

ON THE SOLVABILITY OF $P(D)f = g$ FOR VECTOR VALUED FUNCTIONS

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Let $P(D)$ be an elliptic linear partial differential operator with constant coefficients, $\Omega \subset \mathbb{R}^n$ open and E a complete locally convex space. By $C^\infty(\Omega, E)$ we denote the linear space of all E -valued C^∞ -functions. Our problem is under which conditions the equation $P(D)f = g$ has a solution $f \in C^\infty(\Omega, E)$ for every $g \in C^\infty(\Omega, E)$. This is known to be true for any Fréchet space E due to a result of Grothendieck ([5]). We shall give a necessary and sufficient condition for the case of E being the dual of a Fréchet space, i.e. for $E = F'_b$ where F is a Fréchet space.

We use the standard notation on (F) -spaces and their duals (s. [9], [13]) and on partial differential equations and distributions (s. [6], [15]). We put $\mathcal{N}(\Omega) = \{f \in C^\infty(\Omega) : P(D)f = 0\}$, $\mathcal{N}(\Omega, E) = \{f \in C^\infty(\Omega, E) : P(D)f = 0\}$ and denote by $\mathcal{N}$ and $\mathcal{N}^E$ the corresponding sheaves on $\mathbb{R}^n$. We have a canonical exact sequence

$$0 \rightarrow \mathcal{N}(\Omega, E) \rightarrow C^\infty(\Omega, E) \xrightarrow{P(D)} C^\infty(\Omega, E) \rightarrow H^1(\Omega, \mathcal{N}^E) \rightarrow 0.$$

where $H^1(\Omega, \mathcal{N}^E)$ is the first cohomology group on Ω with values in the sheaf $\mathcal{N}^E$. Hence our problem is equivalent to the question under which conditions $H^1(\Omega, \mathcal{N}^E) = 0$.

1. We start by giving examples which show in a relatively easy way that both cases will occur.

It is a result of Grothendieck (s. [15]) that an elliptic $P(D)$ does not have a right inverse or equivalently that $\mathcal{N}(\Omega)$ is not complemented in $C^\infty(\Omega)$. Hence the equation $P(D)f = g$ with $g(t) = \delta_t$ has no solution $f \in C^\infty(\Omega, \mathcal{E}'(\Omega))$ and therefore $H^1(\Omega, \mathcal{N}^{\mathcal{E}'(\Omega)}) \neq 0$.

If $H^1(\Omega, \mathcal{N}^E) = 0$ for any E then $P(D)$ cannot have a right inverse. Hence the above counterexample follows from any other counterexample and the following can be considered as a proof of Grothendieck's result. It gives also some additional information.

1.1. Proposition: If $E = \bigcup_n E_n$ where $E_n \subsetneq E_{n+1}$ are closed subspaces of E and if $P(D)$ is nonconstant elliptic then $H^1(\Omega, \mathcal{N}^E) \neq 0$.

Proof: We construct by induction a biorthogonal sequence in E, E' (c.f. [5]). We can assume $\dim E_{n+1}/E_n > n$, $E_1 \neq \{0\}$. We start with choosing $e_1 \neq 0$, $e_1 \in E_1$ and $f_1 \in E'$ such that $f_1(e_1) = 1$.

Assume $e_1, \dots, e_n, f_1, \dots, f_n$ being determined then we can choose $e_{n+1} \in (E_{n+1} \setminus E_n) \cap \bigcap_{k=1}^n \ker f_k$, $e_{n+1} \neq 0$ and $f_{n+1} \in E'$ such that $f_{n+1}|_{E_n} = 0$, $f_{n+1}(e_{n+1}) = 1$.

We obtain sequences $e_1, e_2, \dots$ in E , $f_1, f_2, \dots$ in E' such that $f_k(e_n) = \delta_{k,n}$ for all k, n and $f_k|_{E_n} = 0$ for $k > n$.

We can assume that Ω is connected. We choose a sequence $B_n = \{x : |x - x_n| \leq r_n\}$ of compact balls such that $r_n > 0$, $B_n \subset \Omega$, $B_n \cap B_m = \emptyset$ for $n \neq m$ and such that for each $K \subset\subset \Omega$ there exists n_0 with $K \cap B_n = \emptyset$ for $n \geq n_0$. For each n we can find $\varphi_n \in \mathcal{D}(B_n)$, $\varphi_n \notin P(D)\mathcal{D}(B_n)$. We put

$$g(x) = \sum_{n=1}^{\infty} \varphi_n(x) e_n.$$

Clearly $g \in C^\infty(\Omega, E)$. Let us assume the existence of $f \in C^\infty(\Omega, E)$ with $P(D)f = g$. We put $\Omega_1 = \Omega \setminus \bigcup_n B_n$. Ω_1 is open. From Baire's theorem (applied to the locally compact space Ω_1) we obtain the existence of n_0 such that $f^{-1}(E_{n_0}) \cap \Omega_1$ has an inner point, i.e. there exists a non empty open set $\Omega_0 \subset f^{-1}(E_{n_0}) \cap \Omega_1$.

