_0))$ and $\gamma_j \in \partial h_j(x_0)$ such that $\delta + \sum_{j=1}^m \mu_j \gamma_j = 0$ and $\sum_{j=1}^m \mu_j h_j(x_0) = 0$. Since f and $-g$ are ρ -invex at x_0 with respect to η and θ and regular at x_0 ,

then by Theroem 2.1 we have

$$\begin{aligned}
& f(x)/g(x) - f(x_0)/g(x_0) \\
& \geq (-g(x)/g(x_0)) \sum_{j=1}^m \mu_j \gamma_j \eta(x, x_0) + \rho \|(1/g(x))^{1/2} \theta(x, x_0)\|^2 \\
& \quad - \sum_{j=1}^m \mu_j h_j(x_0) + \sum_{j=1}^m \mu_j h_j(x).
\end{aligned}$$

Since h_j is ρ'_j -invex at x_0 with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$\begin{aligned}
& f(x)/g(x) - f(x_0)/g(x_0) \\
& \geq (\rho + \sum_{j=1}^m \mu_j \rho'_j) \|\bar{\theta}(x, x_0)\|^2 \\
& \geq 0.
\end{aligned}$$

Therefore, x_0 is an optimal solution of (NFP).

4 Duality Theorems

We consider the following Mond-Weir dual problem to (NFP):

$$\begin{aligned}
(\text{NFD})_M \quad & \text{Maximize} \quad \frac{f(u)}{g(u)} \\
& \text{subject to} \quad 0 \in \partial \left(f(u)/g(u) \right) + \sum_{j=1}^m \mu_j \partial h_j(u), \\
& \quad \sum_{j=1}^m \mu_j h_j(u) \geq 0, \\
& \quad (\mu_1, \dots, \mu_m) \geq 0.
\end{aligned}$$

Theorem 4.1 (Weak Duality). *Let x be feasible for (NFP) and (u, μ) feasible for (NFD)_M. Assume that f and $-g$ are ρ -invex with respect to η and θ , and f and $-g$ are regular functions, and h_j is ρ'_j -invex with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \sum_{j=1}^m \mu_j \rho'_j \geq 0$.*

Then

$$\frac{f(x)}{g(x)} \geq \frac{f(u)}{g(u)}.$$

Proof. Since f and $-g$ are ρ -invex with respect to η and θ , and regular, and (u, μ) is feasible for $(\text{NFD})_M$, then by Theorem 2.1 we have

$$\begin{aligned} & f(x)/g(x) - f(u)/g(u) \\ & \geq (-g(u)/g(x)) \sum_{j=1}^m \mu_j \gamma_j \eta(x, u) + \rho \|(1/g(x))^{1/2} \theta(x, u)\|^2, \end{aligned}$$

for some $\gamma_j \in \partial h_j(u)$. Since h_j is ρ'_j -invex with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$\begin{aligned} & f(x)/g(x) - f(u)/g(u) \\ & \geq (\rho + \sum_{j=1}^m \mu_j \rho'_j) \|\bar{\theta}(x, u)\|^2 \\ & \geq 0. \end{aligned}$$

Theorem 4.2 (Strong Duality). *Let $\bar{x}$ be an optimal solution for (NFP) at which the Cottle constraint qualification is satisfied. Then there exists $\bar{\mu}$ such that $(\bar{x}, \bar{\mu})$ is feasible for $(\text{NFD})_M$. Moreover, if f , g and h satisfy the conditions of Theorem 4.1, then $(\bar{x}, \bar{\mu})$ is an optimal solution of $(\text{NFD})_M$ and the optimal values of (NFP) and $(\text{NFD})_M$ are equal.*

Proof. From the Karush-Kuhn-Tucker necessary conditions, there exists $\bar{\mu}_j \geq 0$, $j = 1, 2, \dots, m$ such that

$$0 \in \partial \left(\frac{f(\bar{x})}{g(\bar{x})} \right) + \sum_{j=1}^m \bar{\mu}_j \partial h_j(\bar{x}),$$

$$\sum_{j=1}^m \bar{\mu}_j h_j(\bar{x}) = 0.$$

Thus $(\bar{x}, \bar{\mu})$ is feasible for $(\text{NFD})_M$. So, by Theorem 4.1, $(\bar{x}, \bar{\mu})$ is an optimal solution of $(\text{NFD})_M$.