We put $\psi_n = f_n \circ f$. For $n > n_0$ we have $\psi_n|_{\Omega_0} = 0$. Moreover $P(D)\psi_n = f_n \circ g = \varphi_n$. Hence ψ_n is real analytic on $\Omega \setminus B_n$ and therefore vanishes on this set. This means $\psi_n \in \mathcal{D}(B_n)$ which is a contradiction.

Examples of spaces E satisfying the assumption of Proposition 1.1. are $\varphi := \bigoplus_{\mathbb{N}} \mathbb{C}$, $\mathcal{E}'(\mathbb{R})$, $\mathcal{D}(\mathbb{R})$ etc. For $E = F'_b$, F a Fréchet space it means that F does not have a continuous norm. Hence the existence of a continuous norm on F is a necessary condition for $H^1(\Omega, \mathcal{N}^E) = 0$.

We shall now give a positive example. It is of particular significance as we will see in section 4. As usually we denote by

$$s = \{x = (x_1, x_2, \dots) : \|x\|_k = \sum_{j=1}^{\infty} |x_j| j^k < +\infty \text{ for all } k\}$$

the space of all rapidly decreasing scalar sequences. Equipped with the norms $\| \cdot \|_k$ it is a nuclear F-space. By the dual pairing

$$y(x) = \sum_j x_j y_j \text{ its dual can be described as}$$

$$s' = \{y = (y_1, y_2, \dots) : \|y\|_k^* = \sup_j |y_j| j^{-k} < +\infty \text{ for some } k\}.$$

Here $\|y\|_k^* = \sup_{\|x\|_k=1} |y(x)|$ is the Minkowski functional of the polar of $\{x : \|x\|_k \leq 1\}$. The topology of s'_b is given by the seminorms

$$p_x(y) = \sum_j |y_j| |x_j|, x \in s.$$

1.2. Proposition: If $P(D)$ is hypoelliptic and Ω is convex then $H^1(\Omega, \mathcal{N}^{\mathcal{D}'}) = 0$ and $H^1(\Omega, \mathcal{N}^{s'}) = 0$.

Proof: We put $\tilde{\Omega} = \Omega \times \mathbb{R}$ and $\tilde{P}(D) = P(D)$ acting as partial differential operator in the first n variables on functions of $n+1$ variables. $\tilde{\Omega}$ is convex.

We put $\mathcal{D}' = \mathcal{D}'(\mathbb{R})$. An element $g \in C^\infty(\Omega, \mathcal{D}')$ defines in a canonical way a $\tilde{g} \in \mathcal{D}'(\tilde{\Omega})$. We can find $\tilde{f} \in \mathcal{D}'(\tilde{\Omega})$ such that $\tilde{P}(D)\tilde{f} = \tilde{g}$. If we call f the element in $\mathcal{D}'(\Omega, \mathcal{D}')$ which corresponds to $\tilde{f}$ this means $P(D)f = g$. Since $P(D)$ is hypoelliptic we have even $f \in C^\infty(\Omega, \mathcal{D}')$. This proves $H^1(\Omega, \mathcal{N}^{\mathcal{D}'}) = 0$.

To show that $H^1(\Omega, \mathcal{N}^{s'}) = 0$ it obviously suffices to show that s' can be imbedded as a complemented subspace in $\mathcal{D}'$ or equivalently that s can be imbedded as a complemented subspace into $\mathcal{D}$.

We choose $\varphi_0 \in \mathcal{D}$ such that $\sum_{v \in \mathbb{Z}} \varphi_0(x-v) = 1$ for all $x \in \mathbb{R}$. Let $\mathcal{E}_p$ be the space of all periodic functions on $\mathbb{R}$ with period 1. Then $\Phi : f \rightarrow f\varphi_0$ defines a continuous linear map from $\mathcal{E}_p$ into $\mathcal{D}$, which imbeds $\mathcal{E}_p$ as a complemented subspace into $\mathcal{D}$ since $\Psi : \varphi \rightarrow \sum_{v \in \mathbb{Z}} \varphi(x-v)$ defines a left inverse. By Fourier expansion $\mathcal{E}_p \cong s$.

2. We make now use of the theory of the functors $\text{Ext}^1(\cdot, \cdot)$ (s. [11], [23]) to give a necessary and sufficient condition for $H^1(\Omega, \mathcal{N}^E) = 0$ where $E = F'_b$, F a Fréchet space and $P(D)$ elliptic.