Theorem 4.3 (Strict Converse Duality). Let $\bar{x}$ be feasible for (NFP) and $(\bar{u}, \bar{\mu})$ be feasible for (NFD)_M such that $f(\bar{x})/g(\bar{x}) \leq f(\bar{u})/g(\bar{u})$. Assume that f and $-g$ are ρ -invex at $\bar{u}$ with respect to η and θ , and f and $-g$ are regular at $\bar{u}$, and h_j is ρ'_j -invex with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \sum_{j=1}^m \bar{\mu}_j \rho'_j \geq 0$.

Then

$$\bar{x} = \bar{u}.$$

Proof. Since f and $-g$ are ρ -invex at $\bar{u}$ with respect to η and θ , and regular at $\bar{u}$ and $(\bar{u}, \bar{\mu})$ is feasible for (NFD)_M, then by Theorem 2.1 we have

$$\begin{aligned} & f(\bar{u})/g(\bar{u}) - f(\bar{x})/g(\bar{x}) \\ & \leq (g(\bar{u})/g(\bar{x})) \sum_{j=1}^m \bar{\mu}_j \gamma_j \eta(\bar{x}, \bar{u}) - \rho \|(1/g(\bar{x}))^{1/2} \theta(\bar{x}, \bar{u})\|^2 \\ & \quad + \sum_{j=1}^m \bar{\mu}_j h_j(\bar{u}) - \sum_{j=1}^m \bar{\mu}_j h_j(\bar{x}), \end{aligned}$$

for some $\gamma_j \in \partial h_j(\bar{u})$. Since h_j is ρ'_j -invex with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$\begin{aligned} & f(\bar{u})/g(\bar{u}) - f(\bar{x})/g(\bar{x}) \\ & \leq -(\rho + \sum_{j=1}^m \bar{\mu}_j \rho'_j) \|\bar{\theta}(\bar{x}, \bar{u})\|^2 \\ & \leq 0. \end{aligned}$$

Thus $\bar{x} = \bar{u}$.

We propose the following Wolfe dual problem to (NFP):

$$\begin{aligned} \text{(NFD)}_W \quad & \text{Maximize} \quad \frac{f(u)}{g(u)} + \sum_{j=1}^m \mu_j h_j(u) \\ & \text{subject to} \quad 0 \in \partial(f(u)/g(u)) + \sum_{j=1}^m \mu_j \partial h_j(u), \\ & \quad (\mu_1, \dots, \mu_m) \geq 0. \end{aligned}$$

Theorem 4.4 (Weak Duality). *Let x be feasible for (NFP) and (u, μ) feasible for (NFD)_W. Assume that f and $-g$ are ρ -invex with respect to η and θ , and f and $-g$ are regular functions, and h_j is ρ'_j -invex with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \sum_{j=1}^m \mu_j \rho'_j \geq 0$.*

Then

$$\frac{f(x)}{g(x)} \geq \frac{f(u)}{g(u)} + \sum_{j=1}^m \mu_j h_j(u).$$

Proof. Since f and $-g$ are ρ -invex with respect to η and θ , regular and (u, μ) is feasible for (NFD)_W, then by Theorem 2.1 we have

$$\begin{aligned} & f(x)/g(x) - ((f(u)/g(u)) + \sum_{j=1}^m \mu_j h_j(u)) \\ & \geq (-g(u)/g(x)) \sum_{j=1}^m \mu_j \gamma_j \eta(x, u) + \rho \|(1/g(x))^{1/2} \theta(x, u)\|^2 - \sum_{j=1}^m \mu_j h_j(u) \end{aligned}$$

for some $\gamma_j \in \partial h_j(u)$. Since h_j is ρ'_j -invex with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$\begin{aligned} & f(x)/g(x) - ((f(u)/g(u)) + \sum_{j=1}^m \mu_j h_j(u)) \\ & \geq - \sum_{j=1}^m \mu_j h_j(x) + (\rho + \sum_{j=1}^m \mu_j \rho'_j) \|\bar{\theta}(x, u)\|^2 \\ & \geq 0. \end{aligned}$$

Theorem 4.5 (Strong Duality). *Let $\bar{x}$ be an optimal solution for (NFP) at which the Cottle constraint qualification is satisfied. Then there exists $\bar{\mu}$ such that $(\bar{x}, \bar{\mu})$ is feasible for (NFD)_W. Moreover, if f , g and h satisfy the conditions of Theorem 4.4, then $(\bar{x}, \bar{\mu})$ is an optimal solution of (NFD)_W and the optimal values of (NFP) and (NFD)_W are equal.*

Proof. From the Karush-Kuhn-Tucker necessary conditions, there exists $\bar{\mu}_j \geq 0$, $j = 1, 2, \dots, m$ such that

$$0 \in \partial \left(\frac{f(\bar{x})}{g(\bar{x})} \right) + \sum_{j=1}^m \bar{\mu}_j \partial h_j(\bar{x}),$$

$$\sum_{j=1}^m \bar{\mu}_j h_j(\bar{x}) = 0.$$

Thus $(\bar{x}, \bar{\mu})$ is feasible for $(\text{NFD})_W$. So, by Theorem 4.4, $(\bar{x}, \bar{\mu})$ is an optimal solution of $(\text{NFD})_W$.