From the exact sequence

$$0 \rightarrow \mathcal{N}(\Omega) \xrightarrow{\iota} C^\infty(\Omega) \xrightarrow{P(D)} C^\infty(\Omega) \rightarrow 0$$

we obtain an exact sequence

$$0 \rightarrow L(F, \mathcal{N}(\Omega)) \xrightarrow{\iota^*} L(F, C^\infty(\Omega)) \xrightarrow{P(D)^*} L(F, C^\infty(\Omega)) \xrightarrow{\delta^0} \text{Ext}^1(F, \mathcal{N}(\Omega)) \xrightarrow{\iota^1} \text{Ext}^1(F, C^\infty(\Omega)) \rightarrow \dots$$

In general $\text{Ext}^1(F, C^\infty(\Omega))$ will not vanish, but we have:

2.1. Lemma: $\iota^1 = 0$

Proof: We choose a sequence $\Omega_1 \subset \subset \Omega_2 \subset \subset \dots \subset \subset \Omega$ of open sets such that $\Omega = \bigcup_k \Omega_k$ and call G_k the completion of $C^\infty(\Omega)|_{\Omega_k}$ with respect to $\|f\|_k = \sup \{ |f^{(\alpha)}(x)| : x \in \Omega_k, |\alpha| \leq k \}$, H_k the closure of $\mathcal{N}(\Omega)|_{\Omega_k}$ in G_k . We obtain in a natural way a commutative diagram where the lines are canonical resolutions

$$\begin{array}{ccccccc} 0 & \rightarrow & C^\infty(\Omega) & \rightarrow & \prod_k G_k & \xrightarrow{q} & \prod_k G_k \rightarrow 0 \\ & & \uparrow \iota & & \uparrow j & & \uparrow j \\ 0 & \rightarrow & \mathcal{N}(\Omega) & \rightarrow & \prod_k H_k & \xrightarrow{q} & \prod_k H_k \rightarrow 0 \end{array}$$

q is defined by $q((f_k)_k) = (f_k - f_{k+1}|_{\Omega_k})_k$. This gives us the following commutative diagram with exact lines.

$$\begin{array}{ccccccc} 0 & \rightarrow & L(F, C^\infty(\Omega)) & \rightarrow & \prod_k L(F, G_k) & \xrightarrow{q} & \prod_k L(F, G_k) \xrightarrow{\delta^0} \text{Ext}^1(F, C^\infty(\Omega)) \rightarrow 0 \\ & & \uparrow \iota^* & & \uparrow j^* & & \uparrow j^* \\ 0 & \rightarrow & L(F, \mathcal{N}(\Omega)) & \rightarrow & \prod_k L(F, H_k) & \xrightarrow{q} & \prod_k L(F, H_k) \xrightarrow{\delta^0} \text{Ext}^1(F, \mathcal{N}(\Omega)) \rightarrow 0 \end{array}$$

Let $A_k \in L(E, H_k)$, $k = 1, 2, \dots$ be given. We choose $\varphi_k \in \mathcal{D}(\Omega_k)$, $\varphi_k \equiv 1$ on Ω_{k-1} ($\Omega_0 := \emptyset$) and put

$$B_k x = - \sum_{\nu=1}^k \varphi_\nu (A_\nu x) + A_k x.$$

Then $B_k \in L(F, G_k)$ and

$$B_k x - (B_{k+1} x)|_{\Omega_k} = \varphi_{k+1} (A_{k+1} x) + A_k x - A_{k+1} x = A_k x$$

since $\varphi_{k+1} - 1$ vanishes on Ω_k .

We proved $\iota^1 \circ \delta^0 = \delta^0 \circ j^* = 0$, hence $\iota^1 = 0$.

An immediate consequence of 2.1. is that we have an exact sequence

$$0 \rightarrow L(F, \mathcal{N}(\Omega)) \rightarrow L(F, C^\infty(\Omega)) \xrightarrow{P(D)^*} L(F, C^\infty(\Omega)) \rightarrow \text{Ext}^1(F, \mathcal{N}(\Omega)) \rightarrow 0$$

and hence :

2.2. Proposition: $H^1(\Omega, \mathcal{N}^E) \simeq \text{Ext}^1(F, \mathcal{N}(\Omega))$.

We now make use of the results of [23] which give us rather precise information about the conditions under which $\text{Ext}^1(\cdot, \cdot)$ vanishes. We assume therefore that F and H are Fréchet spaces from which one is nuclear, $\| \cdot \|_1 \leq \| \cdot \|_2 \leq \dots$ are fundamental systems of seminorms in F or H respectively, $\|y\|_k = \sup_{\|x\|_k \leq 1} |y(x)|$ for $y \in H$. We shall apply the results for $H = \mathcal{N}(\Omega)$ which is nuclear, so no restrictions on F will remain. We define

$$(S_1^*) \quad \exists n_0 \quad \forall \mu \quad \exists k \quad \forall K, m \quad \exists n, S \quad \forall x \in F \quad y \in H' :$$

$$\|x\|_m \|y\|_k^* \leq S(\|x\|_n \|y\|_K^* + \|x\|_{n_0} \|y\|_\mu^*)$$

$$(S_2^*) \quad \forall \mu \quad \exists n_0, K, k \quad \forall K, m \quad \exists n, s \quad \forall x \in F \quad y \in H' :$$

$$\|x\|_m \|y\|_k^* \leq S(\|x\|_n \|y\|_K^* + \|x\|_{n_0} \|y\|_\mu^*)$$

Then [23], Theorem 4.7. together with [23], Prop. 4.9. says :

2.3. Theorem: $(S_1^*) \Rightarrow \text{Ext}^1(F, H) = 0 \Rightarrow (S_2^*)$.

We shall use certain consequences of theorem 2.2. which will be stated below. To formulate them and our main result we need the following definitions. They describe certain properties for Fréchet spaces which are linear topological invariants:

$$(DN) \quad \exists n_0 \quad \forall m \quad \exists n, C : \| \cdot \|_m^2 \leq C \| \cdot \|_{n_0} \| \cdot \|_n$$

$$(\underline{DN}) \quad \exists n_0 \quad \forall m \quad \exists n, d, C : \| \cdot \|_m^{1+d} \leq C \| \cdot \|_{n_0} \| \cdot \|_n^d$$

$$(\Omega) \quad \forall p \quad \exists q \quad \forall k \quad \exists d, C : \| \cdot \|_q^{*1+d} \leq C \| \cdot \|_k^* \| \cdot \|_p^{*d}$$

These invariants play an important role in the structure theory of nuclear Fréchet spaces. Together with nuclearity properties they are characteristic for the subspaces of infinite type or finite type power series spaces or the quotients of infinite type power series spaces respectively (s. [3], [19], [20], [25]). Together with nuclearity (DN) characterizes the subspaces of $s, (\Omega)$ the quotient spaces of s (s. [17], [24], [18]).

(DN) is equivalent to the existence of a fundamental system of seminorms such that

$$\| \|_k^2 \leq \| \|_{k-1} \| \|_{k+1} ,$$

(Ω) to the existence of a fundamental system of seminorms with

$$\| \|_k^{*2} \leq \| \|_{k-1}^* \| \|_{k+1}^*$$

for all $k = 2, 3, \dots$

The following theorem contains consequences from 2.3. which are proved in [23], § 5 and § 6. 2.4.b was in an equivalent formulation first proved in [17] and [24].

2.4. Theorem: (a) $\text{Ext}^1(s, H) = 0$ iff H has property (Ω).

(b) If F has property (DN) and H has property (Ω) then $\text{Ext}^1(F, H) = 0$.

(c) If H has property (DN) and $\text{Ext}^1(F, H) = 0$, if moreover there exists an increasing sequence $(\alpha_j)_j$ with $\lim_j \alpha_j = +\infty$ and $\sup_j \frac{\alpha_{j+1}}{\alpha_j} < +\infty$ such that the following is true

(P) $\exists \mu_0 \forall \mu \geq \mu_0 \exists K_0 \forall K \geq K_0 \exists D > 0, r > 1, R > 1 \forall v$

$$\frac{1}{D} R^{-\alpha_v} \leq \delta_v(U_K, U_\mu) \leq D r^{-\alpha_v}$$

then F has property (DN).

For the formulation of 2.4. (c) we used the Kolomogorov diameters $\delta_v(U_K, U_\mu)$ for the two convex sets $U_K = \{x : \|x\|_K \leq 1\} \subset U_\mu = \{x : \|x\|_\mu \leq 1\}$. They are defined in the following way: let H be a linear space, $U \subset V$ convex sets. Then we define $\delta_v(U, V)$ as the infimum of all $\delta > 0$ such that there exists a linear subspace $F \subset E$ with dimension at most v and $U \subset \delta V + F$.

The main content of section 3 of this paper will be the proof of the following proposition.

2.5. Proposition: Let $P(D)$ be elliptic and $\Omega \subset \mathbb{R}^n$ open, then :

(a) If Ω is connected then $\mathcal{N}(\Omega)$ has property (DN)

(b) $\mathcal{N}(\Omega)$ has property (Ω).

We are now ready to prove the main result of this paper. We assume $P(D)$ to be elliptic, $\Omega \subset \mathbb{R}^n$ non empty open and $E = F'_b$, where F is a Fréchet space.

2.6. Theorem: $H^1(\Omega, \mathcal{N}^E) = 0$ iff F has property (DN).

Proof: By 2.1. we have $H^1(\Omega, \mathcal{N}^E) \cong \text{Ext}^1(F, \mathcal{N}(\Omega))$. One part of the assertion now follows immediately from 2.4.(b) and 2.5.(b).

For the proof of the other implication we remark that $H^1(\Omega, \mathcal{N}^E) = 0$ implies $H^1(\Omega_0, \mathcal{N}^E) = 0$ for any connected component $\Omega_0 \neq \emptyset$ of Ω . But then the result follows from 2.4.(c) and 2.5.(a) if condition (P) in 2.3.(c) with appropriate $(\alpha_j)_j$ is guaranteed for $\mathcal{N}(\Omega_0)$.

We claim that for elliptic $P(D)$ and an open set $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) (P) is true for $\mathcal{N}(\Omega)$ with $\alpha_\nu = \frac{1}{\nu^{n-1}}$. The right hand inequality is exactly what is shown in the proof of [24], Satz 5.4. (see the definition of $\Lambda_1(\alpha)$ - nuclearity as given e.g. in [24], Def. 1.1.). The assertion holds for arbitrary μ_0 .