Theorem 4.6 (Strict Converse Duality). *Let $\bar{x}$ be an optimal solution for (NFP) at which the Cottle constraint qualification is satisfied. Assume that f and $-g$ are ρ -invex at $\hat{x}$ with respect to η and θ , and f and $-g$ are regular at $\hat{x}$, and h_j is ρ_j' -invex with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \sum_{j=1}^m \hat{\mu}_j \rho_j' > 0$. If $(\hat{x}, \hat{\mu})$ is an optimal solution of $(\text{NFD})_W$, then $\hat{x} = \bar{x}$ and the optimal values of (NFP) and $(\text{NFD})_W$ are equal.*

Proof. Assume that $\hat{x} \neq \bar{x}$. Since $\bar{x}$ is an optimal solution of (NFP), there exists $\bar{\mu} \geq 0$ such that $(\bar{x}, \bar{\mu})$ is an optimal solution of $(\text{NFD})_W$. Then

$$f(\bar{x})/g(\bar{x}) + \sum_{j=1}^m \bar{\mu}_j h_j(\bar{x}) = f(\hat{x})/g(\hat{x}) + \sum_{j=1}^m \hat{\mu}_j h_j(\hat{x}) = \max_{(x, \mu) \in Y} (f(x)/g(x) + \sum_{j=1}^m \mu_j h_j(x))$$

where Y is a feasible set of $(\text{NFD})_W$. Because $(\hat{x}, \hat{\mu}) \in Y$, we have

$$0 \in \partial(f(\hat{x})/g(\hat{x})) + \sum_{j=1}^m \hat{\mu}_j \partial h_j(\hat{x}).$$

Since f and $-g$ are ρ -invex at $\hat{x}$ with respect to η and θ , and regular at $\hat{x}$, then by Theorem 2.1 we have

$$f(\bar{x})/g(\bar{x}) - f(\hat{x})/g(\hat{x})$$

$$\geq (-g(\hat{x})/g(\bar{x})) \sum_{j=1}^m \hat{\mu}_j \gamma_j \eta(\bar{x}, \hat{x}) + \rho \|(1/g(\bar{x}))^{1/2} \theta(\bar{x}, \hat{x})\|^2$$

for some $\gamma_j \in \partial h_j(\hat{x})$. Since h_j is ρ'_j -invex with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$\begin{aligned} & f(\bar{x})/g(\bar{x}) + \sum_{j=1}^m \hat{\mu}_j h_j(\bar{x}) - (f(\hat{x})/g(\hat{x}) + \sum_{j=1}^m \hat{\mu}_j h_j(\hat{x})) \\ & \geq (\rho + \sum_{j=1}^m \hat{\mu}_j \rho'_j) \|\bar{\theta}(\bar{x}, \hat{x})\|^2 > 0. \end{aligned}$$

It follows then that

$$\begin{aligned} & f(\bar{x})/g(\bar{x}) + \sum_{j=1}^m \hat{\mu}_j h_j(\bar{x}) \\ & > f(\hat{x})/g(\hat{x}) + \sum_{j=1}^m \hat{\mu}_j h_j(\hat{x}) = f(\bar{x})/g(\bar{x}) + \sum_{j=1}^m \bar{\mu}_j h_j(\bar{x}) \end{aligned}$$

or that

$$\sum_{j=1}^m \hat{\mu}_j h_j(\bar{x}) > \sum_{j=1}^m \bar{\mu}_j h_j(\bar{x}).$$

But from Theorem 3.2, we have that $\sum_{j=1}^m \bar{\mu}_j h_j(\bar{x}) = 0$, hence $\sum_{j=1}^m \hat{\mu}_j h_j(\bar{x}) > 0$ which contradicts the facts that $\hat{\mu}_j \geq 0$ and $h_j(\bar{x}) \leq 0$. Hence $\hat{x} = \bar{x}$.

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