We choose now a sequence of open sets $\omega_1 \subset \subset \omega_2 \subset \subset \dots$, $\bigcup_k \omega_k = \Omega$ and put $\|f\|_k = \sup_{x \in \omega_k} |f(x)|$, $B_k = \{y \in \mathcal{N}(\Omega)' : \|y\|_k^* \leq 1\}$. For $K > \mu$ we put $\beta_\nu = -\log \delta_\nu(B_\mu, B_K)$. We choose $R \in \mathbb{R} \setminus \{+\infty\}$, $R > \sup_{x \in \omega_K} |x| =: r_1$ and a sequence $r_1 < r_2 < \dots \nearrow R$. For $\mathcal{N}(U_R)$ we take the norms $|f|_0 = \|f\|_\mu$, $|f|_1 = \|f\|_K$, $|f|_k = \sup_{|x| \leq r_k} |f(x)|$ for $k \geq 2$. Theorem 3.1. below implies that $\mathcal{N}(U_R)$ has property $(\underline{\text{DN}})$ with $n_0 = 0$. A slightly modified version for the case of $(\underline{\text{DN}})$ of the proof of [25], Lemma 4.3. tells that for any absolutely convex bounded set $B \subset \mathcal{N}(U_R)'$ we have another such set $\tilde{B}$ with $B \subset \tilde{B}$ and $\delta_\nu(B, \tilde{B}) \leq C \rho^{-\beta_\nu}$ for all ν with some $C > 0$, $\rho > 1$. From [20], Satz 7.7. we know that $\mathcal{N}(U_R) \cong \Lambda_0(\alpha)$ for $R < +\infty$. Easy calculation gives $\sup_{\nu} \frac{\beta_\nu}{\alpha_\nu} < +\infty$ which proves the assertion, since $\delta_\nu(U_K, U_\mu) \geq \frac{1}{\nu+1} \delta_\nu(B_\mu, B_K)$ (s. [14], S. 72). We could also have used the fact that $\mathcal{N}(\mathbb{R}^n) \cong \Lambda_\infty(\alpha)$ (s. [26]).

It should be remarked that the condition in 2.6. is independent of Ω and the elliptic operator $P(D)$. An equivalent result was first proved in [16] for the operator $P(D) = \frac{\partial}{\partial \bar{z}}$. To see the equivalence and also give an intrinsic condition on E not involving the space F we use an in-

variant for (DF)-spaces introduced in [16]. Let $B_1 \subset B_2 \subset \dots$ be a fundamental system of absolutely convex bounded sets in E . We define a property (A) by:

$$(A) \quad \exists n_0 \forall m \exists n, C \forall r > 0 : B_m \subset r B_{n_0} + \frac{C}{r} B_n$$

In [17], 1.4. it is proved that $E = F'_b$, F Fréchet space, has property (A) iff F has property (DN). We obtain for $E = F'_b$, F Fréchet space:

2.6. Theorem: $H^1(\Omega, \mathcal{N}^E) = 0$ iff E has property (A).

Examples will be given in section 4.

3. We have still to prove proposition 2.5. i.e. properties (DN) and (Ω) , for the space $\mathcal{N}(\Omega)$ where $P(D)$ is elliptic. We use a result from [19].

Let X be an arbitrary N - dimensional real-analytic manifold, $\mathcal{A}$ the sheaf of complex valued real analytic functions on X , $\mathcal{F} \subset \mathcal{A}$ a subsheaf such that for every open set $U \subset X$ the space $\mathcal{F}(U)$ is complete in the compact open topology. Then [19], Satz 5.1. says the following:

3.1. Theorem: Let $\emptyset \neq \Omega_1 \subset \Omega_2 \subset \Omega_3 \subset X$ be open sets, Ω_3 connected. For $f \in \mathcal{F}(X)$ put $\|f\|_j = \sup_{x \in \Omega_j} |f(x)|$, $j = 1, 2, 3$. Then there exist $C > 0$, $\lambda > 0$, $\mu > 0$ with $\lambda + \mu = 1$ such that $\|f\|_2 \leq C \|f\|_1^\lambda \|f\|_3^\mu$ for all $f \in \mathcal{F}(X)$.

An immediate consequence is one part of proposition 2.5. .

3.2. Corollary: If Ω is connected the $\mathcal{N}(\Omega)$ has property (DN).

The other part of proposition 2.5. we obtain by applying 3.1. to an appropriate representation of the dual space of $\mathcal{N}(\Omega)$. We recall the facts of Grothendieck's duality theory (s. [4], [1], [26]).

Let $\check{E}$ be a fixed tempered fundamental solution for ${}^tP(D) = P(-D)$. A solution $f \in C^\infty(\mathbb{R}^n \setminus K)$, K kompakt, of ${}^tP(D) f = 0$ is called regular in infinity with respect to $\check{E}$ if

$$(1) \quad f \psi = \check{E} * {}^tP(D)(f\psi)$$

for one (every) $\psi \in C^\infty(\mathbb{R}^n)$ with $\text{supp } \psi \subset \mathbb{R}^n \setminus K$, $\text{supp } (1-\psi)$ compact.

For compact $K \subset \mathbb{R}^n$ we define by $R(K)$ the space of all such functions. $R(K)$ is a Fréchet space in the compact open topology on $\mathbb{R}^n \setminus K$.

We put

$$R(C\Omega) = \varinjlim_{K \subset \subset \Omega} R(K)$$

where K runs through the compact subsets of Ω . For $f \in \mathcal{N}(\Omega)$ and $g \in R(C\Omega)$ we put

$$(f, g) = \int f \, {}^tP(D)(\psi g)$$

where ψ is as above. By the dual pairing $(\cdot, \cdot)$ the space $R(C\Omega)$ with its inductive topology can be identified with $\mathcal{N}(\Omega)'_b$.

Since $\check{E}$ is tempered it follows from (1) that there exist m, C such that for $f \in R(K)$ and ψ as above we have

$$(2) \quad |(f\psi)(x)| \leq C (1+|x|^2)^m \sup_{\substack{\xi \in \mathbb{R}^n \\ |\alpha| \leq m}} |\psi^{(\alpha)}(\xi)|$$

for all $x \in \mathbb{R}^n$, where $\varphi = {}^tP(D)(f\psi) \in \mathcal{D}'(\mathbb{R}^n)$.

We put $K_\nu = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \frac{1}{\nu}, |x| \leq \nu\}$ ($\text{dist}(x, \emptyset) = -\infty$). For given ν we choose $\psi_\nu \in C^\infty(\mathbb{R}^n)$, $\text{supp } \psi_\nu \subset \mathbb{R}^n \setminus K_\nu$ such that $\text{supp } (1-\psi_\nu) \subset \overset{o}{K}_{\nu+1}$, i.e. $\psi_\nu = 1$ in a neighborhood of $\mathbb{R}^n \setminus \overset{o}{K}_{\nu+1}$. Then we obtain from (2) the existence of constants C_ν and compact sets $L_\nu \subset \overset{o}{K}_{\nu+1} \setminus K_\nu$ with

$$(3) \quad \sup_{x \in \mathbb{R}^n \setminus K_{\nu+1}} |f(x)| (1+|x|^2)^{-m} \leq C_\nu \sup_{x \in L_\nu} |f(x)|$$

for all $f \in R(K_\nu)$. L_ν has been chosen an account of the (hypo) ellipticity of tP such that $\sup_{x \in L_\nu} |f(x)|$ estimates the derivatives of f on $\text{supp } \psi_\nu \cap \text{supp } (1-\psi_\nu) \subset \overset{o}{K}_{\nu+1} \setminus K_\nu$.

We put

$$R_\nu = \{f \in R(K_\nu) : \|f\|_\nu^* := \sup_{x \in \mathbb{R}^n \setminus K_\nu} |f(x)| (1+|x|^2)^{-m} < +\infty\}.$$

With the norm $\|\cdot\|_\nu^*$ this is a Banach space. From (3) we obtain that

$$R(K) = \varinjlim_\nu R_\nu$$

topologically. Hence finally we obtain :

3.3. Lemma: $\mathcal{N}(\Omega)'_b \cong \varinjlim_{\nu} R_{\nu}$.

It should be remarked that due to Baire's theorem this implies that the unit balls in R_{ν} , $\nu = 1, 2, \dots$ are a fundamental system of bounded sets in this space. Or equivalently : the $\|\cdot\|_{\nu}^*$ are dual norms of a fundamental system of norms in $\mathcal{N}(\Omega)$.

We are now ready to prove the second part of proposition 2.

3.4. Proposition: For elliptic $P(D)$ the space $\mathcal{N}(\Omega)$ has property (Ω) .

Proof: It suffices to show that for $k > p + 1 > p > 1$ we have $\lambda > 0$, $\mu > 0$, $\lambda + \mu = 1$ and $C > 0$ such that

$$\|f\|_{p+1}^* \leq C \|f\|_k^{\lambda} \|f\|_p^{\mu}$$

for all $f \in R_{p-1}$. For in the inductive lim we can replace R_p by the closure of R_{p-1} in R_p . We use 3.1.

For open $\omega \subset \mathbb{R}^n$ we put

$$G(\omega) = \{(1+|x|^2)^{-m} f : f \in C^{\infty}(\omega), {}^tP(D) f = 0\}$$

and call $\mathcal{G}$ the sheaf generated by the $G(\omega)$. Since ${}^tP(D)$ is elliptic $\mathcal{G}$ is a sheaf of real analytic functions which satisfies the assumption of 3.1..

Let R be so large that $K_k \subset \{x : |x| < R\} =: U_R$. We put $\Omega_1 = (\mathbb{R}^n \setminus K_k) \cap U_R$, $\Omega_2 = (\mathbb{R}^n \setminus K_{p+1}) \cap U_R$, $\Omega_3 = (\mathbb{R}^n \setminus K_p) \cap U_{R+1}$. Then $\Omega_1 \subset \Omega_2 \subset \Omega_3$. Every bounded component of Ω_3 contains a point of $\mathbb{R}^n \setminus \Omega$ hence of Ω_1 . So we can apply 3.1. to each of the finitely many components of Ω_3 and to the sheaf $\mathcal{G}$ on $X = \mathbb{R}^n \setminus K_{p-1}$.

We obtain $\lambda > 0$, $\mu > 0$, $\lambda + \mu = 1$ and $C > 0$ such that

$$\sup_{x \in \Omega_2} (1+|x|^2)^{-m} |f(x)| \leq C \left(\sup_{x \in \Omega_1} (1+|x|^2)^{-m} |f(x)| \right)^{\lambda} \left(\sup_{x \in \Omega_3} (1+|x|^2)^{-m} |f(x)| \right)^{\mu}$$

and hence for $f \in R_{p-1}$.

$$\|f\|_{p+1}^* < C \|f\|_k^{\lambda} \|f\|_p^{\mu},$$

because this inequality is trivial if the sup in $\|f\|_{p+1}^*$ is taken on for $|x| \geq R$.

In the following situation we can get even more. Let us call $\Omega_0 = \mathbb{R}^n \setminus \bar{\Omega}$. If for every v each connected component of $\Omega \setminus K_v$ has nonempty intersection with Ω_0 , then $\Omega_1 = \Omega_0 \cap U_R$, R large enough, fulfills all requirements on Ω_1 we needed in the proof above. This will be the case if Ω_0 is unbounded and $\bar{\Omega}_0 = \mathbb{R}^n \setminus \Omega$. In this case we put for $f \in R(CK)$

$$\|f\|_0 = \sup_{x \in \Omega_0} |f(x)| (1+|x|^2)^{-m}.$$

$\|\cdot\|_0$ is a continuous norm on $R(C\Omega)$. Hence there is a bounded absolutely convex set B in $\mathcal{N}(\Omega)$ such that $\|f\|_0 = \sup \{(u, f) : u \in B\}$.

We define another topological invariant

$$(\tilde{\Omega}) \quad \forall p \quad \exists q, d \quad \forall k \quad \exists C : \| \cdot \|_q^{*1+d} \leq C \| \cdot \|_k^* \| \cdot \|_p^{*d}.$$

This is equivalent to the existence of a bounded subsolutely convex set B in the underlying Fréchet space H such that with $\|y\|_0 = \sup \{y(x) : x \in B\}$ for $y \in H'$ we have:

$$\forall p \quad \exists q, d \quad C : \| \cdot \|_q^{*1+d} \leq C \| \cdot \|_0 \| \cdot \|_p^{*d}$$

Property $(\tilde{\Omega})$ plays an important role in the theory of holomorphic functions on Fréchet spaces (s. [2], [10]). It implies that every continuous linear map into a space with property (DN) is bounded ([22]).

We obtain immediately:

3.5. Proposition: If $\Omega_0 := \mathbb{R}^n \setminus \bar{\Omega}$ is unbounded and $\bar{\Omega}_0 = \mathbb{R}^n \setminus \Omega$ then for elliptic $P(D)$ the space $\mathcal{N}(\Omega)$ has property $(\tilde{\Omega})$.

We close this section by two remarks:

Remark: (1) In special cases there are much sharper results:

- (a) $\mathcal{N}(\mathbb{R}^n) \cong H(\mathbb{C}^{n-1})$ for $n \geq 2$ (s. Wiechert [26]).
- (b) For convex bounded $\Omega \subset \mathbb{R}^n$: $\mathcal{N}(\Omega) \cong H(D^{n-1})$ (s. [20], [26]).

Here $H(\mathbb{C}^{n-1})$ denotes the space of entire functions on $\mathbb{C}^{n-1}$, $H(D^{n-1})$ the space of holomorphic functions on the $(n-1)$ -dimensional polydisc.

- (2) For a convex set $\Omega \subset \mathbb{R}^n$ and hypoelliptic $P(D)$ condition (Ω) for the

space $\mathcal{N}(\Omega)$ follows by 2.4.(a) also from 1.2. Condition (Ω) resp. $(\bar{\Omega})$ for convex resp. convex bounded Ω sets in $\mathbb{R}^n$ and for solutions even of (hypo) elliptic systems of partial differential equations have been proved in Petzsche [12].

4. Examples of spaces with or without (DN) are given at various places. The first list is contained in [5], II, § 4, n° 3, Cor. 2., for the equivalence to (DN) of the conditions used there s. [22], 7.2.. Many significant examples arise from the following type of spaces.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$, $\alpha_n \nearrow +\infty$ be a sequence of real numbers, $r \in \mathbb{R} \setminus \{+\infty\}$. We define

$$\Lambda_r(\alpha) = \{x = (x_1, x_2, \dots) : \|x\|_\rho = \sum_j |x_j| e^{\rho \alpha_j} < +\infty \text{ for } \rho < r\}.$$

Then we know from [17], 2.4.:

4.1. Proposition: $\Lambda_r(\alpha)$ has (DN) iff $r = +\infty$.

For $\alpha_j = \log j$ we have $\Lambda_\infty(\alpha) \cong s$, hence we recovered 1.2. from 2.6. even for arbitrary open Ω . In the nuclear case s is in the following sense universal for the class of F such that with $E = F'_b$ we have $H^1(\Omega, \mathcal{N}^E) = 0$: the nuclear spaces with (DN) are exactly the subspaces of s , the class of all nuclear F with $H^1(\Omega, \mathcal{N}^E) \cong \text{Ext}^1(F, \mathcal{N}(\Omega)) = 0$ is closed under subspaces.

For $\alpha_j = j^{\frac{1}{N}}$ we have $\Lambda_\infty(\alpha) \cong H(\mathbb{C}^N)$, $\Lambda_0(\alpha) \cong H(D^N)$ the space of entire functions in N variables and the space of holomorphic functions on the polydisc respectively. By Köthe's duality we have $\Lambda_\infty(\alpha)' \cong \mathcal{O}^N$, $\Lambda_0(\alpha)' = H(\bar{D}^N)$, the spaces of germs of holomorphic functions in 0, or on $\bar{D}^N$ respectively.

For $\alpha_j = M(j^{\frac{1}{N}})$ we have $\Lambda'_0(\alpha) \cong \mathcal{E}^{\{M_p\}}(K)$ the class of $\{M_p\}$ ultradifferentiable functions in the sense of Roumieu on K (s. [21]). The sequence $\{M_p\}$ is assumed to satisfy conditions (M1), (M2), (M3) of Komatsu [7], K to be a compact in $\mathbb{R}^N$ with sufficiently smooth boundary (s. [8]). As usual we have $M(t) = \sup_p \log \frac{t^{pM_0}}{M_p}$. For $M_p = (p!)^s$, $s > 1$ we obtain the Gevrey classes. In this case we have $\alpha_j = \frac{1}{sN}$.

A function $f \in C^\infty(\Omega, E)$ in the case of the spaces above we can consider as C^∞ -function $f(x, \lambda)$ depending on a parameter λ , which is e.g. a complex parameter in the case of $E = \mathcal{O}$, $H(\bar{D}^N)$ or a Gevrey-parameter in the case of $\mathcal{E}^{(p!^s)}(K)$. $H^1(\Omega, \mathcal{N}^E) = 0$ resp. $\neq 0$ means in these cases that the equation

$P(D) f(x, \lambda) = g(x, \lambda)$ is solvable (resp. not solvable) for any given parametrized family $g(x, \lambda)$ with $f(x, \lambda)$ parametrized in the same way.

With the above definition for instance in the case of $\mathcal{O}^N$ the domain of definition of the parameter may vary not only locally but also with the order of derivative in x . This is avoided by means of the following definition: $C_b^\infty(\Omega, E)$ is defined as the set of all functions $f \in C^\infty(\Omega, E)$ such that for every $x \in \Omega$ there is a neighbourhood $U(x) \subset \Omega$ and a closed absolutely convex bounded set $B \subset E$ such that $f|_U \in C^\infty(U, E_B)$. E_B is the continuously imbedded Banach subspace of E generated by B . $C_b^{\infty, E}$ denotes the corresponding sheaf.

Since for (hypo) elliptic $P(D)$ clearly $\mathcal{N}(\Omega, E) \subset C_b^\infty(\Omega, E)$ and $P(D) : C_b^{\infty, E} \rightarrow C_b^{\infty, E}$ is a surjective map of sheaves, we obtain an exact sequence

$$0 \rightarrow \mathcal{N}(\Omega, E) \rightarrow C_b^\infty(\Omega, E) \xrightarrow{P(D)} C_b^\infty(\Omega, E) \rightarrow H^1(\Omega, \mathcal{N}^E) \rightarrow 0$$

Hence we have that $P(D) : C_b^\infty(\Omega, E) \rightarrow C_b^\infty(\Omega, E)$ is surjective if and only if $H^1(\Omega, \mathcal{N}^E) = 0$. For $E = F'_b$, F Fréchet spaces we obtain:

4.2. Proposition: $P(D) : C_b^\infty(\Omega, E) \rightarrow C_b^\infty(\Omega, E)$ is surjective iff F has property (DN).

Hence $P(D) : C^\infty(\Omega, E) \rightarrow C^\infty(\Omega, E)$ and $P(D) : C_b^\infty(\Omega, E) \rightarrow C_b^\infty(\Omega, E)$ are surjective e.g. for $E = \mathcal{O}^N$, s' , and not surjective e.g. for $E = H(\overline{D}^N)$, $\mathcal{E}_{\{M_p\}}^{\{M_p\}}(K)$, $\{M_p\}$ and K as above.

It is easy to see that it is possible to analogize the theory of [23] for an analysis of $H^1(\Omega, \mathcal{N}^E)$ rather than using it. In this case we obtain conditions analogous to S_1^* , S_2^* where the $\| \cdot \|_k$ are replaced by the dual (semi-) norms for a fundamental system of absolutely convex bounded sets in an arbitrary (DF)-space E , the $\| \cdot \|_k^*$ come from the canonical norms in $\mathcal{N}(\Omega)$. Hence we are able to prove for elliptic $P(D)$ and an arbitrary (DF)-space E that $H^1(\Omega, \mathcal{N}^E) = 0$ iff E has property (A). Due to 2.5. (b) and e.g. [12], 3.3. we know the sufficiency of (A) in this case.

Finally we want to remark that the sufficiency part of 2.6. implies that $P(D)$ has a continuous linear right inverse on every continuously imbedded subspace $F \subset C^\infty(\Omega)$ with property (DN).

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